Lévy Processes
in Credit Risk and Market Models

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Preface

During the last years banks have realized that default risk cannot be neglected. The subjective assessment of credit worthiness has to be replaced with a more objective way of estimating default risk.

Mathematical credit risk models in the literature are mainly models based on Brownian motion although it is known that real-life financial data provides a different statistical behavior than that implied by these models. Lévy processes are an appropriate tool to increase accuracy of models in finance. They have been used to model stock prices, and term structures of interest rates, thus allowing more accurate derivative pricing and risk management. In this study we discuss how general Lévy processes can be applied to credit risk and market models. There are also a few jump-diffusion models where the jump-components have sample paths of bounded variation, but some of the very important subclasses are special types of Lévy processes.

The emphasis in this study lies on the application of Lévy processes in credit risk models. We will mainly deal with the question of pricing defaultable corporate bonds. However we will also consider a portfolio model where the main question is the modeling of the loss distribution. Furthermore, we present a Lévy Libor model which contains the classical Libor model as a special case.

In Chapter 1 we briefly survey some aspects of credit risk, generalized hyperbolic distributions and Lévy processes. In the overview of the structural approach, we show how Lévy processes can be used to generalize the classical structural approach due to Merton (1974). In Chapter 2 we present and generalize the approach of the commercial software package CreditRisk$^{+\text{TM}}$ by Credit Suisse Financial Products. This chapter might be of some interest to practitioners since we introduce an alternative distribution to CreditRisk$^{+\text{TM}}$ which allows for heavier tails and still guarantees analytical tractability.

Then we introduce a credit risk Heath-Jarrow-Morton-framework based on Lévy processes in Chapter 3, where we generalize the Gaussian approach in Bielecki and Rutkowski (1999, 2000). With a view to more realistic modeling we also include the topic of reorganization within this framework, which is based on an idea of Schönbucher (1998, 2000a).

In Chapter 4 we show how the market practice of pricing caplets with the Black formula can be pushed further to a model where the driving process is a

Lévy process.

At this point I want to express my warmest thanks to my academic teacher and supervisor Prof. Dr. Ernst Eberlein. I am most indebted to him for introducing me to financial mathematics and to the theory of Lévy processes and also for numerous encouraging discussions which gave me new ideas and led to new insights. Actually I have learned a lot more than this thesis contains.

I also want to thank Thomas Goll for studying and discussing together some chapters of Jacod and Shiryaev (1987) and other mathematical texts. Further thanks go to my other colleagues from the Institut für Mathematische Stochastik, Albert-Ludwigs-Universität Freiburg, and of the Freiburg Center for Data Analysis and Modeling, especially Thomas Gerds and Jan Beyersmann. Financial support by the Deutsche Forschungsgemeinschaft (DFG) is gratefully acknowledged. I want to thank our secretary Monika "The Eye" Hattenbach for finding inconsistencies in the layout and the bibliography.

Furthermore, I want to thank my mother Ayşe Özkan and my sister Meltem Özkan. Very special thanks go to Verena Trenkner.

Contents

Chapter 1

Introduction

1.1 Mathematical approaches

A default-free zero-coupon bond is a financial security paying its owner one currency unit at a prespecified date T in the future. A defaultable bond, usually a corporate bond, is a financial security *promising* its owner to pay one currency unit at a prespecified maturity date T in the future. In contrast to the case of default-free bonds the issuer of a credit risky bond will default with a certain probability before or at time T. In case of default the holder of such a bond will receive only a fractional amount of the promised currency unit or nothing at all.

Roughly speaking, models for credit risk can be divided into two categories: structural models, see Merton (1974), where the quantity one models is the total value of the firm, and intensity based models, where default itself is modeled, namely typically as a jump-process (see e.g. Jarrow and Turnbull (1995)). Another group are portfolio models, where the loss distribution of the portfolio is of special interest. We give a short overview to these types of models. Of course, the distinction between these approaches is not always strict.

1.1.1 Structural models

Credit risk is the risk that an obligor cannot meet his obligations because the amount of liabilities due surpasses the amount of funds available. Since the funds available as well as the liabilities are stochastic variables, one way of modeling credit risk is the modeling of these variables as stochastic processes.

The value-of-firm based models go back to the famous Nobel-prized article of Black and Scholes (1973) and to Merton (1974). These models are also called 'structural' or 'classical' models – 'structural' because the variables underlying the model are imposed in the firm's balance sheet.

The main idea is that the debt of a firm can be interpreted as a combination of a default-free bond and a put option on the value of the firm. We consider a firm which is financed by equity and zero coupon debt with face value K maturing at date T.

We assume a stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. The measure $\mathbf{P}$ is the real-world measure. The value of the firm at time t is denoted by V_t, which we model as an adapted exponential Lévy process

$$V_t = V_0 \exp(L_t). \tag{1.1}$$

In the original paper of Merton (1974) the Lévy process L is a Brownian motion with drift, more precisely $L_t = (\mu - \sigma^2/2)t + \sigma W_t$, where $\mu \in \mathbb{R}, \sigma > 0$ and W is a standard Brownian motion. However, the calculations can be done analogously for an arbitrary Lévy process. A Lévy process is essentially a process with independent and stationary increments. We give details and references in Section 1.3. For example, we may take L as a generalized hyperbolic Lévy process, see Eberlein and Keller (1995), or Eberlein and Prause (2002), or we may use the variance gamma model as in Madan (2000). Concerning the market we make the same assumptions as in the original paper, see Merton (1974, Assumption A1 – A7).

At maturity T the firm is obliged to pay the amount K to the bond holder. If the value of the firm at time T is less than the debt K, the firm will not be able to meet its obligations, i.e. the firm holders have to abandon the company. This means at time T the bond holder receives

$$\begin{aligned} \min(V_T, K) &= \max(K, 0) - \max(K - V_T, 0) \\ &= K - (K - V_T)^+. \end{aligned} \tag{1.2}$$

Default is only triggered, when the value of the firm at maturity date T, V_T, is less than the firm's debt K. In this basic model default cannot occur before the maturity date T. The payoff to the bond holder at T equals the payoff of a default-free bond with face value K minus a European put option on the value of the firm with strike K. In other words, purchasing a note from a risky firm is the same as purchasing a risk-free bond and simultaneously selling a put option to the borrower that states that the bond holder will buy the firm for the amount of the note. The recovery rate, i.e. the fraction of amount that the bond holder receives in case of default, is V_T/K. Hence, by equation (1.2), the value of the bond at time t is

$$K \exp(-r(T-t)) - \mathrm{Put}(V_t, K, T-t, \mathbf{Q}), \tag{1.3}$$

where r is the interest rate, which is assumed to be constant over time and the second term is the price of the put option, i.e.

$$\mathrm{Put}(V_t, K, T-t, \mathbf{Q}) = \mathbb{E}_{\mathbf{Q}}[\mathrm{e}^{-r(T-t)}(K - V_T)^+ \,|\, \mathcal{F}_t],$$

and $\mathbf{Q}$ is an equivalent martingale measure. In the Merton-approach there is a unique martingale measure $\mathbf{Q}$ and the price of the put is calculated with the classical Black-Scholes option formula. In the general Lévy setting there is usually no unique martingale measure. In the generalized hyperbolic setting, one can take for example the Esscher transform, see e.g. Eberlein and Keller (1995) or Eberlein and Prause (2002).

The value of the put option reflects the risk of default. The higher the default risk, the higher is the value of the option, i.e. the more the firm holder has to pay to the lender to 'convince' him. An important credit risk measure is the probability that a default will occur given the information at time t under the real-world measure P, denoted by $\mathrm{Def}_{\mathbf{P}}(t)$. It can easily be calculated that this probability is given by

$$\mathrm{Def}_{\mathbf{P}}(t) = F_{\mathbf{P}^{L_{T-t}}}(\log(K/V_t)),$$

where $F_{\mathbf{P}^{L_{T-t}}}$ is the cumulative distribution function of the law of L_{T-t}, $\mathbf{P}^{L_{T-t}}$. This can be seen as follows. Due to the independent and stationary increments of L we have

$$\begin{aligned}\mathrm{Def}_{\mathbf{P}}(t) &= \mathbf{P}\left(V_T < K \mid \mathcal{F}_t\right) = \mathbf{P}\left(V_t \exp(L_T - L_t) < K \mid \mathcal{F}_t\right) \\ &= \mathbf{P}(L_T - L_t < \log\left(K/V_t\right)) = F_{\mathbf{P}^{L_{T-t}}}(\log(K/V_t))).\end{aligned}$$

In the Merton model, we have

$$\mathrm{Def}_{\mathbf{P}}(t) = \mathrm{N}\left((\sigma\sqrt{T-t})^{-1}\left(\ln\left(K/V_t\right) - (\mu - \frac{1}{2}\sigma^2)(T-t)\right)\right),$$

where N is the cumulant standard normal distribution function, see e.g. Jarrow and van Deventer (1999) or Crouhy, Galai, and Mark (2000). Note that in the classical setting we have $\mathrm{Def}_{\mathbf{P}}(t) = \mathrm{N}(-d)$, where d is known as the "distance-to-default". In the generalized hyperbolic setting there is no closed form expression for the cumulant distribution function, but we can calculate the distribution of L_{T-t} numerically by efficient fast Fourier methods. If the Lévy process remains a Lévy process under the measure $\mathbf{Q}$, then the risk-neutral default probability can be calculated in exactly the same way as under the real-world measure.

In Figure 1.1 we simulated paths of the value of the firm by using stock data and compare the probability of default in the classical Merton model and the generalized hyperbolic model. Figure 1.2 shows the ratio of the default probabilities. When the time to maturity $T-t$ is large the difference between the Merton and the generalized hyperbolic model can be neglected. However, close to maturity the distribution of the increments of L has much less mass in the tails in the Merton model than in the generalized hyperbolic model. For example the normal distribution is close to maturity more optimistic than the generalized hyperbolic distribution when the value of the firm is strictly greater than K. These facts are well known from option pricing theory, see e.g. Eberlein and Prause (2002).

There are several generalizations of the Merton model. Black and Cox (1976) consider different classes of seniority and also include the possibility of default

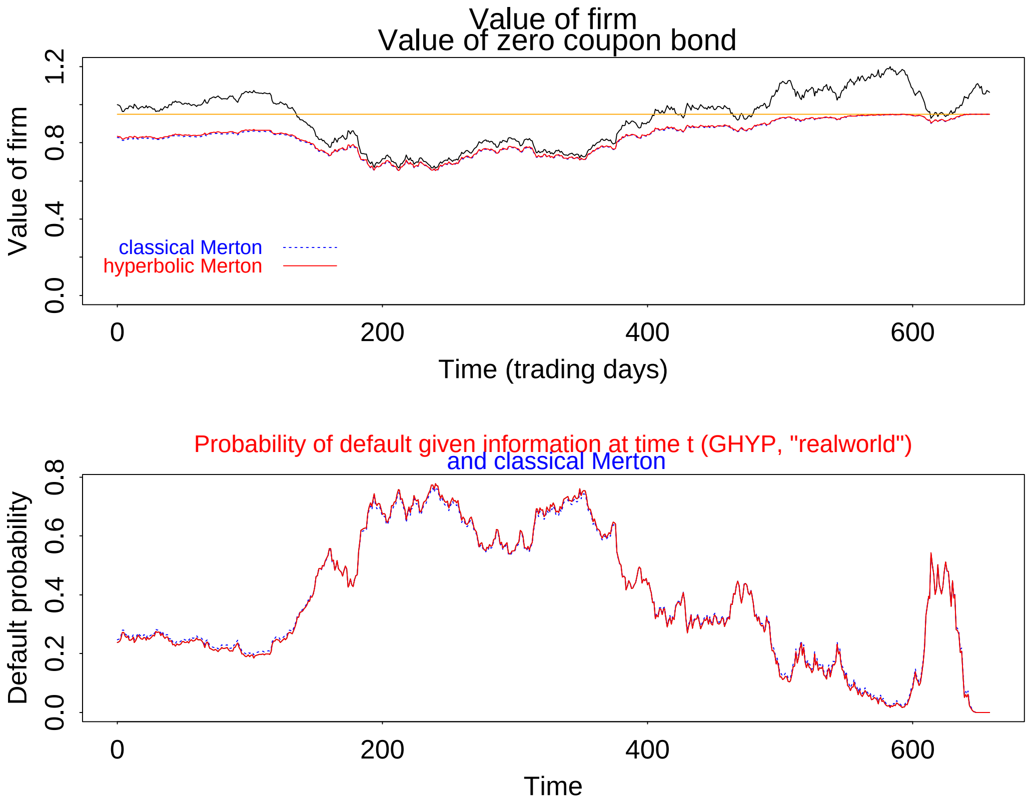


Figure 1.1: Classical Merton vs. generalized hyperbolic Merton.

before the maturity T. The default time is modeled as a first passage time over a given boundary. Unfortunately, it seems to be impossible to determine the distribution of a crossing time for general Lévy processes. The distributions of first hitting times are only known in some very special cases, such as Lévy processes with bounded variation, see e.g. Gusak and Koralyuk (1968), or in case of a Brownian motion with constant drift, see e.g. Harrison (1985).

Geske (1977) prices defaultable bonds paying coupons by using a compound option approach. The cost of deposit insurance is discussed in Merton (1977, 1978), and Ronn and Verma (1986).

Further work in this area has been done by Longstaff and Schwartz (1995). The Longstaff-and-Schwartz model allows for stochastic interest rates and correlation between interest rates and the value of the firm. Their model implies that credit spreads are strongly negatively correlated to the level of interest rates. This is consistent with data placed at disposal by Moody's. This fact is also affirmed in Duffee (1998), where negative correlation between credit spreads and U.S. Treasury yields is documented.

Jump-diffusion approaches substituting the geometric Brownian motion in the Merton model are considered in Schönbucher (1996) and Zhou (1997). The

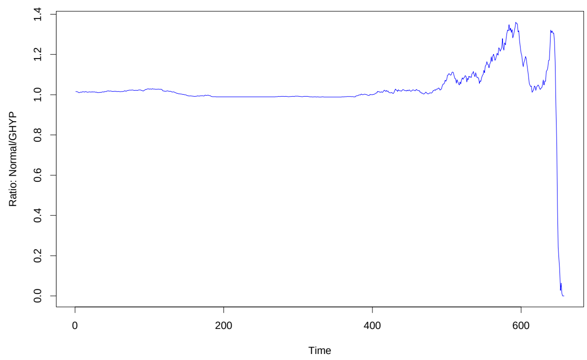


Figure 1.2: Ratio of the default probabilities according to Figure 1.1.

motivation in Zhou is an empirical investigation by Jones, Mason, and Rosenfeld (1984). They show that credit spreads on corporate bonds are too high to be matched by the classical approach.

Structural models can also be applied to value the impact of credit risk on the pricing of options. Johnson and Stulz (1987), and Hull and White (1995), use extended Merton models to price 'vulnerable' options. These are options on the firm's bonds, written by a third party, who could also default. An empirical investigation is done in Anderson and Sundaresan (2000). They found that structural credit risk models track credit spreads very well during some periods but fail during other periods.

The structural approach can also be applied to sovereign risk, see e.g. Karmann and Plate (2000). Brady bonds are used to estimate the probability of default of a country in Claessens and Pennacchi (1996).

Another application is the topic of optimal capital structure of a firm, see e.g. Brennan and Schwartz (1978), Leland (1994), Leland and Toft (1996). Recent articles on this topic are Duffie and Lando (2001), Uhrig-Homburg (2001) and Hilberink and Rogers (2002). The first paper studies implications of imperfect information, the second article endogenizes the firm's default point assuming that equity issuance is costly. The latter includes a generalization where jumps of the firm value are generated by a spectrally negative Lévy process, i.e. a Lévy process with sample paths of bounded variation with no positive jumps.

Credit derivatives and stochastic liabilities are modeled in Ammann (1999).

1.1.2 Intensity-based models

In intensity-based models the time of default is specified usually as the jump time of a one-jump process. The intensity-based approaches are often formulated in terms of the hazard rate and do not necessarily rely on the value of the firm. For the discussion of the hazard rate with a view to credit risk models we refer to

Jeanblanc and Rutkowski (2000) and Blanchet-Scalliet and Jeanblanc (2002).

The first approaches in intensity-based models have been made in Pye (1974), Ramaswamy and Sundaresan (1986), Litterman and Iben (1991), and Jarrow and Turnbull (1995). In the Jarrow-Turnbull approach the interest rate is constant and the default time is exponentially distributed with parameter λ. Jarrow and Turnbull consider the 'promised dollars' of corporate bonds as a foreign currency. Other intensity-based models have been developed in Duffie, Schroder, and Skiadas (1996), Duffie and Singleton (1999a), and Lando (1998). In the latter paper the dependence between stochastic intensities and the spot rate are handled.

The mathematical background for the intensity-based models are in general default-free term structure models. Most of the models, we mentioned above build on short-rate models. But there are also a few models which build on the more general Heath-Jarrow-Morton framework (HJM), see Heath, Jarrow, and Morton (1992). In Björk, Kabanov, and Runggaldier (1997) the HJM-framework which is based on Brownian motions has been pushed further to jump-diffusions already with a view to applications in credit risk. In Eberlein and Raible (1999) the HJM-framework has been developed further to Lévy processes and in Björk, Di Masi, Kabanov, and Runggaldier (1997) to semimartingales. The latter also discusses important questions of martingale measures and completeness of the market. An overview of default-free term structure models can be found for example in Björk (1998), or Musiela and Rutkowski (1998).

Credit ratings are key inputs to the more advanced intensity-based models of Jarrow, Lando, and Turnbull (1997), Thomas, Allen, and Morkel-Kingsbury (1998), Lando (1998, 2000), Schönbucher (2000a), Bielecki and Rutkowski (1999, 2000). The Jarrow-Turnbull approach is refined in the work of Jarrow, Lando, and Turnbull (1997) by taking the fact into account that the credit quality of the obligor may change over time. This is done by including the information of the rating of the company, which is modeled by a Markov chain with state space $\mathcal{K} = \{1, \dots, K\}$, where 1 corresponds to the best possible rating different from default-freeness, and K is the state of default.

There are only a few examples of intensity-based approaches which do not restrict themselves to continuous models based on Brownian motion. Examples are Duffie and Singleton (1999a), Schönbucher (2000a, Section 2.7), or Hoogland, Neumann, and Vellekoop (2001), but the emphasis lies always on jump-diffusions, where the jump component has sample paths of bounded variation. We go one step further in direction of more realistic assumptions and consider models based on Lévy processes.

In Chapter 3 we generalize the Bielecki-Rutkowski approach to construct an intensity-based term structure model based on Lévy processes. We also include the topic of restructuring after default, since in practice a default does not necessarily lead to the liquidation of the company. The topic of reorganization in a mathematical setting has first been introduced in Schönbucher (1998, 2000a).

1.1.3 Portfolio approaches

Portfolio approaches are usually based on simulation techniques. A portfolio model based on the Merton approach which is designed to take care of market risk has been developed in Iscoe, Kreinin, and Rosen (1999) and is discussed in Kafetzaki-Boulamatsis and Tasche (2001). A related work which builds on the Longstaff-and-Schwartz model can be found in Barnhill and Maxwell (2002). A methodology for scenario simulation for credit risk within a multi-currency portfolio can be found in Jamshidian and Zhu (1997). An analytical approximation technique for the value at risk of a portfolio which includes defaultable securities is developed in Duffie and Pan (2001).

Recent papers study different methodologies to include dependent defaults. Jarrow and You (2001) model dependence via changing the default intensity. Correlated jump-intensity processes are considered in Duffie and Singleton (1999b). Dependence structures modeled by copulas can be found e.g. in Frey and McNeil (2001), Schönbucher and Schubert (2001), or Giesecke (2001). A different approach based on Polya's urn scheme is developed in Schmock and Seiler (2002).

This overview of mathematical approaches to credit risk is far from being complete. We refer to the following (incomplete list of) surveys for further references and also for the detailed discussion of the pros and contras of the different approaches: Cooper and Martin (1996), Lando (1997), Altman and Saunders (1998), Ammann (1999), Jeanblanc and Rutkowski (1999), Jarrow and Turnbull (2000), Madan (2000), Schönbucher (2000a), and Bielecki and Rutkowski (2002).

1.2 Practical approaches

In general, commercial software packages for credit risk management estimate the distribution of losses faced by a holder of a portfolio of credit risky securities. The pricing of defaultable securities under a risk-adjusted measure is in general not considered. These approaches are closely related to the portfolio approaches in the last subsection.

Some of the widely-used credit risk assessment software packages are CreditMetrics$^{\mathrm{TM}}$ (by J.P. Morgan), CreditRisk$^{+\mathrm{TM}}$ (by Credit Suisse Financial Products) and the KMV approach including CreditMonitor$^{\mathrm{TM}}$ and PortfolioManager$^{\mathrm{TM}}$. Other commercial credit risk products have been developed by McKinsey, Midas Kapiti International, Kamakura, and many other firms. CreditMetrics$^{\mathrm{TM}}$ is a rating-based model, CreditRisk$^{+\mathrm{TM}}$ is an actuarial or intensity-based model, and KMV builds on the Merton approach, where the distance-to-default plays an important role for appreciating default risk.

Some of the software packages permit a closer look, while others are 'black boxes'. Credit Suisse Financial Products and J.P. Morgan have released their approaches freely to the public. A survey on CreditMetrics^{TM} and CreditRisk^{+TM}, two of the benchmarks for credit risk analysis, can be found for example in Nelken (1999). Broeker and Rolfes (1998) note that CreditRisk^{+TM} is preferable for illiquid loan portfolios because of lower data requirements, while CreditMetrics^{TM} appears more suitable for liquid corporate bonds, because information on ratings can be used. A detailed comparison of CreditMetrics^{TM} and CreditRisk^{+TM} was done by Gordy (2000). Gordy shows that the underlying probabilistic structures of both software packages are similar. A description of the different approaches by J.P. Morgan, Credit Suisse Financial Products, KMV, and McKinsey can be found in Crouhy, Galai, and Mark (2000). Koyluoglu and Hickman (1998) show that the approaches used in these products can be summarized under a general framework if one examines only the default component of portfolio credit risk. Broeker and Rolfes (1998) propose a way to include ratings in CreditRisk^{+TM} and show that the loss distribution is very similar to the one calculated in CreditMetrics^{TM}. Some aspects of the McKinsey-approach can be found in Wilson (1998). An empirical investigation, where the focus is on the credit spreads in the KMV/Merton model, is done in Bohn (1999). Bürgisser, Kurth, and Wagner (2001) and Giese (2002) handle inter alia correlated defaults in the CreditRisk^{+TM}-framework. Kern and Rudolph (2001) present an empirical comparison of CreditMetrics^{TM}, CreditRisk^{+TM}, and CreditPortfolioView^{TM} (McKinsey) with regard to middle market loan portfolios. Haaf and Tasche (2002) investigate contributions of single obligors using the Value-at-Risk and expected shortfall as risk measures in the CreditRisk^{+TM}-approach.

1.3 Lévy processes

Lévy processes play a fundamental role in the theory of mathematical finance as well as in other fields as physics or telecommunications, see e.g. Barndorff-Nielsen, Mikosch, and Resnick (2001). The first stock model ever was a Brownian motion and goes back to Bachelier (1900). The Brownian motion is the only nondeterministic Lévy process with continuous sample paths. Lévy processes include a number of standard processes as special subclasses, e.g. Brownian motion, Poisson and compound Poisson processes, or stable processes. Note that the sum of a Brownian motion and a compound Poisson process is again a Lévy process, so that Lévy processes contain also this very important class of jump-diffusions.

The results that we present here can be found in the monographs Bertoin (1996), Sato (1999), and Applebaum (2002). Related books are for example Jacod and Shiryaev (1987) and Protter (1995).

We assume as given a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, such that the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfies the *usual conditions*, see for example Jacod and Shiryaev (1987).

A Lévy process on $\mathbb{R}^d$ is defined as follows.

Definition 1.1 *An adapted stochastic process $L = (L_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ is a Lévy process if the following conditions are satisfied:*

1. $L_0 = 0$ *a.s.*

2. *L has independent increments, i.e.*
   $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t < \infty$.

3. *L has stationary increments, i.e.*
   for any $s, t > 0$ the distribution of $L_{t+s} - L_t$ does not depend on t.

4. *L is stochastically continuous, i.e.*
   for every $t \geq 0$ and $\varepsilon > 0$: $\lim_{s\to t} \mathbf{P}(|L_s - L_t| > \varepsilon) = 0$.

The definition of a Lévy process can also be formulated without using a filtration explicitly, see Sato (1999, Definition 1.6). There is a unique modification of L which is càdlàg (*continue à droite avec des limites à gauche*, that means: right-continuous sample paths with existing left-hand limits), and which is again a Lévy process, see e.g. Theorem I.30 in Protter (1995). We will always work with this modification.

Due to the stationary and independent increments every Lévy process on $\mathbb{R}^d$ is fully characterized by an infinitely divisible probability distribution which equals $\mathbf{P}^{L_1}$, the law of L_1. The well-known Lévy-Khintchine formula determines the characteristic function of infinitely divisible distributions:

$$\begin{aligned}\phi_{L_1}(z) &:= \mathbb{E}[\exp(i\langle z, L_1\rangle)]\\ &= \exp\left(i\langle b, z\rangle - \frac{1}{2}\langle z, cz\rangle + \int_{\mathbb{R}^d} \left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{1}_D(x)\right) \nu(\mathrm{d}x) \right), \quad z \in \mathbb{R}^d,\end{aligned} \tag{1.4}$$

where $D := \{y \in \mathbb{R}^d : |y| \leq 1\}$, $\langle x, y\rangle := \sum_{j=1}^d x_j y_j$ is the Euclidean scalar product; $b \in \mathbb{R}^d$, c is a symmetric nonnegative-definite $d \times d$ matrix, and ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$. Because

$$\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{1}_{\{|x|\leq 1\}} = \begin{cases} O(|x|^2), & \text{as } |x| \to 0,\\ O(1), & \text{as } |x| \to \infty,\end{cases}$$

it is also required that ν satisfies $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\, \nu(\mathrm{d}x) < \infty$, so it is ensured that $x \mapsto \mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{1}_{\{|x|\leq 1\}}$ is integrable with respect to ν.

Remark: Due to the properties of the Lévy process we have $\phi_{L_t} = (\phi_{L_1})^t$.

Definition 1.2 *The triplet* (b, c, ν) *is called the* generating triplet *of* L_1 *or of the corresponding Lévy process* $L = (L_t)_{t \geq 0}$. *The matrix* c *is called the* Gaussian covariance matrix, *and* ν *is called its* Lévy measure.

Remark: Knowledge of the generating triplet (b, c, ν) is sufficient to determine some very fundamental sample path properties of the Lévy process L. For example, the sample paths of L are continuous (a.s.) if and only if $\nu = 0$. If $\nu(\mathbb{R}^d) = \infty$, then, (a.s.) jumping times are countable and dense in $[0, \infty)$. It can also be seen from the generating triplet, whether a Lévy process has sample paths of bounded variation or not: L has sample paths of infinite variation on any time interval $(0, t]$ if and only if $c \neq 0$ or $\int_{\{|x| \leq 1\}} |x| \, \nu(\mathrm{d}x) = \infty$. These properties as well as further ones can be found in Sato (1999, Section 21).

Obviously, there is a connection between the jumps of L and the Lévy measure ν. Explicitly, if we introduce the integer-valued random measure of jumps associated with L

$$\mu^L(\omega; \mathrm{d}t, \mathrm{d}x) := \sum_s \mathbb{1}_{\{\triangle L_s \neq 0\}}(\omega) \varepsilon_{(s, \triangle L_s(\omega))}(\mathrm{d}t, \mathrm{d}x), \tag{1.5}$$

where $\triangle L_s = L_s - L_{s-}$ and ε_a is the Dirac measure in a. Note that the sum on the right-hand side is well defined because of the càdlàg property of sample paths. Then μ^L is a Poisson random measure and

$$\nu^L(\mathrm{d}t, \mathrm{d}x) := \mathrm{d}t \, \nu(\mathrm{d}x) \tag{1.6}$$

is its intensity measure. For any $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$, $\mu^L(.; B)$ has Poisson distribution with mean $\nu^L(B)$ (possibly 0 or $+\infty$), see Sato (1999, Theorem 19.2) or Jacod and Shiryaev (1987, II.2.6.(iii) and II.4.20).

Note that the definition of μ^L is valid for any adapted càdlàg $\mathbb{R}^d$-valued process, see Jacod and Shiryaev (1987, II.1.16) and that in case that L is an additive process μ^L is a Poisson random measure, too. An additive process satisfies the same conditions as a Lévy process but does not necessarily have stationary increments, see Sato (1999).

Let us mention, that if $\int_{D^c} |x| \, \nu(\mathrm{d}x) < \infty$, or equivalently $\mathbb{E}[|L_1|] < \infty$, the Lévy-Khintchine formula can be rewritten as

$$\phi_{L_1}(z) = \exp\left(i\langle b_1, z\rangle - \frac{1}{2}\langle z, cz\rangle + \int_{\mathbb{R}^d} \left(\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\right) \nu(\mathrm{d}x) \right), z \in \mathbb{R}^d, \tag{1.7}$$

where $b_1 = \mathbb{E}[L_1]$, and c and ν are invariant to equation (1.4). In Chapter 3 and 4 we will work with stochastic integrals with respect to Lévy processes. The mathematical background for stochastic integrals customized to Lévy processes can be found in Applebaum (2002) or for the more general semimartingales in Jacod and Shiryaev (1987).

1.4 Generalized hyperbolic distributions

As mentioned above, it is sufficient to know the distribution of L_1 to determine the Lévy process $L = (L_t)_{t\geq 0}$. We say that L is a generalized hyperbolic Lévy motion, if the law of L_1 is generalized hyperbolic. Generalized hyperbolic distributions have been introduced in Barndorff-Nielsen (1977), and have found many successful applications in the last years.

Definition 1.3 *The Lebesgue density of a one-dimensional generalized hyperbolic distribution is given by*

$$\begin{aligned}\mathrm{ghyp}(x;\lambda,\alpha,\beta,\delta,\mu) &= a(\lambda,\alpha,\beta,\delta)(\delta^2+(x-\mu)^2)^{0.5(\lambda-0.5)}\\ &\quad\times \mathrm{K}_{\lambda-0.5}(\alpha\sqrt{\delta^2+(x-\mu)^2})\exp(\beta(x-\mu)), x\in\mathbb{R},\end{aligned}$$

where $$a(\lambda,\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{0.5\lambda}}{\sqrt{2\pi}\alpha^{\lambda-0.5}\delta^{\lambda}\mathrm{K}_\lambda(\delta\sqrt{\alpha^2-\beta^2})}$$

and where K_λ *is a modified Bessel function of the third kind with index* λ. *The five parameters are assumed to satisfy the following conditions:*

$$\begin{aligned}&\lambda,\mu\in\mathbb{R},\\ &\lambda<0 \quad\Rightarrow\quad \delta>0, |\beta|\leq\alpha,\\ &\lambda=0 \quad\Rightarrow\quad \delta>0, |\beta|<\alpha,\\ &\lambda>0 \quad\Rightarrow\quad \delta\geq 0, |\beta|<\alpha.\end{aligned}$$

The generalized hyperbolic distribution can be obtained as a mean-variance mixture of the normal and the generalized inverse Gaussian distribution. The generalized inverse Gaussian distribution plays an important role in Chapter 2.

Definition 1.4 *The generalized inverse Gaussian distribution is defined by the following Lebesgue density.*

$$\mathrm{gig}(x;\lambda,\psi,\chi) = \frac{(\psi/\chi)^{0.5\lambda}}{2\mathrm{K}_\lambda(\sqrt{\psi\chi})}x^{\lambda-1}\exp\left(-\frac{1}{2}(\chi x^{-1}+\psi x)\right)\mathbb{1}_{\mathbb{R}^+}(x). \tag{1.8}$$

Then we have the following mixture representation for the generalized hyperbolic distribution

$$\mathrm{ghyp}(x;\lambda,\alpha,\beta,\delta,\mu) = \int_0^\infty \mathrm{normal}(x;\mu+\beta y,y)\ \mathrm{gig}(y;\lambda,\delta^2,\alpha^2-\beta^2)\ \mathrm{d}y, \tag{1.9}$$

where $\mathrm{normal}(.;\mu,\sigma^2)$ is the Lebesgue density of the normal distribution with mean μ and variance σ^2.

Important subclasses are the hyperbolic and the normal inverse Gaussian (NIG) distribution.

Definition 1.5 *For $\lambda = 1$ we get the hyperbolic distribution given by its Lebesgue density*

$$\mathrm{hyp}(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta \mathrm{K}_1(\delta\sqrt{\alpha^2-\beta^2}\,)} \exp\left(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right), \tag{1.10}$$

where $\mu \in \mathbb{R}, \delta \geq 0$ and $|\beta| < \alpha$.

Definition 1.6 *For $\lambda = -0.5$ we get the normal inverse Gaussian distribution. Its Lebesgue density is*

$$\mathrm{nig}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi} \exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right) \frac{\mathrm{K}_1(\alpha\sqrt{\delta^2+(x-\mu)^2}\,)}{\sqrt{\delta^2+(x-\mu)^2}}, \tag{1.11}$$

where $\mu \in \mathbb{R}, \delta \geq 0$ and $|\beta| < \alpha$.

The moment generating function $u \mapsto \int \mathrm{e}^{ux}\,\mathrm{ghyp}(x;\lambda,\alpha,\beta,\delta,\mu)\,\mathrm{d}u$, $|\beta+u| < \alpha$, plays an important role in financial mathematics. Prause (1999, Lemma 1.13) shows that the moment generating function in the generalized hyperbolic case is given by

$$\mathrm{M}_{\mathrm{ghyp}}(u;\lambda,\alpha,\beta,\delta,\mu) = \mathrm{e}^{u\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{0.5\lambda} \frac{\mathrm{K}_\lambda(\delta\sqrt{\alpha^2-(\beta+u)^2}\,)}{\mathrm{K}_\lambda(\delta\sqrt{\alpha^2-\beta^2}\,)}. \tag{1.12}$$

The moment generating function of the normal inverse Gaussian distribution is

$$\mathrm{M}_{\mathrm{nig}}(u;\alpha,\beta,\delta,\mu) = \exp\left(\mu u + \delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+u)^2}\right)\right). \tag{1.13}$$

Thus, it is clear that, if $X_1,\ldots,X_n$ are NIG-distributed according to $X_i \sim \mathrm{NIG}(\alpha,\beta,\delta_\mathrm{i},\mu_\mathrm{i})$, then the sum is again normal inverse Gaussian: $X_1+\ldots+X_n \sim \mathrm{NIG}(\alpha,\beta,\sum_{\mathrm{i}=1}^{\mathrm{n}}\delta_\mathrm{i},\sum_{\mathrm{i}=1}^{\mathrm{n}}\mu_\mathrm{i})$.

The characteristic function ϕ of the generalized hyperbolic distribution is given by

$$\phi_{\mathrm{GH}}(u) = \exp\left(i\mu u + \frac{\lambda}{2}\log\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)\right)\frac{\mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{\mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}. \tag{1.14}$$

In the NIG case the characteristic function simplifies enormously:

$$\phi_{\mathrm{NIG}}(u) = \exp\left(i\mu u + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right)\right). \tag{1.15}$$

According to Hawkes, see Rogers and Williams (1994, Chapter I, Theorem (30.3)), the existence of a classical local time, which is defined as the Lebesgue density of the following measure m, where

$$m(A) = \int_0^\infty \mathrm{e}^{-t}\mathbb{1}_A(L_s)\,ds, A \in \mathcal{B}(\mathbb{R})$$

is equivalent to

$$\int_0^\infty Re\left(\frac{1}{1-\psi(u)}\right)\,\mathrm{d}u < \infty, \tag{1.16}$$

where $\phi(u) = \exp(\psi(u))$ and $Re(z)$ is the real part of a complex number $z \in \mathbb{C}$. Obviously, in the NIG case $\psi(u)$ is asymptotically linear in u. Thus, the integral in equation (1.16) does not converge. Note that although the local time in the classical sense does not exist, the local time in the more general sense is defined, see Protter (1995, Chapter IV.5). The non-existence of the classical local time is typical for processes with sample paths of bounded variation. An exception to this rule is, besides of the NIG case, the Cauchy process. Here a link between the generalized hyperbolic distribution and the Cauchy distribution can be suspected. The Lebesgue density of the Cauchy distribution with parameters c and μ is given by $\mathrm{cauchy}(x; c, \mu) = \frac{1}{\pi}\frac{c}{c^2+(x-\mu)^2}, c > 0, \mu \in \mathbb{R}$.

By Blæsild (1999), cited in Prause (1999), it has only be shown that the Cauchy distribution with parameters $c = 1$ is a limiting case. But also the general Cauchy distribution appears to be a limiting distribution.

Proposition 1.7 *The Cauchy distribution with parameters δ and μ is obtained as a limiting case of the generalized hyperbolic distribution for $\lambda = -\frac{1}{2}$, $\alpha, \beta \to 0$.*

Proof: This can be seen directly from the characteristic function in equation (1.15), which converges for $\alpha, \beta \to 0$ to $\phi_{\mathrm{cauchy}}(u) = \exp(i\mu u + \delta|u|)$, the characteristic function of the Cauchy distribution. Alternatively, one can also consider the densities itself. We define

$$a(\lambda,\alpha,\beta,\delta) := \frac{\left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\lambda}}{\sqrt{2\pi}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)},$$

$$b(x;\lambda,\alpha,\beta,\delta,\mu) := \frac{K_{\lambda-0.5}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\frac{1}{\alpha}\sqrt{\delta^2+(x-\mu)^2}\right)^{1/2-\lambda}},$$

$$c(x;\beta,\mu) := \exp\left(\beta\left(x-\mu\right)\right).$$

Thus $\text{ghyp}(x;\lambda,\alpha,\beta,\delta,\mu) = a(\lambda,\alpha,\beta,\delta)b(x;\lambda,\alpha,\beta,\delta,\mu)c(x;\beta,\mu)$. We show that each of the functions on the right-hand side converge. Then the left-hand side converges to the product of the limits of the right-hand side.

It is clear that for $\beta \to 0$ the function $c(x;\beta,\mu)$ converges for all $x \in \mathbb{R}$ to 1.

Next we calculate the limit for $a(\lambda,\alpha,\beta,\delta)$. The Bessel function K_λ is symmetric in λ, i.e.

$$K_\lambda(\gamma) = K_{-\lambda}(\gamma).$$

For $\gamma \searrow 0$ and $\lambda > 0$, we have

$$K_\lambda(\gamma) \simeq \Gamma(\lambda)2^{\lambda-1}\gamma^{-\lambda},$$

see e.g. Antosiewicz (1968). Note that in our case $-\lambda > 0$.

Therefore, $K_\lambda(\delta\sqrt{\alpha^2-\beta^2}) = K_{-\lambda}(\delta\sqrt{\alpha^2-\beta^2})$ and $K_{-\lambda}(\delta\sqrt{\alpha^2-\beta^2}) \simeq \Gamma(-\lambda)2^{-\lambda-1}\delta^\lambda\sqrt{\alpha^2-\beta^2}^{\lambda}$.

Hence

$$\begin{aligned} a(\lambda,\alpha,\beta,\delta) &= \frac{\left(\frac{\sqrt{\alpha^2-\beta^2}}{\delta}\right)^{\lambda}}{\sqrt{2\pi}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \\ &\simeq \frac{(\sqrt{\alpha^2-\beta^2})^\lambda}{\sqrt{2\pi}\delta^\lambda\Gamma(-\lambda)2^{-(\lambda+1)}\delta^\lambda(\sqrt{\alpha^2-\beta^2})^\lambda} \\ &= \frac{2^{\lambda+1}}{\sqrt{2\pi}\delta^{2\lambda}\Gamma(-\lambda)} \\ &= \frac{\delta}{\pi}. \end{aligned}$$

Next, we calculate the limit of $b(x;\lambda,\alpha,\beta,\delta,\mu)$ for $\alpha,\beta \to 0$. Due to the symmetry in λ we have that

$$K_{\lambda-0.5}(\alpha\sqrt{\delta^2+(x-\mu)^2}) = K_{0.5-\lambda}(\alpha\sqrt{\delta^2+(x-\mu)^2}).$$

With the same arguments as above we get

$$
\begin{aligned}
b(x;\lambda,\alpha,\beta,\delta,\mu) &= \frac{K_{0.5-\lambda}(\alpha\sqrt{\delta^2+(x-\mu)^2}\,)}{\left(\frac{1}{\alpha}\sqrt{\delta^2+(x-\mu)^2}\right)^{0.5-\lambda}} \\
&\simeq \frac{\Gamma(0.5-\lambda)2^{-0.5-\lambda}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)^{\lambda-0.5}}{\alpha^{\lambda-0.5}\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{0.5-\lambda}} \\
&= \Gamma\left(0.5-\lambda\right)2^{-(\lambda+0.5)}\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{2\lambda-1} \\
&= \frac{1}{\delta^2+(x-\mu)^2}.
\end{aligned}
$$

Therefore a, b, and c converge for $\alpha, \beta \to 0, \lambda = -0.5$ and hence the density of the generalized hyperbolic distribution converges to

$$
\frac{\delta}{\pi}\frac{1}{\delta^2+(x-\mu)^2},
$$

i.e. the Lebesgue density of the Cauchy distribution with parameters (δ, μ). □

Another interesting connection between one-dimensional NIG-Processes and Cauchy processes is that in both cases there exists no real number that will be hit by the process in finite time with positive probability, this can be seen by combining Theorem 4.1 and 4.2 in Sato (2001).

Other limiting distributions and special cases of the generalized hyperbolic distribution include the normal distribution, the hyperboloid, the normal reciprocal inverse Gaussian, the variance gamma, the Student-t, the generalized inverse Gaussian, or the skewed Laplace distribution. Detailed information can be found in Prause (1999, Chapter 1.1).

Chapter 2

A Portfolio Approach

In this chapter we study the methodology of the software package CreditRisk$^{+\text{TM}}$ and show how it can be generalized in a natural way. In contrast to the approach of Section 1.1.1 and Chapter 3 this chapter deals with a portfolio approach, where only the common behavior of the portfolio is modeled instead of the dynamics of a single defaultable bond.

Observed default rates vary over time. This fact is taken into account in CreditRisk$^{+\text{TM}}$ by modeling default rates as random variables. First we present the CreditRisk$^{+\text{TM}}$ methodology as it is described in the technical document Credit Suisse Financial Products (1997) before we introduce the more flexible distribution in Section 2.3. Several authors have written surveys on CreditRisk$^{+\text{TM}}$, examples are Wilde (2000) or Nelken (1999).

We will change the CreditRisk$^{+\text{TM}}$ terminology slightly to insert it in a more mathematical framework.

2.1 Default events with fixed default rates

Consider a portfolio of $N \in \mathbb{N}$ obligors, and denote the annual probability of default by p_A, $A \in \{1, \ldots, N\}$. Let D be the random variable on a probability space $(\Omega, \mathcal{A}, P)$ describing the number of defaults in the portfolio within the next year. Thus, D is a nonnegative, integer-valued random variable, and we can define the probability generating function of D as follows

$$F(z) = \mathbb{E}[z^D] = \sum_{n=0}^{\infty} P(D = n) z^n, \qquad z \in \mathbb{R}. \tag{2.1}$$

The probability generating function has some especially useful properties. First, the series in equation (2.1) is absolutely convergent at least for $|z| \leq 1$. Consequently, we can calculate the probabilities $P(D = n)$ by differentiating F term by term. Second, if X and Y are two stochastically independent random variables, the probability generating function of $X + Y$ is the product of the probability generating functions of X and Y. Third, if $F_Y(z \,|\, X = x)$ is the probability generating function of Y conditional on $X = x$ and X has distribution with density f, then the unconditional probability generating function of Y is $F_Y(z) = \int F_Y(z \,|\, X = x) f(x)\, \mathrm{d}x$. For a survey on probability generating functions see e.g. Feller (1968) or Gut (1995).

Let us consider the sub-portfolio consisting of only one debtor, namely $A \in \{1, \ldots, N\}$. Let $D(A) := \mathbb{1}_{\{\text{Obligor } A \text{ does default within next year}\}}$ be the Bernoulli random variable describing whether A does default or not. The probability generating function F_A of $D(A)$ is given by $F_A(z) = P(D(A) = 0)z^0 + P(D(A) = 1)z^1 = 1 + (z-1)p_A$, where $p_A = P(D(A) = 1)$ is the default probability of A. Under the assumption that default events in the portfolio are stochastically independent, F is the product of the probability generating functions of the single debtors. Therefore the probability generating function of the number of defaults within the portfolio, F, is given by

$$F(z) = \prod_{A=1}^{N} F_A(z) = \prod_{A=1}^{N} (1 + (z-1)p_A). \tag{2.2}$$

Hence, using the logarithm series and assuming that the probabilities of default are uniformly small, we get the following approximation

$$\begin{aligned}
\log F(z) &= \sum_{A=1}^{N} \log(1 + (z-1)p_A) \\
&= \sum_{A=1}^{N} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} ((z-1)p_A)^k \\
&\approx \sum_{A=1}^{N} (z-1)p_A \\
&= \left(\sum_{A=1}^{N} p_A \right) (z-1) \\
&= \mu(z-1), \text{ where } \mu := \sum_{A=1}^{N} p_A.
\end{aligned}$$

Because the random variables $D(A)$, $A \in \{1, \ldots, N\}$, are Bernoulli distributed with parameter p_A, $\mu = \sum_{A=1}^{N} p_A$ is the expected number of default events in one year within the entire portfolio.

Using the last equation and the Taylor expansion of the exponential function,

we get

$$F(z) \approx \exp(\mu(z-1)) = \exp(-\mu)\exp(\mu z) \quad (2.3)$$
$$= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu}\mu^n}{n!} z^n.$$

Thus, D is approximately Poisson distributed with parameter μ:

$$P(D = n) \approx \frac{\mathrm{e}^{-\mu}\mu^n}{n!}. \quad (2.4)$$

Note that the Poisson distribution is a distribution on $\{0, 1, 2, \ldots\}$, and hence it is possible that a Poisson random variable takes values in $\{N + 1, N + 2, \ldots\}$. This implies that an obligor may default several times. But if the number of obligors N is sufficiently large this will happen only with a neglectable probability. This can not be interpreted as default after restructuring of the firm because in the CreditRisk$^{+\mathrm{TM}}$ -model the recovery rate is zero. A model which includes multiple defaults in a rigorous way is the model by Schönbucher (1998) and the related approach in Section 3.5.

Equation (2.4) is the reason why the CreditRisk$^+$ approach is an intensity based approach. The parameter μ is the intensity of the Poisson process $(N_t)_{0 \leq t \leq T}$, where N_t is the number of defaults up to time t.

The difference between the exact probability distribution and the approximated probability distribution is rather small when default probabilities are small, but for larger default probabilities this approximation gets worse. This can be seen in Figure 2.1, Figure 2.2, and Figure 2.3.

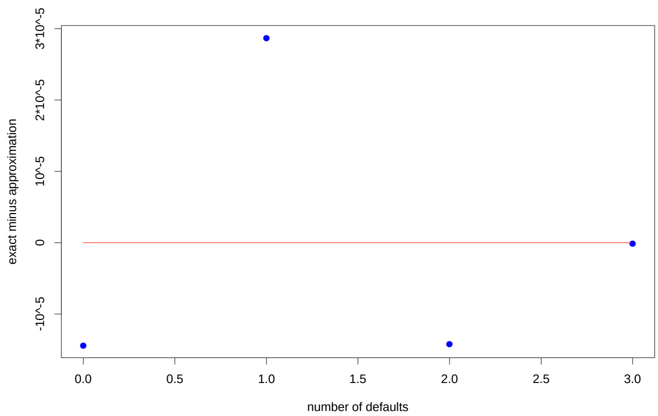


Figure 2.1: Example Portfolio of three obligors: Difference between exact and approximated distribution when default probabilities are small: $p = (0.002, 0.003, 0.004)$.

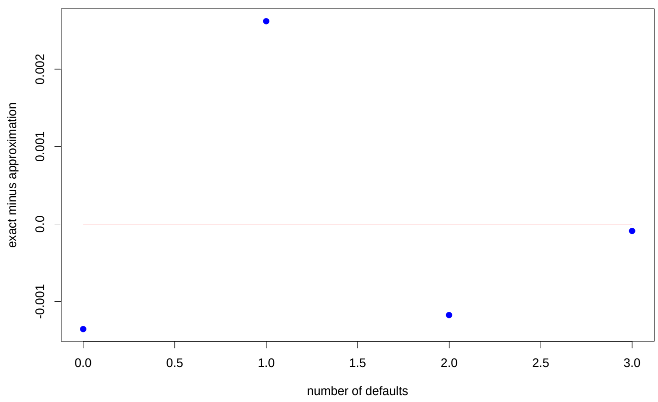


Figure 2.2: Example: Difference between exact and approximated distribution when default probabilities are middle: $p = (0.02, 0.03, 0.04)$.

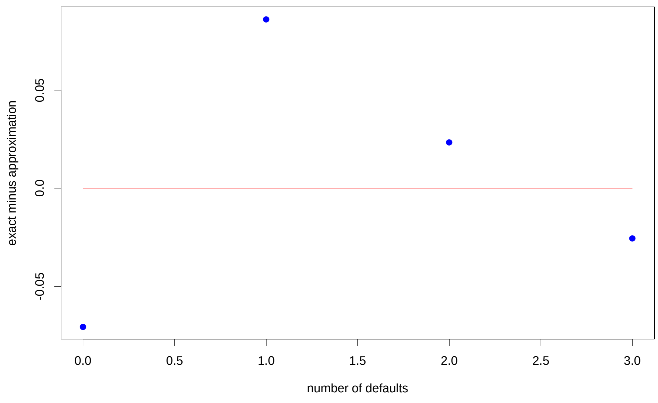


Figure 2.3: Example: Difference between exact and approximated distribution when default probabilities are large: $p = (0.2, 0.3, 0.4)$.

2.2 Loss distribution with fixed default rates

In this section we show how CreditRisk$^{+\mathrm{TM}}$ uses the distribution of default events to calculate the loss distribution of a portfolio. Again, the main instrument is the probability generating function. To make this approach feasible, it is necessary to group, i.e. to discretize, the exposures into bands. The width of bands should be small compared with the average size characteristic of the portfolio.

Again, we need some notation:

A = Obligor index, $A \in \{1, \dots, N\}$,
E_A = Exposure of obligor A,
p_A = Probability of default of obligor A,
λ_A = Expected loss of A.

In order to discretize the exposure and the expected loss on debtor A, it is essential to represent the exposure and the expected loss as multiples of a unit amount L. Gordy (2000) mentions that the loss distribution is quite robust to the choice of L.

$\nu_A = \frac{E_A}{L}$ exposure expressed as a multiple of L, ν_A is rounded to the nearest integer greater than $\frac{E_A}{L}$,

$\varepsilon_A = \frac{\lambda_A}{L}$ expected loss as a multiple of L.

Now the portfolio can be divided into exposure bands $B_j, j \in \{1, \dots, m\}$, where obligors are grouped according to their common exposure value ν_A. Formally, each band B_j is interpreted as a nonempty subset of $\{1, \dots, N\}$, where $B_j \cap B_i = \emptyset$ for $i \neq j$.

Note that one can easily replace the assumption of zero recovery by a fixed recovery rate r_A for each obligor. All one has to do is to replace the exposure E_A by $(1 - r_A)E_A$, and ν_A by $(1 - r_A)\nu_A$. The warranted pay-out $r_A E_A$ has then to be considered separately as the pay-out of a default free bond.

We introduce some further notation.

$\nu_j =$ Common exposure in band B_j in units of L,

$\varepsilon_j =$ Expected loss in B_j in units of L,

$\mu_j =$ Expected number of defaults in B_j.

Thus, $\varepsilon_j = \mu_j \nu_j$. Since $\nu_j = \nu_A$, for all $A \in B_j$, and $\varepsilon_j = \sum_{A \in B_j} \varepsilon_A$, it follows that

$$\mu_j = \frac{\varepsilon_j}{\nu_j} = \sum_{A \in B_j} \frac{\varepsilon_A}{\nu_A}.$$

The total expected number of defaults is denoted by

$$\mu = \sum_{j=1}^{m} \mu_j = \sum_{j=1}^{m} \frac{\varepsilon_j}{\nu_j}.$$

Let $LOSS$ be the random variable describing the aggregate losses in units of L in the portfolio within one year and let $LOSS[B_j], j \in \{1, \ldots, m\}$, be the respective random variable for band B_j. Consequently, $LOSS = \sum_{j=1}^{m} LOSS[B_j]$, where the right-hand side is assumed to be an independent sum. Furthermore, we introduce the random variables $D[B_j], j \in \{1, \ldots, m\}$, representing the number of defaults in band B_j. Then, $LOSS[B_j] = D[B_j]\nu_j, j \in \{1, \ldots, m\}$.

Let $G : z \mapsto G(z)$ and $G_j : z \mapsto G_j(z)$ be the probability generating functions of $LOSS$ and $LOSS[B_j]$, respectively. Therefore,

$$\begin{aligned}
G(z) &= \prod_{j=1}^{m} G_j(z) \\
&= \prod_{j=1}^{m} \sum_{n=0}^{\infty} P(LOSS[B_j] = n) z^n \\
&= \prod_{j=1}^{m} \sum_{n=0}^{\infty} P(D[B_j]\nu_j = n) z^n \\
&= \prod_{j=1}^{m} \sum_{n=0}^{\infty} P(D[B_j] = n) z^{\nu_j n} \qquad (2.5) \\
&\approx \prod_{j=1}^{m} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mu_j} \mu_j^n}{n!} z^{\nu_j n} \\
&= \prod_{j=1}^{m} \exp\left(-\mu_j + \mu_j z^{\nu_j}\right) \\
&= \exp\left(-\mu + \sum_{j=1}^{m} \mu_j z^{\nu_j}\right). \qquad (2.6)
\end{aligned}$$

It will be convenient to define a polynomial P of degree m by setting:

$$P(z) = \frac{\sum_{j=1}^{m} \mu_j z^{\nu_j}}{\mu} = \frac{\sum_{j=1}^{m} \left(\frac{\varepsilon_j}{\nu_j}\right) z^{\nu_j}}{\sum_{j=1}^{m} \frac{\varepsilon_j}{\nu_j}}. \qquad (2.7)$$

The probability generating function G can now be expressed as

$$G(z) = \exp\left(\mu(P(z) - 1)\right) = F(P(z)),$$

where F, given in equation (2.2), is the probability generating function of D, which is the random variable describing the number of defaults within the next year.

One way to get the loss distribution is applying the relation $P(LOSS = n) = \frac{\mathrm{d}^n G}{\mathrm{d}z^n}(0)/(n!)$ to equation (2.6). However it turns out that this approach can be very time consuming. The computation of $A_n := P(LOSS = n), n \geq 0$, can be done more efficiently by the recursive algorithm

$$A_0 = G(0) = \mathrm{e}^{-\mu} \quad (2.8)$$

$$A_n = \sum_{j:\nu_j \leq n} \frac{1}{n} \varepsilon_j A_{n-\nu_j}. \quad (2.9)$$

Once again, it is possible to evaluate the exact distribution of losses using equation (2.2) and (2.5). Although it would take too much time and space to write down the exact probability generating function explicitly, it is still possible to write an algorithm, e.g. in Mathematica™, which allows us to calculate the distribution of $LOSS$ efficiently. Depending on the implementation language the exact approach may be even easier to implement than the recursion algorithm in equations (2.8) and (2.9).

The next example described in Table 2.1 and Figure 2.4 is a "toy example", which shows how small the differences can be.

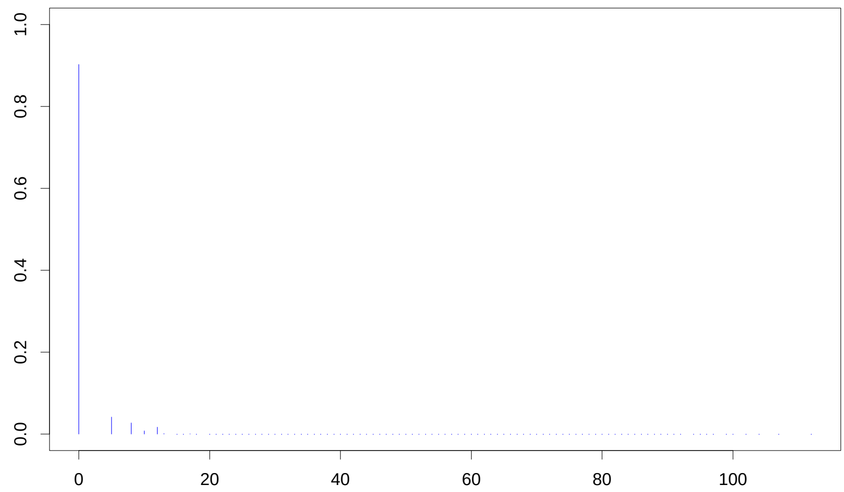


Figure 2.4: Example: Exact loss distribution; see example Table 2.1.

The different numerical approaches are compared in Table 2.2. The recursive algorithm and the approach using equation (2.6) give the same numerical results. The differences are neglectable.

Name	Band	Exposure ν_A	p_A
1	B_1	10	0.001
2	B_1	10	0.002
3	B_1	10	0.005
4	B_2	12	0.008
5	B_2	12	0.003
6	B_2	12	0.006
7	B_2	12	0.002
8	B_3	8	0.01
9	B_3	8	0.009
10	B_3	8	0.011
11	B_4	5	0.02
12	B_4	5	0.025

Table 2.1: Example: Portfolio

Loss k	$P(LOSS = k)$: exact	$P(LOSS = k)$: recursion
0	0.902358	0.90303
5	0.0415528	0.0406363
8	0.027346	0.0270909
10	0.00771824	0.00813855
12	0.0172475	0.0171576
13	0.00125926	0.00121909
15	0.000333675	0.000338805
≥ 16	0.00218486	0.00238922

Table 2.2: Comparison of the numerical approaches: loss distribution

2.3 Sector analysis

One of the main tools in modeling default rates as random variables is partitioning the debtors in different sectors. The idea is that default rates can be related to the influence of a relative small number of background factors. First, it is assumed that each debtor within the portfolio belongs to only one sector, e.g. the country of domicile.

Again, it is necessary to introduce some notation. Let $S_k, 1 \leq k \leq \widetilde{n}$, be the sectors. For now the $S_k, 1 \leq k \leq \widetilde{n}$ are subsets of $\{1, \ldots, N\}$. Each sector can now be divided in bands, as done in the previous section.

Let $D[S_k], 1 \leq k \leq \widetilde{n}$, be the random variable describing the number of defaults in S_k within the time horizon and let $M[S_k], 1 \leq k \leq \widetilde{n}$, be the mean number of defaults in sector S_k, which is also modeled as a random variable with mean μ_k and variance σ_k^2. This means that $P^{D[S_k]}$ is a mixture distribution on $\mathbb{R}_+$, where its mean is distributed according to another distribution $P^{M[S_k]}$.

We summarize the notation as follows:

$$\begin{aligned} M[S_k] &= \text{random variable describing} \\ & \text{the mean number of defaults in sector } S_k, \\ \mu_k &= \mathbb{E}[M[S_k]] \text{ long term annual average number of defaults in } S_k, \\ \sigma_k &= \sqrt{\text{Variance}(M[S_k])}. \end{aligned}$$

Each sector S_k is partitioned into $m(k) \in \mathbb{N}$ bands $B_j^{(k)}, 1 \leq j \leq m(k)$. The common exposure in units of L in band $B_j^{(k)}$ is denoted by $\nu_j^{(k)}, 1 \leq j \leq m(k)$. The expected loss in band $B_j^{(k)}$ in sector S_k in units of L is denoted by $\varepsilon_j^{(k)}$.

$$\begin{aligned} E_j^k &= L\, \nu_j^{(k)}, 1 \leq j \leq m(k), \text{ exposure size in exposure band } B_j^{(k)} \\ & \text{in sector } S_k. \\ \lambda_j^{(k)} &= L\varepsilon_j^{(k)}, 1 \leq j \leq m(k), \text{ expected loss in exposure band } B_j^{(k)} \\ & \text{in sector } S_k. \\ \mu_j^{(k)} &= \text{Expected number of defaults in band } B_j^{(k)} \text{ in sector } S_k. \end{aligned}$$

The expected number of defaults in sector S_k is $\mu_k = \sum_{j=1}^{m(k)} \mu_j^{(k)} = \sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}$.

It is assumed that the random variables $M[S_k], 1 \leq k \leq \widetilde{n}$, are stochastically independent. Conditional on the value of $M[S_k]$, it is possible to write down the probability generating function for the distribution of the number of defaults $D[S_k]$ as follows:

$$F_{D[S_k]|M[S_k]=x}(z) = \mathrm{e}^{x(z-1)},$$

see also equation (2.3).

If the distribution of $M[S_k]$ is absolutely continuous with respect to the Lebesgue measure, i.e. $P^{M[S_k]}(\mathrm{d}x) = f_k(x)\,\mathrm{d}x$, for some suitable density function f_k, we get

$$\begin{aligned}F_{D[S_k]}(z) &= \sum_{n=0}^{\infty} P(D[S_k]=n) z^n \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} P(D[S_k]=n|M[S_k]=x) P^{M[S_k]}(\mathrm{d}x) z^n \\ &= \int_0^{\infty} \underbrace{\sum_{n=0}^{\infty} z^n P(D[S_k]=n|M[S_k]=x)}_{=F_{D[S_k]|M[S_k]=x}(z)} f_k(x)\,\mathrm{d}x \\ &= \int_0^{\infty} \mathrm{e}^{x(z-1)} f_k(x)\,\mathrm{d}x. \end{aligned} \tag{2.10}$$

In order to obtain a closed form expression for the probability generating function in equation (2.10), it is assumed in the CreditRisk$^{+\mathrm{TM}}$ technical document that $M[S_k]$ is Gamma-distributed with mean μ_k and variance σ_k^2. Therefore the density f_k of $P^{M[S_k]}$ is given by

$$f_k(x) = \frac{1}{\beta_k^{\alpha_k}\Gamma(\alpha_k)} \mathrm{e}^{-\frac{x}{\beta_k}} x^{\alpha_k - 1},$$

where $\Gamma(.)$ is the Gamma function, $\alpha_k = \frac{\mu_k^2}{\sigma_k^2}$, and $\beta_k = \frac{\sigma_k^2}{\mu_k}$.

Using this assumption $F_{D[S_k]}$ expresses as follows

$$F_{D[S_k]}(z) = \left(\frac{1 - \frac{\beta_k}{1+\beta_k}}{1 - \frac{\beta_k}{1+\beta_k} z} \right)^{\alpha_k}, \tag{2.11}$$

or

$$F_{D[S_k]}(z) = (1-p_k)^{\alpha_k} (1-p_k z)^{-\alpha_k},$$

where $p_k := \frac{\beta_k}{1+\beta_k}$. Using the Taylor expansion of $F_{D[S_k]}$ we get

$$F_{D[S_k]}(z) = (1-p_k)^{\alpha_k} \sum_{n=1}^{\infty} \begin{pmatrix} n + \alpha_k - 1 \\ n \end{pmatrix} p_k^n z^n.$$

Therefore, the number of defaults in S_k is distributed according to a negative binomial distribution. This is similar to the problem studied in Arbous and Kerrich (1951), where the negative binomial distribution arises when the number of accidents sustained by individuals in a time interval is Poisson distributed with parameter μ and μ is assumed to be Gamma-distributed.

It is interesting to note, that in a different portfolio approach in Schmock and Seiler (2002) where Polya's urn scheme and the theory of exchangeable sequences is used to model dependent credit risk, the number of defaults justifies the Dirichlet-binomial distribution.

One disadvantage of the negative binomial distribution is its lack of flexibility, e.g. it cannot have skewness coefficient greater than two and mode greater than zero, see e.g. Stein, Zucchini, and Juritz (1987). Furthermore, practitioners prefer distributions which allow more mass in the tails of the distributions. A similar problem arises in modeling stock prices. In the classical model in Samuelson (1965), the stock price follows a geometric Brownian motion, and thus log returns are normal-distributed. However, empirical studies show that real-life stock data provides more mass in the tails than normal distributions, see e.g. Mandelbrot (1963) or more recently Eberlein and Keller (1995).

One can overcome this problem and get close to perfect fits of the whole distribution including the tails by using (generalized) hyperbolic distributions, see e.g. Eberlein and Keller (1995), Prause (1999), or Eberlein (2001). The family of generalized hyperbolic distribution has been introduced in Barndorff-Nielsen (1977) to model particle size distributions of wind blown sands. The interesting point is that generalized hyperbolic distributions are normal mean-variance mixture distributions of normal and generalized inverse Gaussian (GIG) distributions, see equation (1.9). So the heavy tails of the generalized hyperbolic distributions arise from the GIG distribution. That is the reason why we consider the mixture of the Poisson distribution with the generalized inverse Gaussian distribution. The resulting mixture distribution is also known as the Sichel distribution.

The assumption that $M[S_k]$ is Gamma-distributed can be replaced by any distribution on $\mathbb{R}^+$, but preserving the analytic tractability of the default distribution is an important point. We assume that $P^{M[S_k]}$ follows a GIG distribution with parameters $\lambda_k, \psi_k, \chi_k$, i.e. the Lebesgue density f_k of $P^{M[S_k]}$ is given by, cf. Definition 1.4,

$$f_k(x) = \frac{(\psi_k/\chi_k)^{\lambda_k/2}}{2\mathrm{K}_{\lambda_k}(\sqrt{\psi_k \chi_k})} x^{\lambda_k - 1} \exp\left(-\frac{1}{2}(\chi_k x^{-1} + \psi_k x)\right) \mathbb{1}_{\mathbb{R}^+}(x),$$

where $\lambda_k \in \mathbb{R}$ and $\psi_k, \chi_k \in \mathbb{R}^+$. In case that $\lambda_k > 0$ and $\chi_k \to 0$ we get the Gamma distribution. Other interesting cases are the inverse Gaussian distribution ($\lambda_k = -\frac{1}{2}$), or the hyperbola distribution ($\lambda_k = 0$). For a detailed survey on the generalized inverse Gaussian distribution we refer to Jørgensen (1982).

The mixture distribution of Poisson and GIG has been introduced first in Sichel (1971). In Sichel (1975) a short survey and an application in linguistics can be found. The special case, where the mixing distribution is inverse Gaussian (IG), i.e. $\lambda_k = -\frac{1}{2}$, is also known as the inverse Gaussian Poisson distribution (IGP). The IGP is of special interest because it leads to relative simple estimators and provides good fits in several applications, see Stein, Zucchini, and Juritz (1987).

If we assume that the mean number of defaults in sector S_k, $M[S_k]$, is GIG distributed with parameters $\lambda_k, \psi_k, \chi_k$ we get for $z < 1 + \frac{\psi_k}{2}$ a closed form expression for the probability generating function of $D[S_k]$

$$F^{GIG}_{D[S_k]}(z) = \frac{(\psi_k/\chi_k)^{\lambda_k/2}}{2\mathrm{K}_{\lambda_k}\left(\sqrt{\psi_k\chi_k}\right)} \int\limits_0^\infty x^{\lambda_k-1} \exp\left(-\frac{1}{2}(\chi_k x^{-1} + (\psi_k - 2z + 2)x)\right) \mathrm{d}x$$

$$= \frac{(\psi_k/\chi_k)^{\lambda_k/2}}{2\mathrm{K}_{\lambda_k}\left(\sqrt{\psi_k\chi_k}\right)} \frac{2\,\mathrm{K}_{\lambda_k}\left(\sqrt{2-2\,z+\psi_k}\,\sqrt{\chi_k}\right)}{(2-2\,z+\psi_k)^{\frac{\lambda_k}{2}}\left(\frac{1}{\chi_k}\right)^{\frac{\lambda_k}{2}}} \tag{2.12}$$

$$= \left(\frac{\psi_k}{2-2z+\psi_k}\right)^{\lambda_k/2} \frac{\mathrm{K}_{\lambda_k}(\sqrt{2-2z+\psi_k}\sqrt{\chi_k})}{\mathrm{K}_{\lambda_k}(\sqrt{\psi_k}\sqrt{\chi_k})}. \tag{2.13}$$

Equation (2.12) can be easily seen from the Lebesgue density f of the GIG distribution. Since $\int_0^\infty f_k(x)\,\mathrm{d}x = 1$ we have

$$\int\limits_0^\infty x^{\lambda_k-1}\mathrm{e}^{-\frac{1}{2}(\chi_k x^{-1}+\psi_k x)}\,\mathrm{d}x = \frac{2\mathrm{K}_{\lambda_k}(\sqrt{\chi_k\psi_k})}{(\psi_k/\chi_k)^{\lambda_k/2}},$$

for $\chi_k, \psi_k > 0$.

By taking derivatives of the moment generating function $F^{GIG}_{D[S_k]}$ and using properties of the modified Bessel functions of the third kind one can derive the probability of $n \in \mathbb{N}_0$ defaults in sector S_k

$$P(D[S_k] = n) = \frac{1}{n!}\left(\frac{\chi_k}{\psi_k}\right)^{n/2}\left(\frac{\psi_k+2}{\psi_k}\right)^{-(\lambda_k+n)/2} \frac{\mathrm{K}_{\lambda_k+n}\left(\sqrt{\chi_k(\psi_k+2)}\right)}{\mathrm{K}_{\lambda_k}\left(\sqrt{\chi_k\psi_k}\right)}.$$

In Sichel (1971) a useful recurrence formula is derived, so that only the modified Bessel functions K_{λ_k} and K_{λ_k+1} need to be calculated. In our parameterization we get for $n \geq 2$

$$P(D[S_k] = n) = \frac{2(n-1)(\lambda_k+n-1)P(D[S_k] = n-1) + \chi_k P(D[S_k] = n-2)}{n(n-1)(\psi_k+2)}.$$

In case that we choose $\lambda_k = -\frac{1}{2}$ in equation (2.13), we get the IGP distribution. We use $\mathrm{K}_{-\frac{1}{2}}(x) = \sqrt{\pi/2}x^{-1/2}\mathrm{e}^{-x}$ to simplify the probability generating function of $D[S_k]$

$$F^{IG}_{D[S_k]}(z) = \left(\frac{\psi_k}{2-2z+\psi_k}\right)^{-1/4} \frac{\sqrt{\pi/2}(2-2z+\psi_k)^{-1/4}\chi_k^{-1/4}\mathrm{e}^{-\sqrt{2-2z+\psi_k}\sqrt{\chi_k}}}{\sqrt{\pi/2}\psi_k^{-1/4}\chi_k^{-1/4}\mathrm{e}^{-\sqrt{\psi_k\chi_k}}}$$
$$= \exp\left(\sqrt{\chi_k}\left(\sqrt{\psi_k} - \sqrt{2-2z+\psi_k}\right)\right).$$

Calculating the derivatives $\frac{1}{n!}\frac{\mathrm{d}^n F^{IG}_{D[S_k]}}{\mathrm{d}z^n}(z)|_{z=0}, n = 0, 1, 2, \ldots,$ yields

$$P(D[S_k] = 0) = F^{IG}_{D[S_k]}(0) = \mathrm{e}^{\sqrt{\chi_k}(\sqrt{\psi_k}-\sqrt{2+\psi_k})},$$

$$P(D[S_k] = n) = \frac{\mathrm{e}^{\sqrt{\chi_k}(\sqrt{\psi_k}-\sqrt{2+\psi_k})}}{n!} \sum_{\ell=0}^{n-1} \frac{(n-1+\ell)!}{(n-1-\ell)!\,\ell!} 2^{-\ell} \left(\chi_k(\psi_k+2)\right)^{-\ell/2}.$$

Next we study the connection between μ_k, σ_k^2 and the parameters λ_k, χ_k, and ψ_k. According to Jørgensen (1982) the moments of the random variable $M[S_k] \sim GIG(\lambda_k, \chi_k, \psi_k)$ are given by

$$\mathbb{E}[M[S_k]^\ell] = \frac{\mathrm{K}_{\lambda_k+\ell}\left(\sqrt{\chi_k\psi_k}\right)}{\mathrm{K}_{\lambda_k}\left(\sqrt{\chi_k\psi_k}\right)} \left(\sqrt{\frac{\chi_k}{\psi_k}}\right)^\ell, \quad \ell \in \mathbb{N}.$$

Therefore in the GIG case one has to know more than $\mu_k = \mathbb{E}[M[S_k]]$ and $\sigma_k^2 = \mathbb{E}[M[S_k]^2] - \mu_k^2$ to determine the parameters λ_k, χ_k, and ψ_k. Under the assumption $M[S_k] \sim IG(\chi_k, \psi_k)$ we have the following expressions for the mean and the variance of $M[S_k]$

$$\begin{aligned} \mu_k &= \frac{\mathrm{K}_{\frac{1}{2}}\left(\sqrt{\chi_k\psi_k}\right)}{\mathrm{K}_{-\frac{1}{2}}\left(\sqrt{\chi_k\psi_k}\right)}\sqrt{\frac{\chi_k}{\psi_k}} \\ &= \sqrt{\frac{\chi_k}{\psi_k}}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \sigma_k^2 &= \mathbb{E}\left[M[S_k]^2\right] - \mu_k^2 \\ &= \frac{\mathrm{K}_{\frac{3}{2}}\left(\sqrt{\chi_k\psi_k}\right)}{\mathrm{K}_{-\frac{1}{2}}\left(\sqrt{\chi_k\psi_k}\right)}\left(\sqrt{\frac{\chi_k}{\psi_k}}\right)^2 - \frac{\chi_k}{\psi_k} \\ &= \left(1 + \frac{1}{\sqrt{\chi_k\psi_k}}\right)\frac{\chi_k}{\psi_k} - \frac{\chi_k}{\psi_k} \\ &= \frac{\sqrt{\chi_k}}{\left(\sqrt{\psi_k}\right)^3}. \end{aligned} \tag{2.15}$$

We have used the following properties of the modified Bessel function K_λ, see e.g. Abramowitz and Stegun (1968),

$$\mathrm{K}_\lambda(x) = \mathrm{K}_{-\lambda}(x),$$
$$\mathrm{K}_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}} \mathrm{e}^{-x},$$
$$\mathrm{K}_{\frac{3}{2}}(x) = \mathrm{K}_{\frac{1}{2}}(x) \left(1 + \frac{1}{x}\right).$$

Solving equations (2.14) and (2.15) gives the following relationship

$$\chi_k = \frac{\mu_k^3}{\sigma_k^2}, \tag{2.16}$$
$$\psi_k = \frac{\mu_k}{\sigma_k^2}. \tag{2.17}$$

Using these two equations (2.16) and (2.17) and the estimators $\widehat{\mu}_k$ and $\widehat{\sigma}_k$ suggested by CreditRisk$^{+\mathrm{TM}}$(Chapter A7.3), we get estimators $\widehat{\chi}_k$ and $\widehat{\psi}_k$, respectively.

Further information on Sichel distributions can be found in Willmot (1986). A multivariate extension has been developed in Stein, Zucchini, and Juritz (1987). The multivariate extension can be used to include sector correlations within the portfolio, where one can follow the approach in Giese (2002).

2.4 Default losses with variable default rates

Remember that $LOSS$ is the random variable describing the aggregate losses in our portfolio in units of L. Analogously, $LOSS[S_k]$ and $LOSS[B_j^{(k)}]$ are the random variables, describing the losses in units of L in Sector S_k and band $B_j^{(k)}$, respectively.

The probability generating function G_{LOSS} can be written as

$$G_{LOSS}(z) = \prod_{k=1}^{\widetilde{n}} G_{LOSS[S_k]}(z),$$

since we have assumed $LOSS[S_k], k = 1, \dots, \widetilde{n}$, to be stochastically independent. The authors of the CreditRisk$^{+\mathrm{TM}}$ technical document show that

$$G_{LOSS[S_k]}(z) = F_{D[S_k]}(P_k(z)), \tag{2.18}$$

where $F_{D[S_k]}(.)$ is the probability generating function of $D[S_k]$, which is the number of defaults in S_k, defined in equation (2.11) and $P_k(.)$ is a polynomial similar to the polynomial $P(.)$ in equation (2.7), defined as

$$P_k(z) = \frac{\sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} z^{\nu_j^{(k)}}}{\sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}}. \tag{2.19}$$

Therefore,

$$G_{LOSS}(z) = \prod_{k=1}^{\tilde{n}} \left(\frac{1-p_k}{1 - \frac{p_k}{\mu_k} \sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} z^{\nu_j^{(k)}}} \right)^{\alpha_k},$$

where p_k and α_k are defined as in Section 2.3.

The distribution of $LOSS$ is then calculated using a recurrence relation, which is applied to the Taylor series expansion of $G_{LOSS}(.)$. A drawback of this recursive algorithm is that it can lead to material rounding errors (sometimes even negative probabilities) if large portfolios and many sectors are being considered. In practical implementations, it is sometimes not at all possible to calculate higher loss quantiles (e.g. 99.9 %) or only at errors as high as 30 %, see Giese (2002). An alternative recursive algorithm using saddle point approximation has been developed by Gordy (2001). Giese (2002) has developed a faster, more stable and precise algorithm by making use of the cumulant generating function of the gamma distribution.

Equation (2.18) can also be used in the GIG and IG case. There, we get closed form expressions as well:

$$G_{LOSS}^{GIG}(z) = \prod_{k=1}^{\tilde{n}} \left(\frac{\psi_k}{2 - 2P_k(z) + \psi_k} \right)^{\alpha_k} \frac{\mathrm{K}_{\lambda_k}\left(\sqrt{\chi_k}\sqrt{2 - 2P_k(z) + \psi_k}\right)}{\mathrm{K}_{\lambda_k}\left(\sqrt{\psi_k \chi_k}\right)},$$

where $P_k(z)$ is defined in equation (2.19). In the IG case we get

$$\begin{aligned} G_{LOSS}^{IG}(z) &= \prod_{k=1}^{\tilde{n}} G_{LOSS[S_k]}^{IG}(z) \\ &= \exp\left(\sum_{k=1}^{\tilde{n}} \sqrt{\chi_k} \left(\sqrt{\psi_k} - \sqrt{2 - \frac{2}{\sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}} \sum_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} z^{\nu_j^{(k)}} + \psi_k} \right) \right). \end{aligned}$$

An example will be discussed in Section 2.6 to compare the CreditRisk$^{+\mathrm{TM}}$ approach with the IG approach. Before we start with the example portfolio, we introduce another approach which may be used to calculate the loss distribution. The idea is, that the assumptions made in Section 2.3 for $D[S_k]$ hold analogously for $D[B_j^{(k)}]$.

Then, we have

$$\begin{aligned} LOSS &= \sum_{k=1}^{\widetilde{n}} LOSS[S_k] \\ &= \sum_{k=1}^{\widetilde{n}} \sum_{j=1}^{m(k)} LOSS[B_j^{(k)}]. \end{aligned}$$

Assuming that all sums are stochastically independent, we have

$$G_{LOSS}(z) = \prod_{k=1}^{\widetilde{n}} \prod_{j=1}^{m(k)} G_{LOSS[B_j^{(k)}]}(z).$$

Therefore, it is sufficient to calculate the probability generating function $G_{B_j^{(k)}}$. Since $LOSS[B_j^{(k)}] = \nu_j^{(k)} D[B_j^{(k)}]$, it follows that

$$\begin{aligned} G_{LOSS[B_j^{(k)}]}(z) &= \sum_{n=0}^{\infty} P(LOSS[B_j^{(k)}] = n) z^n \\ &= \sum_{n=0}^{\infty} P(\nu_j^{(k)} D[B_j^{(k)}] = n) z^n \\ &= \sum_{n=0}^{\infty} P(D[B_j^{(k)}] = n) z^{\nu_j^{(k)} n}. \end{aligned}$$

The power series expansion of a probability generating function is absolutely convergent in $D = \{z : |z| < 1\}$. Therefore, in D it is possible to differentiate with respect to z term by term

$$\begin{aligned} P(LOSS[B_j^{(k)}] = \ell) &= \frac{1}{\ell!} \frac{\mathrm{d}^n G_{LOSS[B_j^{(k)}](z)}}{\mathrm{d}z^\ell} | z = 0 \\ &= \frac{1}{\ell!} \sum_{n=\ell}^{\infty} \frac{(\nu_j^{(k)} n)!}{(\nu_j^{(k)} n - \ell)!} P(D[B_j^{(k)}] = n) z^{\nu_j^{(k)} n - \ell} | z = 0 \\ &= \sum_{n=\ell}^{\infty} \binom{\nu_j^{(k)} n}{\ell} P(D[B_j^{(k)}] = n) z^{\nu_j^{(k)} n - \ell} | z = 0. \end{aligned}$$

Hence, it is sufficient to know the distribution of $D[B_j^{(k)}]$, which has been computed in Section 2.3.

2.5 Generalized sector analysis, risk contribution, and correlation

Generalized sector analysis

The sector analysis introduced in Section 2.3 implies that the default event of each obligor depends on only one systematic background factor. As before, the number of systematic default factors is assumed to be relatively small. But now the default event of each obligor may be affected by several factors. This means that the concept of sectors has to be replaced by another method, e.g. the concept of systematic default factors.

Let $\widetilde{n} \in \mathbb{N}$ be the number of systematic default factors. Let $\theta_{A,k}$ be the proportional influence of the k-th systematic factor, such that $\sum_{k=1}^{\widetilde{n}} \theta_{A,k} = 1$ for each $A \in \{1, \dots, N\}$.

The sector analysis approach is a special case of the systematic factor approach and can be obtained by taking

$$\theta_{A,k} = \delta_{A,k} := \begin{cases} 1 & \text{if } A \in S_k, \\ 0 & \text{if } A \notin S_k. \end{cases}$$

Remember that the probability generating function of the losses in sector S_k was $G_k(z) = F_{D[S_k]}(P_k(z))$, where P_k is the polynomial given in equation (2.19). By simple transformations we get

$$P_k(z) = \frac{\sum\limits_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}} z^{\nu_j^{(k)}}}{\sum\limits_{j=1}^{m(k)} \frac{\varepsilon_j^{(k)}}{\nu_j^{(k)}}} = \frac{\sum\limits_{A \in S_k} \frac{\varepsilon_A}{\nu_A} z^{\nu_A}}{\sum\limits_{A \in S_k} \frac{\varepsilon_A}{\nu_A}} = \frac{\sum\limits_{A=1}^{N} \delta_{A,k} \frac{\varepsilon_A}{\nu_A} z^{\nu_A}}{\sum\limits_{A=1}^{N} \delta_{A,k} \frac{\varepsilon_A}{\nu_A}}.$$

This can be generalized as follows

$$P_k(z) = \frac{\sum\limits_{A=1}^{N} \theta_{A,k} \frac{\varepsilon_A}{\nu_A} z^{\nu_A}}{\sum\limits_{A=1}^{N} \theta_{A,k} \frac{\varepsilon_A}{\nu_A}}.$$

The probability generating function of the one-year-losses in the portfolio is assumed to be of the following form

$$G(z) = \prod_{k=1}^{\widetilde{n}} G_k(z) = \prod_{k=1}^{\widetilde{n}} F_{D_k}(P_k(z)),$$

where F_{D_k} is the probability generating function of D_k, a non-negative random variable with mean M_k, where as in Section 2.3 M_k is a non-negative random variable with mean $\mu_k = \sum_{A=1}^{N} \theta_{A,k}\mu_A$ and standard deviation $\sigma_k = \sum_{A=1}^{N} \theta_{A,k}\sigma_A$.

In case that $\theta_{A,k} = \delta_{A,k}$ we have $D_k = D[S_k]$ and $M_k = M[S_k]$, where μ_A and σ_A are the mean and the standard deviation of the random variable $D_A = \mathbb{1}_{\{\text{Obligor } A \text{ does default within next year}\}}$.

It is sufficient to add another sector to take the firm specific factors into account. This additional band must have N bands, i.e. each band corresponds with one obligor.

Risk contribution

The risk contribution of an individual obligor A' is defined as the marginal effect to the presence of the exposure $E_{A'}$ on the standard deviation σ or a given quantile of the loss distribution.

The technical document suggests a heuristic formula for the risk contribution of obligor A', denoted by $\text{RC}_{A'}$,

$$\text{RC}_{A'} = E_{A'} \frac{\partial \sigma}{\partial E_{A'}}.$$

The last formula has to be interpreted as

$$\begin{aligned}\text{RC}_{A'} &= E_{A'} \frac{\sigma(\sum_{A\in\{1,\dots,n\}} E_A) - \sigma(\sum_{A\in\{1,\dots,n\}\setminus A'} E_A)}{\sum_{A\in\{1,\dots,n\}} E_A - \sum_{A\in\{1,\dots,n\}\setminus A'} E_A} \\ &= \sigma_{\text{entire portfolio}} - \sigma_{\text{portfolio without } A'}\end{aligned}$$

The marginal effect on the α-quantile of the loss distribution can be defined as

$$RC_{A'}(\alpha) = \alpha\text{-quantile}_{\text{loss distr., entire portf.}} - \alpha\text{-quantile}_{\text{loss distr., portf. without } A'}.$$

Using the derivatives of the probability generating function the variance of the loss distribution is calculated as

$$\sigma^2 = \sum_{k=1}^{\tilde{n}} \varepsilon_k^2 \left(\frac{\sigma_k}{\mu_k}\right)^2 + \sum_{A=1}^{N} \varepsilon_A \nu_A.$$

Then the technical document derives an explicit formula for the risk contribution on the standard deviation.

$$\begin{aligned}RC_{A'} &= \frac{E_{A'}}{2\sigma}\left(2E_{A'}\mu_{A'} + \sum_{k=1}^{\tilde{n}} \left(\frac{\sigma_k}{\mu_k}\right)^2 2\varepsilon_k \theta_{A',k}\mu_{A'}\right) \\ &= \frac{E_{A'}\mu_{A'}}{\sigma}\left(E_{A'} + \sum_{k=1}^{\tilde{n}} \left(\frac{\sigma_k}{\mu_k}\right)^2 \varepsilon_k \theta_{A',k}\right).\end{aligned}$$

A more advanced method regarding the Value-at-Risk and expected shortfall contribution of each obligor within the portfolio has been developed in Haaf and Tasche (2002). For more information on the expected shortfall see e.g. Artzner, Delbaen, Eber, and Heath (1999) or Eberlein and Prause (2002).

Pairwise correlation

The method of the generalized sector analysis implies correlation between the different debtors. Let A and B be two obligors of our portfolio. Let ρ_{AB} be the pairwise correlation between the default indicator variables $D(A) = \mathbb{1}_{\{\text{Obligor } A \text{ does default within next year}\}}$ and $D(B) := \mathbb{1}_{\{\text{Obligor } B \text{ does default within next year}\}}$. Again we set $p_A = P(D(A) = 1)$ and $p_B = P(D(B) = 1)$

Let p_{AB} be the probability that both obligors, A and B, default within the time horizon. Then the correlation ρ_{AB} is

$$\rho_{AB} = \frac{p_{AB} - p_A p_B}{(p_A - p_A^2)^{1/2}(p_B - p_B^2)^{\frac{1}{2}}}.$$

Note that since $D(A)$ and $D(B)$ are Bernoulli random variables, we have that $E[D(I)] = p_I, I = A, B$, and since $D(A)$ and $D(B)$ are indicator functions, we have $D(I)^2 = D(I)$, for $I = A, B$. By evaluating a formula for p_{AB} the correlation of the defaults of A and B simplifies to

$$\rho_{AB} = (p_A p_B)^{1/2} \sum_{k=1}^{n} \theta_{A,k}\theta_{B,k} p_A p_B \left(\frac{\sigma_k}{\mu_k}\right)^2.$$

The last equation implies that two debtors which have no sector in common will have zero correlation. Since historical data suggests that $\sigma_k/\mu_k \approx 1$ one can expect default correlations to be of the same order as default probabilities.

2.6 The CreditRisk$^{+\text{TM}}$ example

To compare the GIG approach with the CreditRisk$^{+\text{TM}}$ approach, we study the example given in the technical document. The example portfolio consists of 25 obligors, listed in Table 2.3. In this example all obligors are in one single sector, e.g. the domestic country. All exposures are also presented as multiples of $L = 202389$, see the column with heading ν_A in Table 2.3. The parameters μ_A and σ_A are determined by the rating class of each firm. Because the random variable $D(A)$ — describing whether debtor A does default or not — is Bernoulli distributed, we have that $p_A = \mu_A$.

According to the technical document we estimate $\widehat{\mu}_1 = \sum_{A \in S_1} p_A = 3.266$ and $\widehat{\sigma}_1 = \sum_{A \in S_1} \sigma_A = 1.633$. Furthermore, $p_1 = \frac{\beta_1}{1+\beta_1} = \frac{\sigma_1^2}{\sigma_1^2 + \mu_1} = 0.449491$, and $\alpha_1 = (\widehat{\mu}_1/\widehat{\sigma}_1)^2 = 4$.

Name	Exposure	ν_A	p_A	σ_A
1	358475	2	0.30	0.15
2	1089819	6	0.30	0.15
3	1799710	9	0.1	0.05
4	1933116	10	0.15	0.075
5	2317327	12	0.15	0.075
6	2410929	12	0.15	0.075
7	2652184	14	0.30	0.15
8	2957685	15	0.15	0.075
9	3137989	16	0.05	0.025
10	3204044	16	0.05	0.025
11	4727724	24	0.015	0.0075
12	4830517	24	0.05	0.025
13	4912097	25	0.05	0.025
14	4928989	25	0.30	0.15
15	5042312	25	0.1	0.05
16	5320364	27	0.075	0.0375
17	5435457	27	0.05	0.025
18	5517586	28	0.03	0.015
19	5764596	29	0.075	0.0375
20	5847845	29	0.03	0.015
21	6466533	32	0.30	0.15
22	6480322	33	0.30	0.15
23	7727651	39	0.016	0.008
24	15410906	77	0.1	0.05
25	20238895	100	0.075	0.0375

Aggregate exposure	130513072
Expected Loss	14221863

Table 2.3: Example: CreditRisk$^{+\text{TM}}$ Sector Analysis.

	Γ	IG
Expected Loss	14540148	14643573
90.0 % – Quantile	31977462	31775073
99.0 % – Quantile	56061753	54442641
99.9 % – Quantile	78122154	67395537

Table 2.4: Comparison of the expected loss and quantiles.

Table 2.4 shows that the inverse Gaussian approach is a real alternative to the Gamma approach by CreditRisk+. Figure 2.7 visualizes the difference between the α-quantiles of the loss distribution for $\alpha \in [0.9, 0.999]$.

The loss distribution in the inverse Gaussian setting, shown in Figure 2.5, has heavier tails than the loss distribution in the Γ-setting. Figure 2.6 shows the ratio of the credit loss density in the inverse Gaussian approach and the credit loss density in the Γ case.

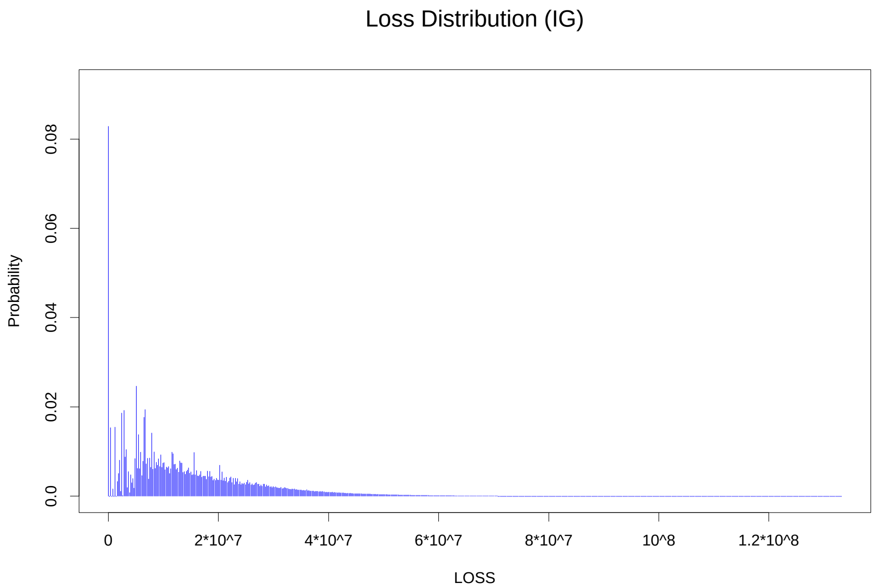


Figure 2.5: Example: Credit loss distribution in the inverse Gaussian approach.

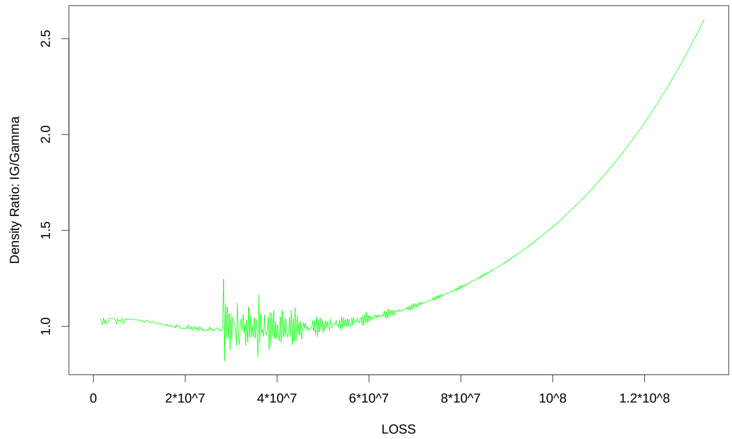


Figure 2.6: Example: Ratio of credit loss densities: IG/Gamma.

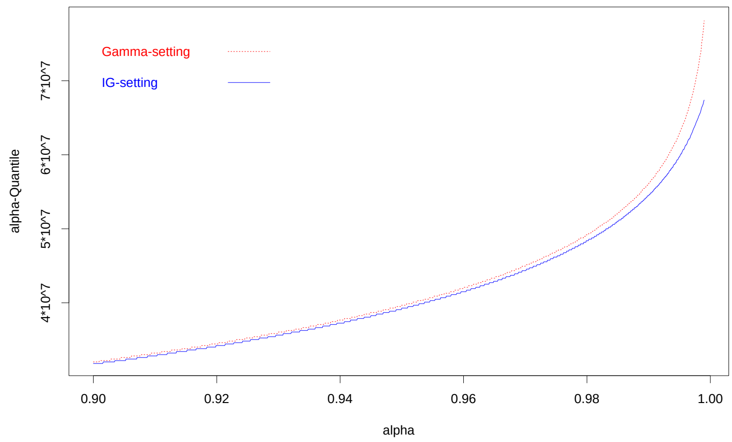


Figure 2.7: Plot of the α-quantiles of the loss distribution, $\alpha \in [0.9, 0.999]$. The quantiles in the Γ-approach are strictly greater than in the IG-setting for these α's.

Chapter 3

The Intensity-based Lévy Credit Risk Model

3.1 Introduction

A new approach to modeling credit risk based on the Heath-Jarrow-Morton methodology (see Heath, Jarrow, and Morton (1992)) was introduced in Bielecki and Rutkowski (1999, 2000, 2002). The main inputs to the Bielecki-Rutkowski model are the information on credit rating migration and the instantaneous forward rates. The result is an arbitrage-free model of defaultable (corporate) bonds. We follow this approach to construct a credit risk framework for the defaultable term structure driven by Lévy processes.

This chapter is organized as follows. First, we will present the Lévy term structure model introduced in Eberlein and Raible (1999) and Raible (2000) in a slightly more general setting, namely under the real-world measure. Next, we will expand this model by introducing defaultable bonds. In order to reflect reality, we will also consider restructuring after default. Finally, we will determine the market price of risk, and give an outline of the implementation of the model. The empirical relevance of the Lévy approach is illustrated by the analysis of a bond price data set. A similar approach to the Bielecki-Rutkowski approach has been developed in Schönbucher (1998). At the end of this chapter we show how the Schönbucher approach can be generalized with regard to Lévy processes.

Let $T^* > 0$ be a fixed time horizon date for all market activities. A default-free zero-coupon bond with maturity date $T \leq T^*$ is a financial security paying to its owner one currency unit at a prespecified date T in the future. A defaultable bond, in general a corporate bond, is a financial security *promising* its owner to pay one currency unit at the prespecified maturity date T in the future. The payment of the promised money unit depends on the financial capability of the issuer to pay the debt at time T. It might happen that the issuer defaults before or at time T, so the holder of the defaultable bond gets only a fractional amount of the promised currency unit or even nothing at all.

We denote by $B(t,T)$ the time t, $0 \leq t \leq T$, price of a default-free T-maturity bond. Note, that $B(T,T) = 1$.

We denote by $D_C(t,T)$ the time t price of a defaultable bond maturing at T. In this case, we have $D_C(T,T) \leq 1$ and $D_C(T,T) = 1$ on $\{\tau > T\}$, where we write τ to denote the time of default, which is a stopping time on a suitable enlargement of the stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \leq t \leq T^*})$, which will be specified in Section 3.2. The subscript C stresses the dependence of the defaultable bond from the credit migration process C, which will be specified in Section 3.3 and Section 3.4, respectively. Some results of this chapter have been presented in Eberlein and Özkan (2002).

3.2 The default-free term structure

A term structure model can be specified through the modeling of bond prices or equivalently, as proposed by Heath, Jarrow, and Morton (1992) (HJM), the (*instantaneous*) *forward rates*. Given the forward rates the bond prices are defined by setting

$$B(t,T) = \exp\left(- \int_t^T f(t,u) \, \mathrm{d}u \right), \text{ for all } t \in [0,T]. \tag{3.1}$$

The *short time interest rate* or *spot rate* is specified as $r(t) := f(t,t)$ when the forward rate is given. A general and more detailed introduction to bond markets can be found in Björk (1998) or Musiela and Rutkowski (1998).

In this section we give a short overview of the Lévy term structure model, which has been introduced by Eberlein and Raible (1999) and which is a generalization of the HJM methodology.

Let $L = (L^1, \ldots, L^d)$ be a d-dimensional Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T^*}$. Let $N := \{A \in \mathcal{F} \,|\, \mathbf{P}(A) = 0\}$ be the set of $\mathbf{P}$-null-sets in $\mathcal{F} = \mathcal{F}_{T^*}$ and $\mathcal{N} := \sigma(N)$ be the σ-field generated by N. We assume that $(\mathcal{F}_t)_{0 \leq t \leq T^*}$ is the smallest right-continuous filtration such that L is adapted and $\mathcal{N} \subset \mathcal{F}_0$ (see Jacod and Shiryaev (1987, III.2.12)). Let $\mathcal{P}$ be the predictable σ-field on $\Omega \times \mathbb{R}_+$. The probability measure $\mathbf{P}$ plays the role of the real-world measure. Note that the HJM approach in Eberlein and Raible (1999) was stated under the martingale measure.

The Lévy measure ν of the distribution of L_1 is assumed to satisfy the following integrability condition: there are constants $M, \varepsilon > 0$, such that

$$\int_{\{|x|>1\}} \exp(u^\top x) \, \nu(\mathrm{d}x) < \infty \quad \text{for all } u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d, \tag{3.2}$$

or equivalently, see Sato (1999, Theorem 25.3),

$$\mathrm{I\!E}[\exp(u^\top L_1)] < \infty \quad \text{for all } u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d.$$

Hence, the Lévy process L is a d-dimensional special semimartingale. Regarding the instantaneous forward rates we make the following assumption.

Assumption ($\mathbb{A}1$): For any fixed maturity $T \leq T^*$ the default free instantaneous forward rate $f(t,T)$ satisfies

$$\mathrm{d}f(t,T) = \partial_2 A(t,T)\ \mathrm{d}t - \partial_2 \Sigma(t,T)^\top\ \mathrm{d}L_t, \tag{3.3}$$

where ∂_2 denotes the derivation operator with respect to the second argument; and the $\mathcal{P} \otimes \mathcal{B}(\mathrm{I\!R}_+)$-measurable stochastic processes Σ and A with values in $\mathrm{I\!R}^d$ and $\mathrm{I\!R}$ respectively are defined on $\Omega \times \Delta$, where $\Delta = \{(s,T) \in \mathrm{I\!R}_+ \times \mathrm{I\!R}_+ \,|\, 0 \leq s \leq T \leq T^*\}$. The stochastic processes Σ and A are path-wise twice continuously differentiable in the second variable, where the partial derivatives of A and Σ are bounded and $\partial_{22}\Sigma(\omega; t, T)$ is a continuous function in t. In addition, we assume that $\Sigma(s,T)^i > 0$ for each $i = 1, \ldots, d$ and for all $(s,T) \in \Delta$ with $s \neq T$, and $A(s,s) = 0$ and $\Sigma(s,s) = (0, \ldots, 0)^\top$.

An equivalent representation of $\mathrm{d}f(t,T)$ can be obtained by the canonical decomposition of L (see e.g. Jacod and Shiryaev (1987, II.2.38)), namely

$$L_t = bt + cW_t + \int_0^t \int_{\mathrm{I\!R}^d} x\,(\mu^L - \nu^L)(\mathrm{d}s, \mathrm{d}x), \tag{3.4}$$

where $b \in \mathrm{I\!R}^d$, c is a $d \times d$ matrix, W is a standard d-dimensional Brownian motion, μ^L is the random measure of jumps, and ν^L is its compensator (see Jacod and Shiryaev (1987) for the definitions of μ^L and ν^L). Note that $\nu^L(\mathrm{d}t, \mathrm{d}x) = \nu(\mathrm{d}x)\ \mathrm{d}t$. Without loss of generality we can assume $b = 0 \in \mathrm{I\!R}^d$, so that $L_t = cW_t + \int_0^t \int_{\mathrm{I\!R}^d} x\,(\mu^L - \nu^L)(\mathrm{d}s, \mathrm{d}x)$.

The instantaneous forward rate dynamics (3.3) is equivalent to

$$\begin{aligned} \mathrm{d}f(t,T) = {} & \partial_2 A(t,T)\ \mathrm{d}t - \partial_2 \Sigma(t,T)^\top c\ \mathrm{d}W_t \\ & - \int_{\mathrm{I\!R}^d} \partial_2 \Sigma(t,T)^\top x\ \left(\mu^L - \nu^L\right)(\mathrm{d}t, \mathrm{d}x). \end{aligned} \tag{3.5}$$

The existence of the integral $\int_0^t \int_{\mathrm{I\!R}^d} (\partial_2 \Sigma(s,T)^\top x)(\mu^L - \nu^L)(\mathrm{d}s, \mathrm{d}x)$ is ensured by Jacod and Shiryaev (1987, II.1.30).

As shown in Raible (2000, Section 6.3.2), it can be easily seen that the Lévy term structure is a special case of the Björk/Di Masi/Kabanov/Runggaldier framework presented in Björk, Di Masi, Kabanov, and Runggaldier (1997).

Proposition 3.1 *The bond price $B(t,T)$ is given by*

$$B(t,T) = B(0,T)\exp\left(\int_0^t (r(s) - A(s,T))\,\mathrm{d}s + \int_0^t \Sigma(s,T)^\top\,\mathrm{d}L_s\right) \tag{3.6}$$

$$\begin{aligned}&= B(0,T)\exp\left(\int_0^t (r(s) - A(s,T))\,\mathrm{d}s + \int_0^t \Sigma(s,T)^\top c\,\mathrm{d}W_s\right.\\ &\qquad\left. + \int_0^t\int_{\mathbb{R}^d} \Sigma(s,T)^\top x\,(\mu^L - \nu^L)(\mathrm{d}s,\mathrm{d}x)\right).\end{aligned} \tag{3.7}$$

Proof: The proof of this equation is mainly the same as in Björk, Di Masi, Kabanov, and Runggaldier (1997, Proposition 5.2). Equation (3.3) yields for $0 \le t \le u \le T^*$

$$f(t,u) - f(0,u) = \int_0^t \partial_2 A(s,u)\,\mathrm{d}s - \int_0^t \partial_2\Sigma(s,u)^\top\,\mathrm{d}L_s \qquad \text{[Pr3.1.a]}$$

and

$$f(u,u) = f(0,u) + \int_0^u \partial_2 A(s,u)\,\mathrm{d}s - \int_0^u \partial_2\Sigma(s,u)^\top\,\mathrm{d}L_s. \qquad \text{[Pr3.1.b]}$$

Equations (3.1) and [Pr3.1.a] yield

$$\begin{aligned}\ln B(t,T) &= -\int_t^T f(t,u)\,\mathrm{d}u \;=\; -\int_t^T (f(0,u) + f(t,u) - f(0,u))\,\mathrm{d}u\\ &= -\int_t^T \left(f(0,u) + \int_0^t \partial_2 A(s,u)\,\mathrm{d}s - \int_0^t \partial_2\Sigma(s,u)^\top\,\mathrm{d}L_s\right)\mathrm{d}u.\end{aligned}$$

Hence by Fubini's Theorem, see Protter (1995, Theorem IV.4.45),

$$\begin{aligned}\ln B(t,T) &= -\int_t^T f(0,u)\,\mathrm{d}u - \int_t^T\int_0^t \partial_2 A(s,u)\,\mathrm{d}s\,\mathrm{d}u + \int_t^T\int_0^t \partial_2\Sigma(s,u)^\top\,\mathrm{d}L_s\,\mathrm{d}u\\ &= -\int_t^T f(0,u)\,\mathrm{d}u - \int_0^t\int_t^T \partial_2 A(s,u)\,\mathrm{d}u\,\mathrm{d}s + \int_0^t\int_t^T \partial_2\Sigma(s,u)^\top\,\mathrm{d}u\,\mathrm{d}L_s.\end{aligned}$$

Therefore

$$\begin{aligned}\ln B(t,T) &= -\left(\int_0^T f(0,u)\ \mathrm{d}u - \int_0^t f(0,u)\ \mathrm{d}u\right)\\ &\quad - \int_0^t \left(\int_s^T \partial_2 A(s,u)\ \mathrm{d}u - \int_s^t \partial_2 A(s,u)\ \mathrm{d}u\right)\ \mathrm{d}s\\ &\quad + \int_0^t \left(\int_s^T \partial_2 \Sigma(s,u)^\top\ \mathrm{d}u - \int_s^t \partial_2 \Sigma(s,u)^\top\ \mathrm{d}u\right)\ \mathrm{d}L_s\\ &= -\int_0^T f(0,u)\ \mathrm{d}u - \int_0^t\int_s^T \partial_2 A(s,u)\ \mathrm{d}u\,\mathrm{d}s + \int_0^t\int_s^T \partial_2\Sigma(s,u)^\top\ \mathrm{d}u\,\mathrm{d}L_s\\ &\quad + \int_0^t f(0,u)\ \mathrm{d}u + \int_0^t\int_s^t \partial_2 A(s,u)\ \mathrm{d}u\,\mathrm{d}s - \int_0^t\int_s^t \partial_2\Sigma(s,u)^\top\ \mathrm{d}u\,\mathrm{d}L_s.\end{aligned}$$

Using Fubini's Theorem, we get

$$\begin{aligned}\ln B(t,T) &= \ln B(0,T) - \int_0^t A(s,T)\ \mathrm{d}s + \int_0^t \Sigma(s,T)^\top\ \mathrm{d}L_s\\ &\quad + \int_0^t f(0,u)\,\mathrm{d}u + \int_0^t\int_0^u \partial_2 A(s,u)\ \mathrm{d}s\,\mathrm{d}u - \int_0^t\int_0^u \partial_2\Sigma(s,u)^\top\ \mathrm{d}L_s\,\mathrm{d}u.\end{aligned}$$

With equation [Pr3.1.b] we get

$$\ln B(t,T) = \ln B(0,T) + \int_0^t (r(s) - A(s,T))\ \mathrm{d}s + \int_0^t \Sigma(s,T)^\top\ \mathrm{d}L_s.$$

This proves equation (3.6). By combining equations (3.6) and (3.4) we get equation (3.7). Note that we have assumed $b = 0$. □

Proposition 3.2 *The dynamics of $B(t,T)$ for fixed $T \in [0,T^*]$ is given by*

$$\mathrm{d}B(t,T) = B(t-,T)\Bigg((r(t) + \frac{1}{2}|\Sigma(t,T)^\top c|^2 - A(t,T))\ \mathrm{d}t + \Sigma(t,T)^\top c\ \mathrm{d}W_t + \int_{\mathbb{R}^d} \Sigma(t,T)^\top x \left(\mu^L - \nu^L\right)(\mathrm{d}t,\mathrm{d}x) + \int_{\mathbb{R}^d} \left(\exp\left(\Sigma(t,T)^\top x\right) - 1 - \Sigma(t,T)^\top x\right) \mu^L(\mathrm{d}t,\mathrm{d}x)\Bigg). \tag{3.8}$$

Proof: Equation (3.7) yields

$$\ln B(t,T) = \ln B(0,T) + \int_0^t (r(s) - A(s,T))\ \mathrm{d}s + \int_0^t \Sigma(s,T)^\top c\ \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} \Sigma(s,T)^\top x\, (\mu^L - \nu^L)(\mathrm{d}s,\mathrm{d}x).$$

On the other hand the Itô formula, as given e.g. in Goll and Kallsen (2000, Lemma A.5), results in

$$\ln B(t,T) = \ln B(0,T) + \int_0^t \frac{1}{B(s-,T)}\,\mathrm{d}B(s,T) - \frac{1}{2}\int_0^t \frac{1}{B(s-,T)^2}\,\mathrm{d}\langle B^c(.,T), B^c(.,T)\rangle_s + \int_0^t \int_{\mathbb{R}} \left(\ln\left(\frac{B(s-,T)+x}{B(s-,T)}\right) - \frac{x}{B(s-,T)}\right) \mu^{B(.,T)}(\mathrm{d}s,\mathrm{d}x).$$

It is simple to get $B^c(.,T)$ from the Itô formula applied to $f(X_t) = \exp(\ln B(t,T))$, see Corollary A.6 in Goll and Kallsen (2000) and its proof where one can find formulae for the semimartingale characteristics in a more general case. We can easily calculate

$$\langle B^c(.,T), B^c(.,T)\rangle_t = \int_0^t \left(B(s-,T)^2 |\Sigma(s,T)^\top c|^2\right) \, \mathrm{d}s.$$

Note that the jumps of $B(.,T)$ satisfy the following relationship

$$\triangle B(t,T) = B(t-,T)\left(\mathrm{e}^{\triangle \ln B(t,T)} - 1\right) = B(t-,T)\left(\mathrm{e}^{\Sigma(t,T)^\top \triangle L_t} - 1\right).$$

Using the definitions of the random jump measures $\mu^{B(.,T)}$ and μ^L we get

$$\int_0^t \int_{\mathbb{R}} \left(\ln\left(\frac{B(s-,T)+x}{B(s-,T)}\right) - \frac{x}{B(s-,T)}\right) \mu^{B(.,T)}(\mathrm{d}s, \mathrm{d}x)$$

$$= -\sum_{0<s\leq t} \left(\exp\left(\Sigma(s,T)^\top \triangle L_s\right) - 1 - \Sigma(s,T)^\top \triangle L_S\right)$$

$$= -\int_0^t \int_{\mathbb{R}^d} \left(\exp\left(\Sigma(s,T)^\top x\right) - 1 - \Sigma(s,T)^\top x\right) \mu^L(\mathrm{d}s, \mathrm{d}x).$$

Therefore,

$$\begin{aligned}\int_0^t \frac{1}{B(s-,T)} \, \mathrm{d}B(s,T) = &\int_0^t \left(r(s) + \frac{1}{2}|\Sigma(s,T)^\top c|^2 - A(s,T)\right) \mathrm{d}s \\ &+ \int_0^t \Sigma(s,T)^\top c \, \mathrm{d}W_s \\ &+ \int_0^t \int_{\mathbb{R}^d} \Sigma(s,T)^\top x \left(\mu^L - \nu^L\right)(\mathrm{d}s, \mathrm{d}x) \\ &+ \int_0^t \int_{\mathbb{R}^d} \left(\exp\left(\Sigma(s,T)^\top x\right) - 1 - \Sigma(s,T)^\top x\right) \mu^L(\mathrm{d}s, \mathrm{d}x).\end{aligned}$$

The last equation proves our proposition. □

The short-term interest rate $r(t) = f(t,t)$ helps us to introduce the adapted process $B = (B_t)_{t\in[0,T^*]}$ of finite variation with continuous sample paths given by

$$B_t := \exp\left(\int_0^t r(s) \, \mathrm{d}s\right). \tag{3.9}$$

In financial interpretation B is the risk-free savings account. Further details on the risk-free savings account can be found for example in Musiela and Rutkowski (1998, Section 11.1.3).

Proposition 3.3 *The discounted bond price* $Z(t,T) := \frac{1}{B_t} B(t,T)$ *satisfies*

$$Z(t,T) = Z(0,T) \exp \left(- \int_0^t A(s,T) \, \mathrm{d}s + \int_0^t \Sigma(s,T)^\top c \, \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} \Sigma(s,T)^\top x \left(\mu^L - \nu^L \right) (\mathrm{d}s, \mathrm{d}x) \right) \quad (3.10)$$

and follows the stochastic dynamics

$$\mathrm{d}Z(t,T) = Z(t-,T) \left(a(t,T) \, \mathrm{d}t + \int_{\mathbb{R}^d} \Sigma(t,T)^\top x \left(\mu^L - \nu^L \right) (\mathrm{d}t, \mathrm{d}x) + \Sigma(t,T)^\top c \, \mathrm{d}W_t + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma(t,T)^\top x} - 1 - \Sigma(t,T)^\top x \right) \mu^L(\mathrm{d}t, \mathrm{d}x) \right), \quad (3.11)$$

where

$$a(t,T) = -A(t,T) + \frac{1}{2} |\Sigma(t,T)^\top c|^2. \quad (3.12)$$

Proof: This is a direct consequence of Proposition 3.2 or of Björk, Di Masi, Kabanov, and Runggaldier (1997, Proposition 5.3). □

3.2.1 Absence of arbitrage

The Lévy term structure model presented in Eberlein and Raible (1999) operates directly under a martingale measure. Here, we model the forward rate under the real-world measure $\mathbf{P}$, see Assumption ($\mathbb{A}1$), as in Björk, Di Masi, Kabanov, and Runggaldier (1997, Section 5.4).

Let $\mathbf{P}_t := \mathbf{P}|_{\mathcal{F}_t}$ be the restriction of $\mathbf{P}$ to the σ-algebra $\mathcal{F}_t$ and

$$\mathcal{Q} := \Big\{ \widetilde{\mathbf{P}} \in \mathcal{M}^1(\Omega, \mathcal{F}) \, : \, \widetilde{\mathbf{P}}_t \sim \mathbf{P}_t \text{ for all } 0 \leq t < \infty \text{ and } (Z(t,T))_{0 \leq t \leq T} \text{ is a local } \widetilde{\mathbf{P}} - \text{martingale for every } T \in [0, T^*] \Big\},$$

where $\mathcal{M}^1(\Omega, \mathcal{F})$ denotes the set of all probability measures on the measurable space $(\Omega, \mathcal{F})$. Following Björk, Di Masi, Kabanov, and Runggaldier (1997) we say that the EMM-property holds if $\mathcal{Q}$ is nonempty. EMM is the abbreviation for "equivalent martingale measure". In the setting of Eberlein and Raible (1999) $\mathbf{P}$ is a martingale measure, and therefore the EMM-property holds. Raible (2000) shows that $\mathcal{Q}$ is a singleton, and therefore the Lévy term structure model in Eberlein and Raible (1999) is approximately complete (see Björk, Di Masi, Kabanov, and Runggaldier (1997, Section 6)).

The EMM-property implies that the coefficients of the model in Assumption ($\mathbb{A}1$) are interrelated and cannot be chosen in an arbitrary way. The following result is based on Björk, Di Masi, Kabanov, and Runggaldier (1997, Proposition 5.3) and is a generalization of the HJM-equality.

Proposition 3.4

$$\mathbf{P} \in \mathcal{Q} \Leftrightarrow \quad (i) \quad \int_0^t \int_{\mathbb{R}} \left(e^{\Sigma(s,T)^\top x} - 1 - \Sigma(s,T)^\top x \right) \nu(\mathrm{d}x)\, \mathrm{d}s < \infty$$

$$(ii) \quad a(t,T) + \int_{\mathbb{R}} \left(e^{\Sigma(t,T)^\top x} - 1 - \Sigma(t,T)^\top x \right) \nu(\mathrm{d}x) = 0 \quad \mathrm{d}\mathbf{P} \otimes \mathrm{d}t\text{–a.e.}$$

If $\mathbf{P} \in \mathcal{Q}$, then the stochastic dynamics of $Z(t,T)$ in equation (3.11) can be written as

$$\mathrm{d}Z(t,T) = Z(t-,T) \left(\Sigma(t,T)^\top c \, \mathrm{d}W_t + \int_{\mathbb{R}^d} \left(e^{\Sigma(t,T)^\top x} - 1 \right) \left(\mu^L - \nu^L \right) (\mathrm{d}s, \mathrm{d}x) \right).$$

To ensure that $\mathcal{Q}$ is nonempty we have to assume the following condition, see Theorem 5.7 in Björk, Di Masi, Kabanov, and Runggaldier (1997).

Condition ($\mathbb{M}1$): Suppose that there exists a predictable process $\phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a predictable function $\widetilde{Y} : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+ \backslash \{0\}$ such that

1. $\int_0^t |\phi(s)|^2 \, \mathrm{d}s < \infty \qquad \mathbf{P}\text{-a.s.},$
2. $\int_0^t \int_{\mathbb{R}^d} \left(1 - \sqrt{\widetilde{Y}(s,x)} \right)^2 \nu(\mathrm{d}x)\, \mathrm{d}s < \infty \qquad \mathbf{P}\text{-a.s.},$
3. For any $T \in [0, T^*]$

$$\int_0^t \int_{\mathbb{R}^d} \left(e^{\Sigma(s,T)^\top x} - 1 \right) \mathbb{1}_{\{\Sigma(s,T)^\top x > \ln 2\}} \widetilde{Y}(s,x)\, \nu(\mathrm{d}x)\, \mathrm{d}s < \infty \qquad \mathbf{P}\text{-a.s.},$$

4. For any $T \in [0, T^*]$

$$a(t,T) + \Sigma(t,T)^\top c\,\phi(t) + \int_{\mathbb{R}^d} \left(\left(e^{\Sigma(t,T)^\top x} - 1 \right) \widetilde{Y}(t,x) - \Sigma(t,T)^\top x \right) \nu(\mathrm{d}x) = 0, \quad \mathrm{d}\mathbf{P} \otimes \mathrm{d}t\text{−a.s.} \tag{3.13}$$

5. $\mathbb{E}_{\mathbf{P}}[\rho_t] = 1$ for all $t \in [0, T^*]$, where $\rho = (\rho_t)_{t \in [0,T^*]}$ is the process given by

$$\rho_t = \exp\left(\int_0^t \phi(s)^\top \,\mathrm{d}W_s - \frac{1}{2}\int_0^t |\phi(s)|^2 \,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \left(\widetilde{Y}(s,x) - 1 \right) \left(\mu^L - \nu^L \right)(\mathrm{d}s, \mathrm{d}x) - \int_0^t \int_{\mathbb{R}^d} \left(\widetilde{Y}(s,x) - 1 - \ln \widetilde{Y}(s,x) \right) \mu^L(\mathrm{d}s, \mathrm{d}x) \right). \tag{3.14}$$

Conditions ($\mathbb{M}$1.1.) – ($\mathbb{M}$1.3.) are the usual integrability conditions for ϕ and $\widetilde{Y}$. Condition ($\mathbb{M}$1.4.) is a generalization of the well-known HJM-equality. Condition ($\mathbb{M}$1.5.) ensures that the measure $\mathbf{P}^*$ which is defined by $\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}} = \rho_{T^*}$ is a local martingale measure which is equivalent to $\mathbf{P}$. It follows from Proposition 5.7 in Björk, Di Masi, Kabanov, and Runggaldier (1997) that $\mathbf{P}^*$ is an element of $\mathcal{Q}$. Note that its density is explicitly given by

$$\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}} = \exp\left(\int_0^{T^*} \phi(s)^\top \,\mathrm{d}W_s - \frac{1}{2}\int_0^{T^*} |\phi(s)|^2 \,\mathrm{d}s + \int_0^{T^*} \int_{\mathbb{R}^d} \left(\widetilde{Y}(s,x) - 1 \right) \left(\mu^L - \nu^L \right)(\mathrm{d}s, \mathrm{d}x) - \int_0^{T^*} \int_{\mathbb{R}^d} \left(\widetilde{Y}(s,x) - 1 - \ln \widetilde{Y}(s,x) \right) \mu^L(\mathrm{d}s, \mathrm{d}x) \right) \tag{3.15}$$

Remark: In case that the discontinuous part of L has sample paths of finite variation, i.e. $\int_{\{|x| \le 1\}} |x|\, \nu(\mathrm{d}x) < \infty$, then the density process ρ simplifies notably to

$$\rho_t = \exp\left(\int_0^t \phi(s)^\top \, \mathrm{d}W_s - \frac{1}{2}\int_0^t |\phi(s)|^2 \, \mathrm{d}s \right.$$
$$\left. + \int_0^t \int_{\mathbb{R}^d} \ln \widetilde{Y}(s,x)\, \mu^L(\mathrm{d}s,\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^d} \left(1 - \widetilde{Y}(s,x)\right) \nu^L(\mathrm{d}s,\mathrm{d}x)\right).$$

The dynamics of the discounted bond price process $Z(.,T)$ under the probability measure $\mathbf{P}^*$ can be expressed as, see Björk, Di Masi, Kabanov, and Runggaldier (1997, equation (5.41)),

$$\begin{aligned}
\mathrm{d}Z(t,T) = Z(t-,T) \Bigg(\Bigg(& a(t,T) + \Sigma(t,T)^\top c\phi(t) \\
& + \int_{\mathbb{R}^d} \left(\left(\mathrm{e}^{\Sigma(t,T)^\top x} - 1 \right) \widetilde{Y}(t,x) - \Sigma(t,T)^\top x \right) \nu(\mathrm{d}x) \Bigg) \, \mathrm{d}t \\
& + \Sigma(t,T)^\top c \, \mathrm{d}\widetilde{W}_t \\
& + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma(t,T)^\top x} - 1 \right) \left(\mu^L - \widetilde{Y}(t,x)\nu^L \right) (\mathrm{d}t,\mathrm{d}x) \Bigg) \\
= Z(t-,T) \Bigg(& \Sigma(t,T)^\top c \, \mathrm{d}\widetilde{W}_t + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma(t,T)^\top x} - 1 \right) \left(\mu^L - \widetilde{\nu}^L \right) (\mathrm{d}t,\mathrm{d}x) \Bigg),
\end{aligned} \tag{3.16}$$

where $\widetilde{W}_t := W_t - \int_0^t \phi(s)\,\mathrm{d}s$ is a d-dimensional standard Brownian process under $\mathbf{P}^*$ and the random measure $\widetilde{\nu}^L(\omega;\mathrm{d}t,\mathrm{d}x) := \widetilde{Y}(\omega,t,x)\nu^L(\mathrm{d}t,\mathrm{d}x)$ is the $\mathbf{P}^*$-compensator of μ^L.

Remark: Due to equation (3.2) L_1 has a moment generating function $u \mapsto \mathbb{E}[\mathrm{e}^{u^\top L_1}]$ on some open interval $(-a,b)^d \supset [-M,M]^d$, $a,b,M > 0$. Therefore, the logarithm of the moment generating function κ is well-defined by

$$\kappa(u) := \ln \mathbb{E}[\mathrm{e}^{u^\top L_1}], \quad u \in (-a,b)^d. \tag{3.17}$$

If $A(t,T) = \kappa(\Sigma(t,T))$, then equation (3.13) holds for $\phi = 0$ and $\widetilde{Y} = 1$. Thus $\mathbf{P} = \mathbf{P}^*$. This means that the Lévy term structure model is considered directly under the martingale measure. More information on this case can be found in Eberlein and Raible (1999) and Raible (2000).

3.2.2 Construction of the density process

In this subsection we want to motivate the assumption made in Condition ($\mathbb{M}$1.5.). Instead of assuming the existence of ϕ and $\widetilde{Y}$ as done above, we can also construct explicitly an equivalent martingale measure for fixed time horizon. We follow Shiryaev (1999, Section VII.3g), where martingale measures are constructed in the semimartingale framework.

For fixed $T \in [0, T^*]$ we define the process $M = (M_t)_{0 \le t \le T}$ by

$$M_t = \int_0^t (Z(s-, T))^{-1} \,\mathrm{d}Z(s, T), \tag{3.18}$$

which is a special semimartingale. Instead of considering M we could consider Z itself, but this is technically more sumptuary. The semimartingale characteristics are the main input to the construction of the martingale measure, see Jacod and Shiryaev (1987, Section II.2). The determination of the semimartingale characteristics of Z can be achieved with some effort by using Corollary A.6 in Goll and Kallsen (2000). In contrast, the characteristics of M can be easily deduced from the dynamics of M. Note that both approaches lead exactly to the same density process.

The dynamics of M is given by, cf. equation (3.11),

$$\begin{aligned}\mathrm{d}M_t = a(t,T)\ \mathrm{d}t &+ \int_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma(t,T)^\top x} - 1 - \Sigma(t,T)^\top x\right)\ \nu^L(\mathrm{d}t, \mathrm{d}x) \\ &+ \Sigma(t,T)^\top c\ \mathrm{d}W_t + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma(t,T)^\top x} - 1\right)\ \left(\mu^L - \nu^L\right)(\mathrm{d}t, \mathrm{d}x).\end{aligned}$$

The jumps of M, $\triangle M$, satisfy

$$\triangle M_t = \int_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma(t,T)^\top x} - 1\right)\ \mu^L(\{t\}, \mathrm{d}x) = \left(\mathrm{e}^{\Sigma(t,T)^\top \triangle L_t} - 1\right) \mathbb{1}_{\{\triangle L_t \neq 0\}} > -1.$$

Thus,

$$\int_0^t \int_{-1}^{\infty} f(s,u)\,\mu^M(\mathrm{d}s, \mathrm{d}u) = \int_0^t \int_{\mathbb{R}^d} f(s, \mathrm{e}^{\Sigma(s,T)^\top x} - 1)\,\mu^L(\mathrm{d}s, \mathrm{d}x),$$

for any positive measurable function f on $\mathbb{R}_+ \times (-1, \infty)$ and also

$$\int_0^t \int_{-1}^{\infty} f(s,u)\,\nu^M(\mathrm{d}s, \mathrm{d}u) = \int_0^t \int_{\mathbb{R}^d} f(s, \mathrm{e}^{\Sigma(s,T)^\top x} - 1)\,\nu^L(\mathrm{d}s, \mathrm{d}x).$$

As a result of the last two equations the dynamics $\mathrm{d}M_t$ can be written as

$$\begin{aligned}\mathrm{d}M_t &= a(t,T)\ \mathrm{d}t + \Sigma(t,T)^\top c\ \mathrm{d}W_t \\ &\quad + \int_{-1}^{\infty} (u - \ln(u+1))\ \nu^M(\mathrm{d}t, \mathrm{d}u) + \int_{-1}^{\infty} u\left(\mu^M - \nu^M\right)(\mathrm{d}t, \mathrm{d}u).\end{aligned} \tag{3.19}$$

Equation (3.19) gives the canonical decomposition of M. The semimartingale characteristics (B^M, C^M, ν^M) of M are given by

$$B_t^M = \int_0^t a(s,T)\ \mathrm{d}s + \int_0^t \int_{-1}^{\infty} (u - \ln(u+1))\ \nu^M(\mathrm{d}s, \mathrm{d}u), \tag{3.20}$$

where $a(s,T)$ is given in equation (3.12),

$$C_t^M = \langle M^c, M^c \rangle_t = \int_0^t |\Sigma(s,T)^\top c|^2\ \mathrm{d}s, \tag{3.21}$$

and finally

$$\nu^M([0,t] \times G) = \int_0^t \int_{\mathbb{R}^d} 1\!\!1_{[0,t]\times G}(s, \mathrm{e}^{\Sigma(s,T)^\top x} - 1)\, \nu^L(\mathrm{d}s, \mathrm{d}x). \tag{3.22}$$

The change of measure is fully characterized by two processes (or measurable functions) β and Y, as can be seen in Girsanov's Theorem, which can be found in a general setting in Jacod and Shiryaev (1987, III.3.24).

Due to the fact that M is a special semimartingale, the measure we construct is a martingale measure, if we can find two predictable processes or functions $\beta : \mathbb{R} \to \mathbb{R}$ and $Y : \mathbb{R}_+ \times (-1, \infty) \to \mathbb{R}_+ \setminus \{0\}$, see Shiryaev (1999), such that

$$B_t^M + \int_0^t \beta(s)\, \mathrm{d}C_s^M + \int_0^t \int_{-1}^{\infty} u\,(Y(s,u) - 1)\, \nu^M(\mathrm{d}s, \mathrm{d}u) = 0, \tag{3.23}$$

or equivalently,

$$\begin{aligned}&\int_0^t a(s,T)\, \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma(s,T)^\top x} - 1 - \Sigma(s,T)^\top x\right)\, \nu^L(\mathrm{d}s, \mathrm{d}x) + \int_0^t \beta(s) c \Sigma^2(s,T)\, \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma(s,T)^\top x} - 1\right)\left(Y\left(s, \mathrm{e}^{\Sigma(s,T)^\top x} - 1\right) - 1\right)\, \nu^L(\mathrm{d}s, \mathrm{d}x) = 0.\end{aligned} \tag{3.24}$$

One solution to the integral equation (3.24) is given by

$$\begin{aligned}\beta(s) &= \frac{A(s,T) - \frac{1}{2}|\Sigma(s,T)^\top c|^2}{|\Sigma(s,T)^\top c|^2}, \quad s < T \\ \beta(T) &= 0,\end{aligned} \tag{3.25}$$

and

$$\widetilde{Y}(s,x) = Y(s, \mathrm{e}^{\Sigma(s,T)^\top x} - 1) = \begin{cases} \frac{\Sigma(s,T)^\top x}{\mathrm{e}^{\Sigma(s,T)^\top x} - 1}, & \text{if } \Sigma(s,T)^\top x \neq 0, \\ 1, & \text{if } \Sigma(s,T)^\top x = 0, \end{cases} \quad x \in \mathbb{R}^d, \tag{3.26}$$

where

$$Y(s,u) = \begin{cases} \frac{\ln(u+1)}{u}, & \text{if } u \neq 0, \\ 1, & \text{if } u = 0, \end{cases} \quad u > -1. \tag{3.27}$$

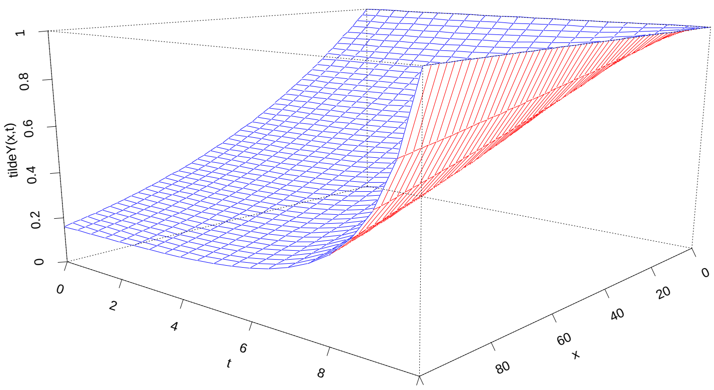


Figure 3.1: Two-dimensional plot of $\widetilde{Y}$, equation (3.26). This example: Vasiček volatility structure $\Sigma(t,T) = \frac{\widehat{\sigma}}{a}(1 - \mathrm{e}^{-a(T-t)}), \widehat{\sigma} = 0.015, a = 0.5, T = 10$.

In Figure 3.1 we visualize this special choice of Y. Note the typical shape of Y, $Y(s,u)$ is close to one for u close to 0, intuitively speaking, the behavior of very small jump sizes is very much the same under both equivalent measures. There is an interesting connection to Raible (2000, Proposition 2.20), he shows that in case of a generalized hyperbolic Lévy processes with parameters $(\lambda, \alpha, \beta, \delta, \mu)$, an equivalent change of measure such that the process remains generalized hyperbolic with parameters $(\lambda', \alpha', \beta', \delta', \mu')$ is fully characterized by the condition that $\delta = \delta'$ and $\mu = \mu'$. And exactly these parameters δ and μ are characterized by the small jumps, see Corollary 2.23 and Proposition 2.24 in Raible (2000). The choice of

β and Y is not unique. If β and Y are deterministic then the semimartingale characteristics remain deterministic, i.e. the process has still independent – but not necessarily stationary – increments. If we make assumptions concerning the drift A, then the change of measure can be accomplished in such a way that the driving Lévy process is again a Lévy process under the new measure. In this case $\widetilde{Y}$ is a deterministic function that does not depend on the time parameter, see e.g. Raible (2000, Proposition 2.19) or Jacod and Shiryaev (1987, Theorem IV.4.39). Some more aspects regarding the change of measure in the context of Lévy processes can be found in Sato (2000).

Equation (3.23) is the analogue of equation (3.13) with $\phi(s)^\top := \beta(s)\Sigma(s,T)^\top c$. In addition, note that

$$\int_0^t \int_{-1}^{\infty} f(s,u) Y(s,u)\, \nu^M(\mathrm{d}s,\mathrm{d}u) = \int_0^t \int_{\mathbb{R}^d} f(s, \mathrm{e}^{\Sigma(s,T)^\top x} - 1)\widetilde{Y}(s,x)\, \nu^L(\mathrm{d}s,\mathrm{d}x)$$

for any positive measurable function f.

We assume that

$$\int_0^t \beta^2(s) \mathrm{d}C_s^M < \infty, \tag{3.28}$$

$$\int_0^t \int_{-1}^{\infty} \left(1 - \sqrt{Y(s,u)}\right)^2 \nu^M(\mathrm{d}s,\mathrm{d}u) < \infty, \quad \text{for every } t \in [0,T]. \tag{3.29}$$

Note that $(|x|^2 \wedge |x|)\, Y(t,x) * \nu_t^M \in \mathcal{A}_{\text{loc}}$, see Jacod and Shiryaev (1987, I.3.8), because $Y(t,x) \leq 1$. Consequently, M remains a special semimartingale under the new measure. (Wolfgang Kluge shows in his diploma thesis that this is true also in the setting where the change of measure is such that L remains a Lévy process under the new measure.)

Provided that the inequalities (3.28) and (3.29) hold, we can apply Theorem 3 in Shiryaev (1999, page 703) to construct the density process ρ, which is the Doléans-Dade exponential (see Jacod and Shiryaev (1987, I.4)) of the process

$N = (N_t)$. The process N is given by

$$
\begin{aligned}
N_t &= \int_0^t \beta(s)\ \mathrm{d}M_s^c + \int_0^t \int_{-1}^{\infty} (Y(s,u)-1)\left(\mu^M - \nu^M\right)(\mathrm{d}s, \mathrm{d}u) \\
&= \int_0^t \underbrace{\frac{A(s,T) - \frac{1}{2}|\Sigma(s,T)^\top c|^2}{|\Sigma(s,T)^\top c|^2}\Sigma(s,T)^\top c}_{=\phi(s)^\top}\,\mathrm{d}W_s \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} \left(Y(s, \mathrm{e}^{\Sigma(s,T)^\top x} - 1) - 1\right)\left(\mu^L - \nu^L\right)(\mathrm{d}s, \mathrm{d}x) \\
&= \int_0^t \phi(s)^\top\ \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} \left(\frac{\Sigma(s,T)^\top x}{\mathrm{e}^{\Sigma(s,T)^\top x} - 1} - 1\right)\left(\mu^L - \nu^L\right)(\mathrm{d}s, \mathrm{d}x). \qquad (3.30)
\end{aligned}
$$

Therefore

$$
\triangle N_t = \int_{\mathbb{R}^d} \left(\frac{\Sigma(t,T)^\top x}{\mathrm{e}^{\Sigma(t,T)^\top x} - 1} - 1\right)\mu^L(\{t\}, \mathrm{d}x) > -1
$$

and for any positive measurable function f

$$
\int_0^t \int_{-1}^{\infty} f(s,u)\,\mu^N(\mathrm{d}s, \mathrm{d}u) = \int_0^t \int_{\mathbb{R}^d} f\left(s, \frac{\Sigma(s,T)^\top x}{\mathrm{e}^{\Sigma(s,T)^\top x} - 1} - 1\right)\mu^L(\mathrm{d}s, \mathrm{d}x).
$$

The density process ρ is given by

$$
\begin{aligned}
\rho_t &= \mathcal{E}(N)_t \\
&= \exp\left(N_t - \frac{1}{2}\int_0^t \beta^2(s)\,\mathrm{d}C_s^M + \int_0^t \int_{-1}^{\infty} \left(\ln(u+1) - u\right)\,\mu^N(\mathrm{d}s, \mathrm{d}u)\right).
\end{aligned}
$$

Hence

$$\rho_t = \exp\left(\int_0^t \phi(s)^\top \, \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} \frac{\Sigma(s,T)^\top x}{\mathrm{e}^{\Sigma(s,T)^\top x} - 1} - 1 \left(\mu^L - \nu^L\right)(\mathrm{d}s, \mathrm{d}x)\right.$$

$$- \frac{1}{2}\int_0^t \frac{(A(s,T) - \frac{1}{2}|\Sigma(s,T)^\top c|^2)^2}{|\Sigma(s,T)^\top c|^4} |\Sigma(s,T)^\top c|^2 \, \mathrm{d}s$$

$$\left. + \int_0^t \int_{\mathbb{R}^d} \left(\ln\left(\frac{\Sigma(s,T)^\top x}{\mathrm{e}^{\Sigma(s,T)^\top x} - 1}\right) - \frac{\Sigma(s,T)^\top x}{\mathrm{e}^{\Sigma(s,T)^\top x} - 1} + 1\right) \mu^L(\mathrm{d}s, \mathrm{d}x)\right)$$

$$= \exp\left(\int_0^t \phi(s)^\top \, \mathrm{d}W_s - \frac{1}{2}\int_0^t |\phi(s)|^2 \, \mathrm{d}s\right.$$

$$+ \int_0^t \int_{\mathbb{R}^d} \left(\widetilde{Y}(s,x) - 1\right) \left(\mu^L - \nu^L\right)(\mathrm{d}s, \mathrm{d}x)$$

$$\left. + \int_0^t \int_{\mathbb{R}^d} \left(\ln \widetilde{Y}(s,x) - \widetilde{Y}(s,x) + 1\right) \mu^L(\mathrm{d}s, \mathrm{d}x)\right). \tag{3.31}$$

In case that L is a Lévy processes where the discontinuous part has sample paths of finite variation the density process simplifies to

$$\rho_t = \exp\left(\int_0^t \phi(s)^\top \, \mathrm{d}W_s - \frac{1}{2}\int_0^t |\phi(s)|^2 \, \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \ln\left(\frac{\Sigma(s,T)^\top x}{\mathrm{e}^{\Sigma(s,T)^\top x} - 1}\right) \mu^L(\mathrm{d}s, \mathrm{d}x)\right.$$

$$\left. - \int_0^t \int_{\mathbb{R}^d} \left(\frac{\Sigma(s,T)^\top x}{\mathrm{e}^{\Sigma(s,T)^\top x} - 1} - 1\right) \nu^L(\mathrm{d}s, \mathrm{d}x)\right)$$

$$= \exp\left(\int_0^t \phi(s)^\top \, \mathrm{d}W_s - \frac{1}{2}\int_0^t |\phi(s)|^2 \, \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \ln \widetilde{Y}(s,x) \, \mu^L(\mathrm{d}s, \mathrm{d}x)\right.$$

$$\left. - \int_0^t \int_{\mathbb{R}^d} \left(\widetilde{Y}(s,x) - 1\right) \nu^L(\mathrm{d}s, \mathrm{d}x)\right). \tag{3.32}$$

The form of the density process ρ in equation (3.31) that we have constructed in the last section coincides with the form of the density process given in equation (3.15).

In the next section we will extend the default-free bond market and consider defaultable bonds.

3.3 Pre-default term structure

Rating agencies, like Moody's or Standard & Poor's, provide valuable information on the credit worthiness of major firms. Furthermore, banks have tailor-made internal rating systems for their credit portfolios. More information on this topic can be found e.g. in Crouhy, Galai, and Mark (2001). Our credit risk model is able to exploit this rating information. The possible rating classes are described by the set $\mathcal{K} = \{1, 2, \dots, K\}$, where class 1 corresponds to the best possible rating different from default-freeness, and the class K corresponds to the default event. For example, in the Second Consultative Paper of the Basel Committee on Banking Supervision (Basel II) the number of ratings is at least eight, including the state of default.

Furthermore, we assume that recovery rates $\delta_i \in [0, 1)$ are constant in each class. The recovery rate δ_i is the fraction of the face value which the bond holder receives at the maturity date T in the case of default if the bond was in rating class i before default. An interesting way of summarizing different recovery conventions can be found in Belanger, Shreve, and Wong (2001).

In this section we consider the default-free term structure given in Section 3.2, and $K - 1$ different term structures each corresponding to one rating class $i \in \mathcal{K} \setminus \{K\}$.

Assumption ($\mathbb{A}2$): For any $T \in [0, T^*]$ the instantaneous forward rate corresponding to the rating class $i \in \{1, \dots, K - 1\}$ satisfies under the real-world probability measure $\mathbf{P}$

$$\mathrm{d}g_i(t, T) = \partial_2 A_i(t, T)\ \mathrm{d}t - \partial_2 \Sigma_i(t, T)^\top\ \mathrm{d}L_t^{(i)}, \tag{3.33}$$

where the $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable stochastic processes A_i and Σ_i satisfy the regularity conditions of ($\mathbb{A}1$). We assume that the canonical decomposition of $L^{(i)}$ is $L_t^{(i)} = b_i t + c_i W_t + \int_0^t \int_{\mathbb{R}^d} p_i x\, (\mu^L - \nu^L)(\mathrm{d}s, \mathrm{d}x)$, where $b_i \in \mathbb{R}^d$ may be assumed to be zero again, and p_i and c_i are $d \times d$-matrices.

If $d \geq K$ we may take $c_i = c$ and $p_i = \mathrm{Id}$, but if one prefers to design a low-dimensional Lévy term structure model, the c_i and p_i will allow for more flexibility.

Using the canonical decomposition of $L^{(i)}$ equation (3.33) can be written as

$$\begin{aligned} \mathrm{d}g_i(t, T) = {} & \partial_2 A_i(t, T)\ \mathrm{d}t - \partial_2 \Sigma_i(t, T)^\top c_i\ \mathrm{d}W_t \\ & - \int_{\mathbb{R}^d} \partial_2 \Sigma_i(t, T)^\top p_i x\ \left(\mu^L - \nu^L\right)(\mathrm{d}t, \mathrm{d}x). \end{aligned} \tag{3.34}$$

Furthermore, to exclude arbitrage we assume that

$$g_{K-1}(t,T) > g_{K-2}(t,T) > \ldots > g_1(t,T) > f(t,T).$$

Thus risky corporate bonds have higher forward rates than less risky bonds. This reflects the fact that the higher the default risk, the lower is the price of the bond. For conditions which generate monotonicity in the Gaussian case see the comparison theorems in Chapter IX of Revuz and Yor (1999).

We set

$$D_i(t,T) = \exp\left(-\int_t^T g_i(t,u)\,\mathrm{d}u\right) \tag{3.35}$$

for $i = 1,\ldots,K-1$. By analogy with the bond price process $B(.,T)$, see equation (3.8), we obtain the following dynamics of $D_i(.,T)$:

$$\begin{aligned}
&\mathrm{d}D_i(t,T)\\
&= D_i(t-,T)\Bigg((a_i(t,T)+g_i(t,t))\,\mathrm{d}t + \int_{\mathbb{R}^d} \Sigma_i(t,T)^\top p_i x\left(\mu^L - \nu^L\right)(\mathrm{d}t,\mathrm{d}x)\\
&\qquad + \Sigma_i(t,T)^\top c_i\,\mathrm{d}W_t + \int_{\mathbb{R}^d}\left(\mathrm{e}^{\Sigma_i(t,T)^\top p_i x} - 1 - \Sigma_i(t,T)^\top p_i x\right)\mu^L(\mathrm{d}t,\mathrm{d}x)\Bigg),
\end{aligned} \tag{3.36}$$

where

$$a_i(t,T) = \frac{1}{2}|\Sigma_i(t,T)^\top c_i|^2 - A_i(t,T). \tag{3.37}$$

We define $Z_i(t,T) := \frac{1}{B_t}D_i(t,T)$. Under P the process $Z_i(.,T)$ satisfies

$$\begin{aligned}
\mathrm{d}Z_i(t,T) = Z_i(t-,T)\Bigg(&(a_i(t,T)+g_i(t,t)-r(t))\,\mathrm{d}t\\
&+\int_{\mathbb{R}^d}\left(\mathrm{e}^{\Sigma_i(t,T)^\top p_i x} - 1 - \Sigma_i(t,T)^\top p_i x\right)\nu^L(\mathrm{d}t,\mathrm{d}x)\\
&+\Sigma_i(t,T)^\top c_i\,\mathrm{d}W_t + \int_{\mathbb{R}^d}\left(\mathrm{e}^{\Sigma_i(t,T)^\top p_i x} - 1\right)\left(\mu^L - \nu^L\right)(\mathrm{d}t,\mathrm{d}x)\Bigg).
\end{aligned}$$

By analogy with equation (3.16) we deduce that $Z_i(.,T)$ satisfies under $\mathbf{P}^*$

$$
\begin{aligned}
\mathrm{d}Z_i(t,T) = Z_i(t-,T) \Bigg(&\left(a_i(t,T) + g_i(t,t) - r(t) + \Sigma_i(t,T)^\top c_i \phi(t)\right) \mathrm{d}t \\
&+ \int\limits_{\mathbb{R}^d} \left(\left(\mathrm{e}^{\Sigma_i(t,T)^\top p_i x} - 1 \right) \widetilde{Y}(t,x) - \Sigma_i(t,T)^\top p_i x \right) \; \nu^L(\mathrm{d}t, \mathrm{d}x) \\
&+ \Sigma_i(t,T)^\top c_i \, \mathrm{d}\widetilde{W}_t + \int\limits_{\mathbb{R}^d} \left(\mathrm{e}^{\Sigma_i(t,T)^\top p_i x} - 1 \right) \left(\mu^L - \widetilde{Y}(t,x) \nu^L \right) (\mathrm{d}t, dx) \Bigg).
\end{aligned}
$$

Therefore, $Z_i(.,T)$ is not a local $\mathbf{P}^*$-martingale in general.

Remark: The process $D_i(.,T)$ (respectively $Z_i(.,T)$) is not the genuine price (respectively discounted price) of a defaultable bond. It is its conditional price at time t given that the defaultable zero coupon bond is in the rating class $i \in \mathcal{K} \setminus \{K\}$ during the time interval $[0,t]$.

For the construction of the unconditional zero coupon bond price we have to take the migrations between the different rating classes into consideration. Furthermore, the migration intensities will account for credit spreads

$$\gamma_i(t) := g_i(t,t) - r(t).$$

The intensity matrix of the rating migration process will have to satisfy some condition to guarantee that the discounted defaultable bond price is a martingale under the appropriate measure. This problem will be discussed in the next section.

The credit migration process is modeled by a conditional Markov process C^1 on the space of rating classes $\mathcal{K} = \{1, \dots, K\}$. The formal construction of C^1 is done in Appendix A by enlarging the probability space from $(\Omega, \mathcal{F}, \mathbf{P}^*, (\mathcal{F}_t)_{0 \le t \le T^*})$ to $(\widetilde{\Omega}, \mathcal{G}, \mathbf{Q}^*, (\mathcal{G}_t)_{0 \le t \le T^*})$.

To avoid unnecessary and confusing notation we retain the names of the stochastic processes when we consider them on the enlarged probability space, e.g. we set $W_t(\widetilde{\omega}) = W_t((\omega, \omega^U, \overline{\omega})) = W_t(\omega)$, that is we denote the Brownian motion on $\widetilde{\Omega}$ by W, too.

The conditional infinitesimal generator of C^1 under $\mathbf{Q}^*$ at time t given $\mathcal{G}_t$ is

$$
\Lambda_t = \begin{pmatrix}
\lambda_{1,1}(t) & \lambda_{1,2}(t) & \cdots & \lambda_{1,K-1}(t) & \lambda_{1,K}(t) \\
\lambda_{2,1}(t) & \lambda_{2,2}(t) & \cdots & \lambda_{2,K-1}(t) & \lambda_{2,K}(t) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_{K-1,1}(t) & \lambda_{K-1,2}(t) & \cdots & \lambda_{K-1,K-1}(t) & \lambda_{K-1,K}(t) \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix},
$$

where in case $i \neq j$, $\lambda_{i,j} : \widetilde{\Omega} \times \mathbb{R}_+ \to \mathbb{R}_+$ are adapted, nonnegative and $\mathbf{Q}^*$-a.s. integrable on every interval $[0,t]$ and for $i = 1, \ldots, K-1$, $\lambda_{i,i}(t) = -\sum_{j \in \mathcal{K}\setminus\{i\}} \lambda_{i,j}(t)$.

The following proposition is taken from Bielecki and Rutkowski (1999, Proposition 2.1).

Proposition 3.5 *For every $f : \mathcal{K} \to \mathbb{R}$ the process M^f is a $(\mathcal{G}_t)$-martingale, where*

$$M_t^f := f(C_t^1) - \int_0^t \Lambda_u f(C_t^1) \, \mathrm{d}u, \quad \textit{for all } t \geq 0.$$

Now we introduce two important auxiliary processes. Let us define $H_i(t) := \mathbb{1}_{\{s \geq 0 : C_s^1 = i\}}(t)$, for $i \in \mathcal{K}$ and let $H_{i,j}(t), i \neq j$, be the number of transitions from rating i to rating j over the time interval $[0,t]$.

Remark: For all $i \in \{1, \ldots, K-1\}$ and $t \in [0,T]$ the following equation holds,

$$H_i(t) = H_i(0) - \sum_{j=1; j\neq i}^{K-1} H_{i,j}(t) + \sum_{j=1; j\neq i}^{K-1} H_{j,i}(t) - H_{i,K}(t). \tag{3.38}$$

This means that, if we know the number of transitions from and to the rating class i over the time interval $[0,t]$ and whether the rating class at time 0 has been i then we know whether the rating class at time t is i or different from i. The same is then true for the left-hand limits in time.

The process $M_{i,j}$ given by

$$M_{i,j}(t) := H_{i,j}(t) - \int_0^t \lambda_{i,j}(u) H_i(u) \, \mathrm{d}u, \quad \forall t \geq 0, \tag{3.39}$$

follows a $(\mathcal{G}_t)$-martingale under $\mathbf{Q}^*$, see Bielecki and Rutkowski (1999, Corollary 2.2).

We define the process $C = (C^1, C^2)$, where C^1 is the current rating at time t, and C_t^2 denotes the previous rating before C_t^1 if a rating changes has happened on $[0,t]$. Otherwise $C_t^2 = C_t^1$.

The default time $\tau : \widetilde{\Omega} \to \mathbb{R}_+$ is the $(\mathcal{G}_t)$-stopping time given by

$$\tau := \inf \left\{ t \geq 0 \, : \, C_t^1 = K \right\}, \tag{3.40}$$

where we set as usual $\inf \emptyset := +\infty$.

3.4 Defaultable term structure

In this section we introduce the price of a defaultable bond. The main idea is to construct the discounted value of the defaultable bond, which changes its dynamics according to the credit migration process $C = (C^1, C^2)$.

For abbreviation purposes we introduce for $i \in \mathcal{K} \setminus \{K\}$

$$\widetilde{\phi}_i(t,T) := \int_{\mathbb{R}^d} \left(\left(\mathrm{e}^{\Sigma_i(t,T)^\top p_i x} - 1 \right) \widetilde{Y}(t,x) - \Sigma_i(t,T)^\top p_i x \right) \nu(\mathrm{d}x),$$

$$\widetilde{\alpha}_i(t,T) := a_i(t,T) + g_i(t,t) - f(t,t) + \Sigma_i(t,T)^\top c_i \phi(t) + \widetilde{\phi}_i(t,T),$$

$$\widetilde{\beta}_i(t,T) := \Sigma_i(t,T)^\top c_i,$$

$$\widetilde{\psi}_i(x,t,T) := \mathrm{e}^{\Sigma_i(t,T)^\top p_i x} - 1,$$

and

$$\widetilde{\beta}(t,T) := \Sigma(t,T)^\top c, \quad \widetilde{\psi}(x,t,T) := \mathrm{e}^{\Sigma(t,T)^\top x} - 1,$$

$$\widetilde{\mu} := \mu^L, \qquad \widetilde{\nu} := \widetilde{Y} \nu^L.$$

Consequently, the dynamics of $Z(.,T)$ and $Z_i(.,T), i = 1, \ldots, K-1$ under $\mathbf{Q}^*$ can now be expressed as

$$\mathrm{d}Z(t,T) = Z(t-,T) \left(\widetilde{\beta}(t,T) \; \mathrm{d}\widetilde{W}_t + \int_{\mathbb{R}^d} \widetilde{\psi}(x,t,T) \; (\widetilde{\mu} - \widetilde{\nu})(\mathrm{d}t, \mathrm{d}x) \right) \tag{3.41}$$

and

$$\mathrm{d}Z_i(t,T) = Z_i(t-,T) \left(\widetilde{\alpha}_i(t,T) \; \mathrm{d}t + \widetilde{\beta}_i(t,T) \; \mathrm{d}\widetilde{W}_t + \int_{\mathbb{R}^d} \widetilde{\psi}_i(x,t,T) \, (\widetilde{\mu} - \widetilde{\nu})(\mathrm{d}t, \mathrm{d}x) \right). \tag{3.42}$$

The natural candidate for the discounted price of a defaultable bond at time t with maturity T is

$$\sum_{i=1}^{K-1} (H_i(t) Z_i(t,T) + \delta_i H_{i,K}(t) Z(t,T)).$$

This is the definition in Eberlein and Özkan (2002). In this thesis we follow an alternative approach proposed in Bielecki and Rutkowski (1999, 2000, 2002). The starting point is less intuitive, but the local martingale property is more obvious at the beginning.

We define the process $(\hat{Z}(t,T))_{0\le t\le T}$ by the following stochastic differential equation

$$\begin{aligned}
\mathrm{d}\hat{Z}(t,T) := &\sum_{i,j=1;i\neq j}^{K-1} (Z_j(t,T) - Z_i(t,T))\,\mathrm{d}M_{i,j}(t) + \sum_{i=1}^{K-1} (\delta_i Z(t,T) - Z_i(t,T))\,\mathrm{d}M_{i,K}(t) \\
&+ \sum_{i=1}^{K-1} H_i(t-) Z_i(t-,T) \widetilde{\beta}_i(t,T)\,\mathrm{d}\widetilde{W}_t + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t-) Z(t-,T) \widetilde{\beta}(t,T)\,\mathrm{d}\widetilde{W}_t \\
&+ \sum_{i=1}^{K-1} H_i(t-) Z_i(t-,T) \int_{\mathbb{R}^d} \widetilde{\psi}_i(x,t,T)\,(\widetilde{\mu} - \widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x) \\
&+ \sum_{i=1}^{K-1} \delta_i H_{i,K}(t-) Z(t-,T) \int_{\mathbb{R}^d} \widetilde{\psi}(x,t,T)\,(\widetilde{\mu} - \widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x),
\end{aligned} \tag{3.43}$$

with initial condition

$$\hat{Z}(0,T) = \sum_{i=1}^{K-1} H_i(0) Z_i(0,T). \tag{3.44}$$

Before we can state the next lemma, we need a consistency condition for the intensity matrix Λ_t.

Condition $(\mathbb{M}2)$**:** For every $i \in \{1,\dots,K-1\}$ and $t \in [0,T]$ the following equality holds

$$\begin{aligned}
\sum_{j=1;j\neq i}^{K-1} (Z_j(t,T) - Z_i(t,T))\lambda_{i,j}(t) + (\delta_i Z(t,T) - Z_i(t,T))\lambda_{i,K}(t) \\
+ Z_i(t-,T)\widetilde{\alpha}_i(t,T) = 0.
\end{aligned} \tag{3.45}$$

The next lemma is a generalization of Lemma 2.2 in Bielecki and Rutkowski (2000).

Lemma 3.6 *If Condition* $(\mathbb{M}2)$ *holds, then* $\hat{Z}(t,T)$ *can be expressed as*

$$\hat{Z}(t,T) = \sum_{i=1}^{K-1} H_i(t) Z_i(t,T) + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t) Z(t,T). \tag{3.46}$$

Moreover, the process $(\hat{Z}(t,T))_{0\le t\le T}$ is a $\mathbf{Q}^*$-martingale and satisfies the following SDE

$$
\begin{aligned}
\mathrm{d}\hat{Z}(t,T) = &\sum_{i,j=1;i\neq j}^{K-1} \left(Z_j(t,T) - H_i(t)\hat{Z}(t,T)\right)\, \mathrm{d}M_{i,j}(t)\\
&+ \sum_{i=1}^{K-1} \left(\delta_i Z(t,T) - H_i(t)\hat{Z}(t,T)\right)\, \mathrm{d}M_{i,K}(t)\\
&+ \sum_{i=1}^{K-1} H_i(t-)\hat{Z}(t-,T)\widetilde{\beta}_i(t,T)\, \mathrm{d}\widetilde{W}_t + H_K(t-)\hat{Z}(t-,T)\widetilde{\beta}(t,T)\,\mathrm{d}\widetilde{W}_t\\
&+ \sum_{i=1}^{K-1} H_i(t-)\hat{Z}(t-,T)\int_{\mathbb{R}} \widetilde{\psi}_i(x,t,T)\,(\widetilde{\mu}-\widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x)\\
&+ H_K(t-)\hat{Z}(t-,T)\int_{\mathbb{R}} \widetilde{\psi}(x,t,T)\,(\widetilde{\mu}-\widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x)
\end{aligned}
\tag{3.47}
$$

with the initial condition

$$\hat{Z}(0,T) = \sum_{i=1}^{K-1} H_i(0)Z_i(0,T).$$

Proof: First, we show that $\hat{Z}(t,T) = \sum_{i=1}^{K-1} H_i(t)Z_i(t,T) + \delta_i H_{i,K}(t)Z(t,T)$.

Using the following equation, cf. equation (3.39),

$$\mathrm{d}M_{i,j}(t) = \mathrm{d}H_{i,j}(t) - \lambda_{i,j}(t)H_i(t)\ \mathrm{d}t = \mathrm{d}H_{i,j}(t) - \lambda_{i,j}(t)H_i(t-)\ \mathrm{d}t,$$

we get

$$
\begin{aligned}
\mathrm{d}\hat{Z}(t,T) = &\sum_{i,j=1;i\neq j}^{K-1} \left(Z_j(t,T) - Z_i(t,T)\right) \mathrm{d}H_{i,j}(t) \\
&- \sum_{i,j=1;i\neq j}^{K-1} (Z_j(t,T) - Z_i(t,T))\lambda_{i,j}(t)H_i(t-)\ \mathrm{d}t \\
&+ \sum_{i=1}^{K-1} (\delta_i Z(t,T) - Z_i(t,T))\ \mathrm{d}H_{i,K}(t) \\
&- \sum_{i=1}^{K-1} (\delta_i Z(t,T) - Z_i(t,T))\lambda_{i,K}(t)H_i(t-)\ \mathrm{d}t \\
&+ \sum_{i=1}^{K-1} H_i(t-)Z_i(t-,T)\widetilde{\beta}_i(t,T)\ \mathrm{d}\widetilde{W}_t \\
&+ \sum_{i=1}^{K-1} \delta_i H_{i,K}(t-)Z(t-,T)\widetilde{\beta}(t,T)\ \mathrm{d}\widetilde{W}_t \\
&+ \sum_{i=1}^{K-1} H_i(t-)Z_i(t-,T) \int_{\mathbb{R}^d} \widetilde{\psi}_i(x,t,T)\,(\widetilde{\mu} - \widetilde{\nu})\,(\mathrm{d}t, \mathrm{d}x) \\
&+ \sum_{i=1}^{K-1} \delta_i H_{i,K}(t-)Z(t-,T) \int_{\mathbb{R}^d} \widetilde{\psi}(x,t,T)\,(\widetilde{\mu} - \widetilde{\nu})\,(\mathrm{d}t, \mathrm{d}x).
\end{aligned}
$$

Now, we add and subtract two auxiliary expressions (line three in the next equation) to make use of the dynamics of $Z_i(.,T)$. We do not need to add further expressions for the dynamics of $Z(.,T)$, cf. equations (3.41) and (3.42).

$$
\begin{aligned}
\mathrm{d}\hat{Z}(t,T) = &\sum_{i,j=1;i\neq j}^{K-1} (Z_j(t,T) - Z_i(t,T))\ \mathrm{d}H_{i,j}(t) \\
&+ \sum_{i=1}^{K-1} (\delta_i Z(t,T) - Z_i(t,T))\ \mathrm{d}H_{i,K}(t) \\
&- \sum_{i=1}^{K-1} H_i(t-) Z_i(t-,T)\widetilde{\alpha}_i(t,T)\ \mathrm{d}t + \sum_{i=1}^{K-1} H_i(t-) Z_i(t-,T)\widetilde{\alpha}_i(t,T)\ \mathrm{d}t \\
&+ \sum_{i=1}^{K-1} H_i(t-) \left(Z_i(t-,T)\widetilde{\beta}_i(t,T)\,\mathrm{d}\widetilde{W}_t + Z_i(t-,T)\int_{\mathbb{R}^d} \widetilde{\psi}_i(x,t,T)\,(\widetilde{\mu}-\widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x) \right) \\
&+ \sum_{i=1}^{K-1} \delta_i H_{i,K}(t-) Z(t-,T) \left(\widetilde{\beta}(t,T)\ \mathrm{d}\widetilde{W}_t + \int_{\mathbb{R}^d} \widetilde{\psi}(x,t,T)\,(\widetilde{\mu}-\widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x) \right) \\
&- \sum_{i=1}^{K-1} H_i(t-) \left(\sum_{j=1;j\neq i}^{K-1} (Z_j(t,T) - Z_i(t,T))\lambda_{i,j}(t) \right. \\
&\qquad\qquad \left. + (\delta_i Z(t,T) - Z_i(t,T))\lambda_{i,K}(t) \right) \mathrm{d}t \\
= &\sum_{i,j=1;i\neq j}^{K-1} (Z_j(t,T) - Z_i(t,T))\ \mathrm{d}H_{i,j}(t) + \sum_{i=1}^{K-1} H_i(t-)\ \mathrm{d}Z_i(t,T) \\
&+ \sum_{i=1}^{K-1} (\delta_i Z(t,T) - Z_i(t,T))\ \mathrm{d}H_{i,K}(t) + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t-)\ \mathrm{d}Z(t,T) \\
&- \sum_{i=1}^{K-1} H_i(t-) \left(\sum_{j=1;j\neq i}^{K-1} (Z_j(t,T) - Z_i(t,T))\lambda_{i,j}(t) + Z_i(t-,T)\widetilde{\alpha}_i(t,T) \right. \\
&\qquad\qquad \left. + (\delta_i Z(t,T) - Z_i(t,T))\lambda_{i,K}(t) \right) \mathrm{d}t.
\end{aligned}
$$

It can be easily seen that under Condition ($\mathbb{M}2$) the process $\hat{Z}(.,T)$ satisfies

$$
\begin{aligned}
\mathrm{d}\hat{Z}(t,T) = &\sum_{i,j=1;i\neq j}^{K-1} (Z_j(t,T) - Z_i(t,T))\ \mathrm{d}H_{i,j}(t) + \sum_{i=1}^{K-1} H_i(t-)\ \mathrm{d}Z_i(t,T) \\
&+ \sum_{i=1}^{K-1} (\delta_i Z(t,T) - Z_i(t,T))\ \mathrm{d}H_{i,K}(t) + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t-)\ \mathrm{d}Z(t,T).
\end{aligned}
$$

If we change from the stochastic differential equation representation to the stochastic integral representation we get

$$\begin{aligned}
\hat{Z}(t,T) - \hat{Z}(0,T) = &\sum_{i,j=1;i\neq j}^{K-1} \int_0^t (Z_j(s,T) - Z_i(s,T))\ \mathrm{d}H_{i,j}(s) \\
&+ \sum_{i=1}^{K-1} \int_0^t (\delta_i Z(s,T) - Z_i(s,T))\ \mathrm{d}H_{i,K}(s) \\
&+ \sum_{i=1}^{K-1} \int_0^t H_i(s-)\ \mathrm{d}Z_i(s,T) + \sum_{i=1}^{K-1} \int_0^t \delta_i H_{i,K}(s-)\ \mathrm{d}Z(s,T).
\end{aligned}$$

It can be shown that for all $i \in \{1,\ldots,K-1\}$ and $t \in [0,T]$

$$H_i(t) = H_i(0) - \sum_{j=1;j\neq i}^{K-1} H_{i,j}(t) + \sum_{j=1;j\neq i}^{K-1} H_{j,i}(t) - H_{i,K}(t).$$

That is, if we know the number of transitions from and to the rating class i over the time interval $[0,t]$ and whether the rating class at time 0 was i then we know that the rating class at time t is i or different from i. The same is then true for the left-hand limits in time.

Thus,

$$\begin{aligned}
\hat{Z}(t,T) - \hat{Z}(0,T) = &\sum_{i,j=1;i\neq j}^{K-1} \int_0^t (Z_j(s,T) - Z_i(s,T))\ \mathrm{d}H_{i,j}(s) \\
&+ \sum_{i=1}^{K-1} \int_0^t (\delta_i Z(s,T) - Z_i(s,T))\ \mathrm{d}H_{i,K}(s) \\
&+ \sum_{i=1}^{K-1} \int_0^t H_i(0)\ \mathrm{d}Z_i(s,T) - \sum_{i=1}^{K-1} \sum_{j=1;j\neq i}^{K-1} \int_0^t H_{i,j}(s-)\ \mathrm{d}Z_i(s,T) \\
&+ \sum_{i=1}^{K-1} \sum_{j=1;j\neq i}^{K-1} \int_0^t H_{j,i}(s-)\ \mathrm{d}Z_i(s,T) \\
&- \sum_{i=1}^{K-1} \int_0^t H_{i,K}(s-)\ \mathrm{d}Z_i(s,T) + \sum_{i=1}^{K-1} \int_0^t \delta_i H_{i,K}(s-)\ \mathrm{d}Z(s,T).
\end{aligned}$$

Hence,

$$\hat{Z}(t,T) = \sum_{i,j=1;i\neq j}^{K-1} \int_0^t (Z_j(s,T) - Z_i(s,T))\ \mathrm{d}H_{i,j}(s)$$
$$+ \sum_{i=1}^{K-1} \int_0^t (\delta_i Z(s,T) - Z_i(s,T))\ \mathrm{d}H_{i,K}(s)$$
$$+ \sum_{i=1}^{K-1} \int_0^t H_i(0)\ \mathrm{d}Z_i(s,T) - \sum_{i,j=1;i\neq j}^{K-1} \int_0^t H_{i,j}(s-)\ \mathrm{d}Z_i(s,T)$$
$$+ \sum_{i,j=1;i\neq j}^{K-1} \int_0^t H_{j,i}(s-)\ \mathrm{d}Z_i(s,T)$$
$$- \sum_{i=1}^{K-1} \int_0^t H_{i,K}(s-)\ \mathrm{d}Z_i(s,T) + \sum_{i=1}^{K-1} \int_0^t \delta_i H_{i,K}(s-)\ \mathrm{d}Z(s,T) + \hat{Z}(0,T).$$

Note that

$$\sum_{i,j=1;i\neq j}^{K-1} \int_0^t Z_j(s,T)\ \mathrm{d}H_{i,j}(s) = \sum_{i,j=1;i\neq j}^{K-1} \int_0^t Z_i(s,T)\ \mathrm{d}H_{j,i}(s).$$

Therefore,

$$\hat{Z}(t,T) = \sum_{i,j=1;i\neq j}^{K-1} \left(\int_0^t Z_i(s,T)\ \mathrm{d}H_{j,i}(s) + \int_0^t H_{j,i}(s-)\ \mathrm{d}Z_i(s,T) \right)$$
$$- \sum_{i,j=1;i\neq j}^{K-1} \left(\int_0^t Z_i(s,T)\ \mathrm{d}H_{i,j}(s) + \int_0^t H_{i,j}(s-)\ \mathrm{d}Z_i(s,T) \right)$$
$$+ \sum_{i=1}^{K-1} \delta_i \left(\int_0^t Z(s,T)\ \mathrm{d}H_{i,K}(s) + \int_0^t H_{i,K}(s-)\ \mathrm{d}Z(s,T) \right) \qquad \text{[L3.6.a]}$$
$$- \sum_{i=1}^{K-1} \left(\int_0^t Z_i(s,T)\ \mathrm{d}H_{i,K}(s) + \int_0^t H_{i,K}(s-)\ \mathrm{d}Z_i(s,T) \right)$$
$$+ \sum_{i=1}^{K-1} \int_0^t H_i(0)\ \mathrm{d}Z_i(s,T) + \hat{Z}(0,T).$$

We take a closer look at the expression in the first brackets in equation [L3.6.a]. The process Z_i is a semimartingale for every $i \in \{1, \ldots, K-1\}$ and $H_{j,i}$ is an

$(\mathcal{G}_t)$-adapted, cádlág process with paths of bounded variation starting in zero, i.e. $H_{j,i}(0) = 0$. Therefore, we can apply Proposition I.4.49 in Jacod and Shiryaev (1987) and obtain

$$\int_0^t Z_i(s,T)\ \mathrm{d}H_{j,i}(s) + \int_0^t H_{j,i}(s-)\ \mathrm{d}Z_i(s,T) = Z_i(t,T)H_{j,i}(t).$$

By analogy we get the other expressions in the remaining brackets, and finally we have

$$\begin{aligned}\hat{Z}(t,T) = &\sum_{i,j=1;i\neq j}^{K-1} Z_i(t,T)H_{j,i}(t) - \sum_{i,j=1;i\neq j}^{K-1} Z_i(t,T)H_{i,j}(t)\\ &+ \sum_{i=1}^{K-1} \delta_i Z(t,T)H_{i,K}(t) - \sum_{i=1}^{K-1} Z_i(t,T)H_{i,K}(t)\\ &+ \sum_{i=1}^{K-1} H_i(0)Z_i(t,T) - \sum_{i=1}^{K-1} H_i(0)Z_i(0,T) + \hat{Z}(0,T).\end{aligned}$$

Remember that the initial condition for $\hat{Z}(.,T)$ is $\hat{Z}(0,T) = \sum_{i=1}^{K-1} H_i(0)Z_i(0,T)$.

$$\begin{aligned}\hat{Z}(t,T) &= \sum_{i=1}^{K-1} Z_i(t,T)\left(H_i(0) - \sum_{j=1;j\neq i}^{K-1} H_{i,j}(t) + \sum_{j=1;j\neq i}^{K-1} H_{j,i}(t) - H_{i,K}(t)\right)\\ &\quad + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t)Z(t,T)\\ &= \sum_{i=1}^{K-1} \left(H_i(t)Z_i(t,T) + \delta_i H_{i,K}(t)Z(t,T)\right).\end{aligned}$$

This proves the first part of the lemma. For the second part it is enough to state that

$$\begin{aligned}H_i(t)\hat{Z}(t,T) &= H_i(t)\sum_{j=1}^{K-1} \left(H_j(t)Z_j(t,T) + \delta_j H_{j,K}(t)Z(t,T)\right)\\ &= H_i(t)Z_i(t,T), \quad i = 1,\ldots,K-1, \qquad \text{[L3.6.b]}\end{aligned}$$

and

$$\begin{aligned}H_K(t)\hat{Z}(t,T) &= \left(\sum_{i=1}^{K-1} H_{i,K}(t)\right)\hat{Z}(t,T)\\ &= \sum_{i=1}^{K-1} \delta_i H_{i,K}(t)Z_i(t,T). \qquad \text{[L3.6.c]}\end{aligned}$$

It is sufficient to put the last two equations [L3.6.b] and [L3.6.c] in equation (3.47) to get equation (3.43), i.e. the definition of $\hat{Z}(.,T)$.

The local martingale property follows immediately from Proposition 3.5 combined with Jacod and Shiryaev (1987, I.4.34(b) and II.1.8.(ii)). By construction it is bounded by 0 and 1, hence it is of class $[D]$. By Jacod and Shiryaev (1987, I.1.47 c)) we can conclude that $\hat{Z}(.,T)$ is a martingale under $\mathbf{Q}^*$. □

Remark: $\hat{Z}(t,T) = \sum_{i=1}^{K-1} H_i(t) Z_i(t,T) + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t) Z(t,T)$ can be rewritten in a more intuitive way

$$\hat{Z}(t,T) = \mathbb{1}_{\{C_t^1 \neq K\}} Z_{C_t^1}(t,T) + \mathbb{1}_{\{C_t^1 = K\}} \delta_{C_t^2} Z(t,T).$$

We are now in the position to define the price of a defaultable T-maturity bond associated to the credit migration process $C = (C^1, C^2)$.

We set

$$D_C(t,T) := B_t \hat{Z}(t,T) \tag{3.48}$$

$$= \sum_{i=1}^{K-1} H_i(t) D_i(t,T) + \sum_{i=1}^{K-1} H_{i,K}(t) \delta_i B(t,T) \tag{3.49}$$

or more intuitively,

$$D_C(t,T) = \mathbb{1}_{\{C_t^1 \neq K\}} D_{C_t^1}(t,T) + \mathbb{1}_{\{C_t^1 = K\}} \delta_{C_t^2} B(t,T).$$

The next theorem shows that the process $D_C(t,T)$ introduced in equation (3.48) has an intuitive interpretation in terms of recovery rate and default time.

Note that the process $\hat{Z}(.,T)$ given by equation (3.46) follows a $(\mathcal{G}_t)$-martingale under $\mathbf{Q}^*$.

Theorem 3.7 *The time t price $D_C(t,T), 0 \leq t \leq T$, of a defaultable bond maturing at time T, which is currently rated as C_t, equals*

$$D_C(t,T) = \sum_{i=1}^{K-1} H_i(t) \mathrm{e}^{-\int_t^T g_i(t,u)\, \mathrm{d}u} + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t) \mathrm{e}^{-\int_t^T f(t,u)\, \mathrm{d}u}, \tag{3.50}$$

or equivalently

$$D_C(t,T) = B(t,T) \left(\sum_{i=1}^{K-1} H_i(t) \mathrm{e}^{-\int_t^T \gamma_i(t,u)\, \mathrm{d}u} + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t) \right), \tag{3.51}$$

where $\gamma_i(t,u) := g_i(t,u) - f(t,u)$, $0 \leq t \leq u \leq T, i = 1, \ldots, K-1$, is the i-th forward credit spread.

Moreover, the defaultable bond price $D_C(t,T)$ satisfies the following version of the risk-neutral valuation formula

$$D_C(t,T) = B_t \mathbb{E}_{\mathbf{Q}^*}\left[\frac{1}{B_T}\left(1\!\!1_{\{\tau>T\}} + \delta_{C_T^2} 1\!\!1_{\{\tau\leq T\}}\right) \mid \mathcal{G}_t\right], \tag{3.52}$$

where τ is the time of default as given in equation (3.40).

Proof: Equation (3.50) is an immediate consequence of equation (3.49) and the definitions of $D_i(t,T)$ and $B(t,T)$, see equation (3.35) and equation (3.1), respectively. Equation (3.51) is a simple consequence of equation (3.50).

Since the terminal payoff of the defaultable bond is

$$D_C(T,T) = 1\!\!1_{\{\tau>T\}} + \delta_{C_T^2} 1\!\!1_{\{\tau\leq T\}},$$

and since $\hat{Z}(.,T)$ is a $\mathbf{Q}^*$-martingale such that $\hat{Z}(T,T) = \frac{1}{B_T} D_C(T,T)$, we have

$$\begin{aligned} D_C(t,T) &= B_t \hat{Z}(t,T) \\ &= B_t \mathbb{E}_{\mathbf{Q}^*}\left[\hat{Z}(T,T) \mid \mathcal{G}_t\right] \\ &= B_t \mathbb{E}_{\mathbf{Q}^*}\left[\frac{1}{B_T}\left(1\!\!1_{\{\tau>T\}} + \delta_{C_T^2} 1\!\!1_{\{\tau\leq T\}}\right) \mid \mathcal{G}_t\right]. \end{aligned}$$

□

All three representations in Theorem 3.7 explain the defaultable price in a very intuitive way. The first of the two summands in equation (3.50) is the bond price expressed in terms of the defaultable forward rate given the state of the credit migration process. The second summand gives the reduced value in the case of default. Equation (3.51) shows explicitly the relation between the default-free and the defaultable bond price. The factor by which the default-free price is reduced is determined by the actual forward credit spread and the recovery rate. Equation (3.52) is the appropriate extension of the usual risk-neutral valuation formula to the case of defaultable instruments. See also Jarrow, Lando, and Turnbull (1997), where a similar valuation formula is the starting point for the derivation of the forward rates, or the references in Lando (1997) and Bielecki and Rutkowski (2002).

3.5 Reorganization and multiple defaults

In many cases firms which have to declare default are not liquidated but are restructured. An empirical investigation of firms in reorganization can be found in Franks and Torous (1989, 1994). After reorganization, companies are again subject to default risk, i.e. potentially they might default again in the future. Schönbucher (1998, 2000a) has proposed an intensity-based model which includes multiple defaults. The framework he uses is closely related to Duffie and Singleton (1999a).

As in Schönbucher (1998, 2000a) we shall assume that if a default occurs, the firm will be reorganized and the effect on the bond is that the bond holder will loose a certain amount, which is represented by a (now possibly random) quantity $q \in [0,1)$. As a consequence default is no longer an absorbing state of the conditional Markov chain C^1 which describes the ratings migration. We assume that the state space of C^1 is the set $\{1,2,\ldots,K-1\}$.

The idea described above is summarized in the next assumption.

Assumption ($\mathbb{A}3$): Define $\tau_0 := 0$, and $q_0 := 0$.

1. Defaults occur at (totally inaccessible) stopping times $\tau_1 < \tau_2 < \ldots$, where $\lim_{i\to\infty} \tau_i = \infty$.
2. At time $\tau_i, i \geq 1$, the face value of the bond is reduced by a factor of $1-q_i$, where $0 \leq q_i < 1$.
3. The double sequence $(\tau_i, q_i)_{i\geq 0}$ defines a marked point process

$$X_t := \sum_{i\geq 1} q_i \mathbb{1}_{\{\tau_i\}}(t)$$

with defining measure

$$\mu^X(\omega; \mathrm{d}t, \mathrm{d}q) = \sum_s \mathbb{1}_{\{\triangle X_s \neq 0\}}(\omega) \varepsilon_{(s,\triangle X_s(\omega))}(\mathrm{d}t, \mathrm{d}q)$$

and predictable $\mathbf{Q}^*$-compensator

$$\nu^X(\omega; \mathrm{d}t, \mathrm{d}q) = K_{\omega,t}(\mathrm{d}q)\, \lambda_t(\omega)\ \mathrm{d}t,$$

where λ_t is the intensity process of the jump times of X and $K_{\omega,t}(\mathrm{d}q)$ is a transition kernel from $(\widetilde{\Omega} \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, cf. Jacod and Shiryaev (1987, II.2.10).

We define the process $Q = (Q_t)_{t\geq 0}$ by

$$Q_t := \prod_{0\leq s\leq t} (1 - X_s) = \prod_{\tau_i \leq t} (1 - q_i) \tag{3.53}$$

and the deflated pseudo-default price is defined by

$$\widetilde{Z}_i(t,T) := Q_t Z_i(t,T).$$

Figure 3.2 shows a sample path of Q. For the simulation we have assumed that the waiting time between two defaults is exponentially distributed with rate 3, and that each q_i is uniformly distributed on $[0, 1]$. Another possible distribution which allows to consider different seniorities is the β-distribution, see e.g. Gupton, Finger, and Bhatia (1997).

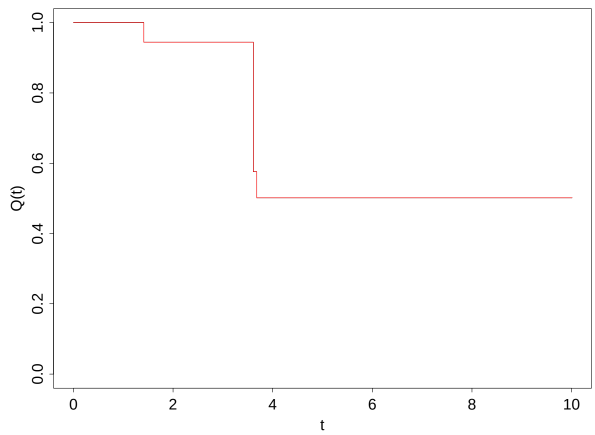


Figure 3.2: Simulated sample paths of Q, see equation (3.53).

Due to Assumption ($\mathbb{A}$3.3.) we can conclude that the $\mathbf{Q}^*$-compensator of Q, denoted by Q^K, has the following form

$$Q_t^K = \int_0^t \Lambda_s^Q \, \mathrm{d}s,$$

i.e. if we set $\widetilde{Q}_t := Q_t - Q_t^K$ then $(\widetilde{Q}_t)_{0\le t\le T}$ is a $\mathbf{Q}^*$-martingale.
In Section 3.6 we will determine the intensity process Λ^Q more precisely. But in this section there is no need of its explicit form.

We define

$$\widetilde{\widetilde{\lambda}}_i(t,T) := a_i(t,T) + \Sigma_i(t,T)^\top c_i\phi(t) + \int_{\mathbb{R}^d} \widetilde{\phi}_i(x,t,T)\,\nu(\mathrm{d}x).$$

Then the $\mathbf{Q}^*$-dynamics of $Z_i(.,T)$ is

$$\mathrm{d}Z_i(t,T) = Z_i(t-,T)\left((\widetilde{\widetilde{\lambda}}_i(t,T) + \gamma_i(t))\ \mathrm{d}t + \widetilde{\beta}_i(t,T)\ \mathrm{d}\widetilde{W}_t \right.$$
$$\left. + \int_{\mathbb{R}^d} \widetilde{\psi}_i(x,t,T)\,(\widetilde{\mu} - \widetilde{\nu})\,(\mathrm{d}t, \mathrm{d}x) \right).$$

Because the process Q is of bounded variation and $Z_i(.,T)$ is a (special) semimartingale, we have, cf. Jacod and Shiryaev (1987, I.4.49),

$$
\begin{aligned}
\mathrm{d}\widetilde{Z}_i(t,T) &= Q_{t-}\,\mathrm{d}Z_i(t,T) + Z_i(t,T)\,\mathrm{d}Q_t \\
&= \widetilde{Z}_i(t-,T)\Bigg((\widetilde{\widetilde{\lambda}}_i(t,T) + \gamma_i(t))\,\mathrm{d}t + \widetilde{\beta}_i(t,T)\,\mathrm{d}\widetilde{W}_t \\
&\qquad + \int_{\mathbb{R}^d} \widetilde{\psi}_i(x,t,T)\,(\widetilde{\mu}-\widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x)\Bigg) + Z_i(t,T)\,\mathrm{d}Q_t.
\end{aligned}
$$

Similar to Section 3.4, we define the following local $\mathbf{Q}^*$-martingale $\widehat{\widetilde{Z}}(.,T)$ by

$$
\begin{aligned}
\mathrm{d}\widehat{\widetilde{Z}}(t,T) := &\sum_{i,j=1;i\neq j}^{K-1} \left(\widetilde{Z}_j(t,T) - \widetilde{Z}_i(t,T)\right)\mathrm{d}M_{i,j}(t) \\
&+ \sum_{i=1}^{K-1} H_i(t-)\widetilde{Z}_i(t-,T)\widetilde{\beta}_i(t,T)\,\mathrm{d}\widetilde{W}_t \\
&+ \sum_{i=1}^{K-1} H_i(t-)\widetilde{Z}_i(t-,T)\int_{\mathbb{R}^d} \widetilde{\psi}_i(x,t,T)\,(\widetilde{\mu}-\widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x) \\
&+ \sum_{i=1}^{K-1} H_i(t-)Z_i(t,T)\,\mathrm{d}\widetilde{Q}_t.
\end{aligned}
\tag{3.54}
$$

Within the framework of multiple defaults Condition ($\mathbb{M}2$) changes to

Condition ($\mathbb{M}2'$): For every $i \in \{1,\ldots,K-1\}$ and $t \in [0,T]$ the following equation holds

$$
\widetilde{Z}_i(t-,T)(\widetilde{\widetilde{\lambda}}_i(t,T) + \gamma_i(t)) + Z_i(t,T)\Lambda_t^Q + \sum_{j=1,j\neq i}^{K-1} (\widetilde{Z}_j(t,T) - \widetilde{Z}_i(t,T))\lambda_{i,j}(t) = 0.
\tag{3.55}
$$

In Section 3.6 we will have a closer look at conditions ($\mathbb{M}2$) and ($\mathbb{M}2'$) to get a better understanding of both conditions and the relation to the short rate spread $\gamma_i(t)$. The next lemma is similar to Lemma 3.6.

Lemma 3.8 *Under Condition* ($\mathbb{M}2'$) $\widehat{\widehat{Z}}(t,T)$ *can be written as*

$$\widehat{\widehat{Z}}(t,T) = Q_t \sum_{i=1}^{K-1} H_i(t) Z_i(t,T). \tag{3.56}$$

Furthermore, $\widehat{\widehat{Z}}(.,T)$ *is a martingale under* $\mathbf{Q}^*$ *and is the unique solution of the following stochastic differential equation*

$$\begin{aligned}
\mathrm{d}\widehat{\widehat{Z}}(t,T) = & \sum_{i,j=1;i\neq j}^{K-1} \left(\widetilde{Z}_j(t,T) - H_i(t)\widehat{\widehat{Z}}(t,T) \right) \mathrm{d}M_{i,j}(t) \\
& + \sum_{i=1}^{K-1} H_i(t-)\widehat{\widehat{Z}}(t-,T)\widetilde{\beta}_i(t,T) \ \mathrm{d}\widetilde{W}_t \\
& + \sum_{i=1}^{K-1} H_i(t-)\widehat{\widehat{Z}}(t-,T) \int_{\mathbb{R}^d} \widetilde{\psi}_i(x,t,T) \left(\widetilde{\mu} - \widetilde{\nu}\right)(\mathrm{d}t, \mathrm{d}x) \\
& + \sum_{i=1}^{K-1} H_i(t-)\widehat{\widehat{Z}}(t-,T) \ \mathrm{d}\widetilde{Q}_t
\end{aligned}$$

with initial condition

$$\widehat{\widehat{Z}}(0,T) = \sum_{i=1}^{K-1} H_i(0) Z_i(0,T).$$

Proof: Using the following equation

$$\mathrm{d}M_{i,j}(t) = \mathrm{d}H_{i,j}(t) - \lambda_{i,j}(t)H_i(t) \ \mathrm{d}t = \mathrm{d}H_{i,j}(t) - \lambda_{i,j}(t)H_i(t-) \ \mathrm{d}t,$$

we get

$$\begin{aligned}
\mathrm{d}\widehat{\widehat{Z}}(t,T) = & \sum_{i,j=1;i\neq j}^{K-1} \left(\widetilde{Z}_j(t,T) - \widetilde{Z}_i(t,T) \right) \mathrm{d}H_{i,j}(t) + \sum_{i=1}^{K-1} H_i(t-) \ \mathrm{d}\widetilde{Z}_i(t,T) \\
& + \sum_{i=1}^{K-1} H_i(t-) \Bigg(\widetilde{Z}_i(t-,T) \left(\widetilde{\widetilde{\lambda}}_i(t,T) + \gamma_i(t) \right) + Z_i(t,T)\Lambda_t^Q \\
& \qquad + \sum_{j=1;j\neq i}^{K-1} (\widetilde{Z}_j(t,T) - \widetilde{Z}_i(t,T))\lambda_{i,j}(t) \Bigg) \ \mathrm{d}t.
\end{aligned}$$

Under Condition ($\mathbb{M}2'$) we have

$$\widehat{\widehat{Z}}(t,T) - \widehat{\widehat{Z}}(0,T) = \sum_{i,j=1;i\neq j}^{K-1} \int_0^t \left(\widetilde{Z}_j(s,T) - \widetilde{Z}_i(s,T)\right) \mathrm{d}H_{i,j}(s) + \sum_{i=1}^{K-1} H_i(s-)\, \mathrm{d}\widetilde{Z}_i(s,T).$$

It can be easily seen that

$$H_i(t) = H_i(0) - \sum_{j=1,j\neq i}^{K-1} H_{i,j}(t) + \sum_{j=1,j\neq i}^{K-1} H_{j,i}(t).$$

Therefore

$$\begin{aligned}
\widehat{\widehat{Z}}(t,T) &= \sum_{i=1}^{K-1} \int_0^t H_i(0)\, \mathrm{d}\widetilde{Z}_i(s,T) + \widehat{\widehat{Z}}(0,T) \\
&\quad + \sum_{i,j=1,j\neq i}^{K-1} \left(\int_0^t \widetilde{Z}_i(s,T)\, \mathrm{d}H_{j,i}(s) + \int_0^t H_{j,i}(s-)\, \mathrm{d}\widetilde{Z}_i(s,T) \right) \\
&\quad - \sum_{i,j=1,j\neq i}^{K-1} \left(\int_0^t \widetilde{Z}_i(s,T)\, \mathrm{d}H_{i,j}(s) + \int_0^t H_{i,j}(s-)\, \mathrm{d}\widetilde{Z}_i(s,T) \right) \\
&= \sum_{i=1}^{K-1} H_i(0)\widetilde{Z}_i(t,T) + \sum_{i,j=1,j\neq i}^{K-1} H_{j,i}(t)\widetilde{Z}_i(t,T) - \sum_{i,j=1,j\neq i}^{K-1} H_{i,j}(t)\widetilde{Z}_i(t,T) \\
&= \sum_{i=1}^{K-1} H_i(t)\widetilde{Z}_i(t,T) \quad = \quad Q_t \sum_{i=1}^{K-1} H_i(t) Z_i(t,T).
\end{aligned}$$

Similar to Section 3.4, the discounted bond price follows a bounded local $\mathbf{Q}^*$-martingale. Hence it is a $\mathbf{Q}^*$-martingale.

For the second part of the lemma we refer to the proof of Lemma 3.6. □

Theorem 3.9 *The time t price $D_C(t,T), 0 \leq t \leq T$, of a defaultable bond maturing at time T, which is currently rated as C_t, equals*

$$D_C(t,T) = Q_t \sum_{i=1}^{K-1} H_i(t) \mathrm{e}^{-\int_t^T g_i(t,u)\, \mathrm{d}u}$$

or equivalently

$$D_C(t,T) = B(t,T)Q_t \sum_{i=1}^{K-1} H_i(t)\mathrm{e}^{-\int_t^T \gamma_i(t,u)\ \mathrm{d}u}.$$

Furthermore, we have the following version of the risk-neutral valuation formula

$$D_C(t,T) = B_t \mathbb{E}_{\mathbf{Q}^*}\left[\frac{1}{B_T} Q_T \,|\, \mathcal{G}_t\right].$$

Proof: The proof of this theorem is analogous to the proof of Theorem 3.7.

3.6 Credit spreads and default intensities

In this section we take a closer look to Condition ($\mathbb{M}2$) and ($\mathbb{M}2'$) to investigate the relationship between default intensities and short rate credit spreads.

Let us consider the zero-recovery case and assume that we are only interested whether the company has defaulted or not. In order to avoid arbitrage, we assume that $D_C(t,T) < B(t,T)$.

In our notation we have $\mathcal{K} = \{1,2\}$, where the state 1 means that the company is still able to pay its debts and state 2 stands for the state of bankruptcy. The recovery rate is $\delta_1 = 0$.

Furthermore, we assume that the following relationship holds

$$a_1(t,T) = -\Sigma_1(t,T)^\top c_1\phi(t) - \int_{\mathbb{R}^d} \left(\left(\mathrm{e}^{\Sigma_1(t,T)^\top p_1 x} - 1\right)\widetilde{Y}(t,x) - \Sigma_1(t,T)^\top p_1 x\right)\nu(\mathrm{d}x). \tag{3.57}$$

Hence, $\widetilde{\alpha}_1(t,T) = g_1(t,t) - f(t,t) = \gamma_1(t,t)$ and $\widetilde{\alpha}_1(t,T)$ does not depend on the maturity T. Equation (3.57) corresponds to the no-arbitrage drift condition for the default free bond, see equation (3.13), page 47. The dynamics of $Z_1(.,T)$ satisfies in this case, cf. equation (3.42) on page 59,

$$\mathrm{d}Z_1(t,T) = Z_1(t-,T)\left(\gamma_1(t,t)\ \mathrm{d}t + \widetilde{\beta}_1(t,T)\ \mathrm{d}\widetilde{W}_t + \int_{\mathbb{R}^d} \widetilde{\psi}_1(x,t,T)\,(\widetilde{\mu} - \widetilde{\nu})\,(\mathrm{d}t,\mathrm{d}x)\right).$$

Under these assumptions Condition ($\mathbb{M}2$) simplifies to

$$Z_1(t,T)\lambda_{1,2}(t) = Z_1(t-,T)\gamma_1(t,t).$$

Because $Z_1(.,T)$ is càdlàg we have

$$\lambda_{1,2}(t) = \gamma_1(t,t) \qquad \mathrm{Leb} - \text{a.s.}$$

This means that under the assumptions above the default intensity equals the short rate credit spread. This relationship has been previously derived in Maksymiuk and Gatarek (1999).

Before we can consider the non-zero-recovery case which corresponds to Section 3.5 we have to take a closer look to the compensator $Q^K = \int_0^\cdot \Lambda_s^Q \,\mathrm{d}s$ of $Q = \prod_{s\leq \cdot}(1 - X_s)$. First, we calculate the stochastic logarithm of Q. Since $Q_t > 0$ we can write Q as a Doléans-Dade exponential.

Proposition 3.10 *The stochastic logarithm of Q, $\mathcal{L}(Q)_t = \int_0^t \frac{1}{Q_{s-}} \,\mathrm{d}Q_s$, is given by*

$$\mathcal{L}(Q)_t = -\int_0^t \int_0^1 q \, \mu^X(\mathrm{d}s, \mathrm{d}q). \tag{3.58}$$

Proof: Since the process Q is purely discontinuous, and therefore its continuous martingale part is zero, Lemma 2.4 in Kallsen and Shiryaev (2000) yields

$$\mathcal{L}(Q)_t = \log\left(\frac{Q_t}{Q_0}\right) - \int_0^t \int_0^1 \left(\log\left(1 + \frac{u}{Q_{s-}}\right) - \frac{u}{Q_{s-}}\right) \mu^Q(\mathrm{d}s, \mathrm{d}u).$$

Using Corollary A.6 in Goll and Kallsen (2000), one can show that the integral of a μ^Q-integrable function f is

$$\int_0^t \int_{\mathbb{R}} f(s,u) \, \mu^Q(\mathrm{d}s, \mathrm{d}u) = \int_0^t \int_0^1 f(s, -qQ_{s-}) \, \mu^X(\mathrm{d}s, \mathrm{d}q).$$

The last equation can be interpreted in an intuitive way: if at time s the process X jumps up with size q, then Q jumps down with size $-qQ_{s-}$ at the same time. Note that $Q_0 = 1$. Therefore,

$$\begin{aligned}\mathcal{L}(Q)_t &= \log\left(\frac{Q_t}{Q_0}\right) - \int_0^t \int_0^1 (\log(|1-q|) + q) \; \mu^X(\mathrm{d}s, \mathrm{d}q) \\ &= \log(Q_t) - \sum_{\tau_i \leq t} \log(1 - q_i) - \int_0^t \int_0^1 q \, \mu^X(\mathrm{d}s, \mathrm{d}q).\end{aligned}$$

Since $\log Q_t = \sum_{\tau_i \leq t} \log(1 - q_i)$, see equation (3.53), we get $\mathcal{L}(Q)_t = -\int_0^t \int_0^1 q \, \mu^X(\mathrm{d}s, \mathrm{d}q)$. □

The process $N_t := \int_0^t \int_0^1 q\, \mu^X(\mathrm{d}s, \mathrm{d}q)$ is interpreted as the loss summation function in Schönbucher (1998, 2000a). Note that $N_t = \sum_{0 \leq s \leq t} X_s = -\mathcal{L}(Q)_t$, or equivalently $\mathrm{d}Q_t = -Q_{t-}\, \mathrm{d}N_t$.

The $\mathbf{Q}^*$-compensator of N is given by

$$A_t := \int_0^t \int_0^1 q\, \nu^X(\mathrm{d}s, \mathrm{d}q) = \int_0^t \int_0^1 q(s)\lambda(s)\ \mathrm{d}s,$$

where $q(\omega, s) := \int_0^1 qK_{\omega,s}(\mathrm{d}q)$. The compensator of $Q = -\int_0^{\cdot} Q_{s-}\ \mathrm{d}N_s$ is hence given by $-\int_0^{\cdot} Q_{s-}\ \mathrm{d}A_s = -\int_0^{\cdot} Q_{s-}q(s)\lambda(s)\ \mathrm{d}s$. Hence $\Lambda_t^Q = -Q_{t-}q(t)\lambda(t)$.

We consider again the setting that we have described above with the same assumptions. Now we have $\mathcal{K} = \{1\}$, since after a default the company is reorganized. Condition ($\mathbb{M}2'$) then simplifies to

$$Q_{t-}Z_1(t-, T)\gamma_1(t) - Q_{t-}Z_1(t, T)q(t)\lambda(t) = 0.$$

And hence

$$q(t)\lambda(t) = g_1(t, t) - f(t, t) \quad \text{Leb} - \text{a.s.}$$

The last equation corresponds to equation (2.71) in Schönbucher (2000a). Note that this relationship can not be transfered directly to the case of several rating classes by considering each rating class for its own. This procedure would imply that the only possible rating changes are the transitions from non-default to default. However, one has to take care of rating transitions from one non-default class to other non-default rating classes. This is why Condition ($\mathbb{M}2$) and ($\mathbb{M}2'$) involve the transition intensities λ_{ij}.

3.7 Market price of risk

In our approach we start with the real-world measure $\mathbf{P} \in \mathcal{M}^1(\Omega, \mathcal{F})$ and construct an equivalent martingale measure $\mathbf{P}^*$ on the same underlying space. In order to include the information given by the credit migration process $C = (C^1, C^2)$ we had to go to an enlarged probability space $(\widetilde{\Omega}, \mathcal{G}, \mathbf{Q}^*)$. The explicit construction of this enlarged space is given in the appendix.

Now we construct an equivalent measure $\mathbf{Q} \sim \mathbf{Q}^*$ such that the projection $\Pi_\Omega(\mathbf{Q}) = \mathbf{P}$, i.e. the probability measure $\mathbf{Q}$ plays the role of the real-world measure on the enlarged measurable space $(\widetilde{\Omega}, \mathcal{G})$. Furthermore, the conditional Markov process C shall be a conditional Markov process under $\mathbf{Q}$ again. Figure 3.3 shows the relationship between the probability spaces.

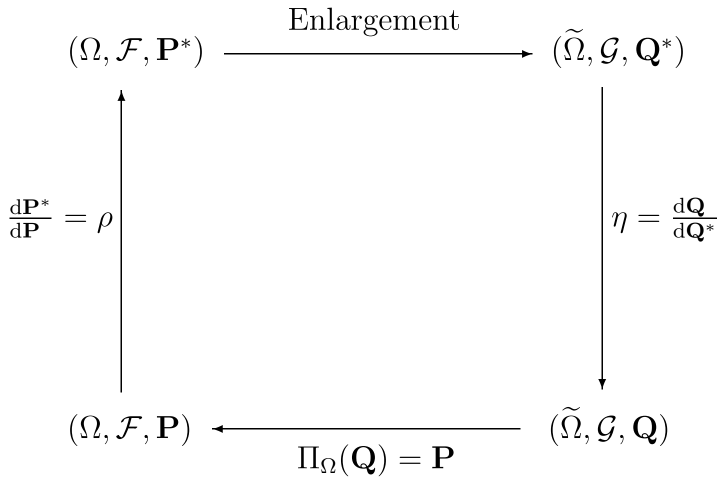


Figure 3.3: Relationship between the probability spaces.

Let $\phi_{i,j} : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be an arbitrary $(\mathcal{F}_t)_{0 \leq t \leq T^*}$-predictable function which according to our convention can be considered as a function on the extended space $\widetilde{\Omega} \times \mathbb{R}_+$ as well. We assume that

$$\int_0^{T^*} \phi_{i,j}(t) \lambda_{i,j}(t) \; \mathrm{d}t < \infty \qquad \mathbf{Q}^*\text{-a.s.}$$

Introduce the $\mathbf{Q}^*$-local martingale R by setting

$$R_t = \sum_{i,j=1, i \neq j}^{K-1} \int_0^t (\phi_{i,j}(s) - 1) \; \mathrm{d}M_{i,j}(s), \tag{3.59}$$

and define a process $\eta = (\eta_t)_{0 \leq t \leq T^*}$ by the following equation

$$\begin{aligned} \mathrm{d}\eta_t = \eta_{t-} \Bigg(& -\phi^\top(t) \; \mathrm{d}\widetilde{W}_t - \int_{\mathbb{R}^d} \left(\widetilde{Y}(t,x) - 1\right) \left(\mu^L - \nu^L\right) (\mathrm{d}t, \mathrm{d}x) \\ & + \int_{\mathbb{R}^d} \frac{1}{\widetilde{Y}(t,x)} \left(\widetilde{Y}(t,x) - 1\right)^2 \mu^L(\mathrm{d}t, \mathrm{d}x) + \mathrm{d}R_t \Bigg), \end{aligned} \tag{3.60}$$

with the initial condition $\eta_0 = 1$.

Note that η is a local martingale with respect to $\mathbf{Q}^*$. The only two expressions which are not obvious, are the expressions concerning the jumps of L. But here, we have that

$$
\begin{aligned}
&-\int\limits_{\mathbb{R}^d} \left(\widetilde{Y}(t,x)-1\right)\left(\mu^L-\nu^L\right)(\mathrm{d}t,\mathrm{d}x)+\int\limits_{\mathbb{R}^d} \frac{1}{\widetilde{Y}(t,x)}\left(\widetilde{Y}(t,x)-1\right)^2 \mu^L(\mathrm{d}t,\mathrm{d}x)\\
&=-\int\limits_{\mathbb{R}^d}(\widetilde{Y}(t,x)-1)(\mu^L-\widetilde{\nu}^L)(\mathrm{d}t,\mathrm{d}x)-\int\limits_{\mathbb{R}^d}(\widetilde{Y}(t,x)-1)(\widetilde{\nu}^L-\nu^L)(\mathrm{d}t,\mathrm{d}x)\\
&\quad+\int\limits_{\mathbb{R}^d}\frac{1}{\widetilde{Y}(t,x)}(\widetilde{Y}(t,x)-1)^2(\mu^L-\widetilde{\nu}^L)(\mathrm{d}t,\mathrm{d}x)+\int\limits_{\mathbb{R}^d}\frac{1}{\widetilde{Y}(t,x)}(\widetilde{Y}(t,x)-1)^2\widetilde{\nu}^L(\mathrm{d}t,\mathrm{d}x).
\end{aligned}
$$

Remember that $\widetilde{\nu}^L(\mathrm{d}t,\mathrm{d}x)=\widetilde{Y}(t,x)\nu^L(\mathrm{d}t,\mathrm{d}x)$. Hence the integrals with respect to the compensators vanish, because

$$
\begin{aligned}
&-\int\limits_{\mathbb{R}^d}(\widetilde{Y}(t,x)-1)(\widetilde{\nu}^L-\nu^L)(\mathrm{d}t,\mathrm{d}x)+\int\limits_{\mathbb{R}^d}\frac{1}{\widetilde{Y}(t,x)}(\widetilde{Y}(t,x)-1)^2\widetilde{\nu}^L(\mathrm{d}t,\mathrm{d}x)\\
&=-\int\limits_{\mathbb{R}^d}(\widetilde{Y}(t,x)-1)^2\,\nu^L(\mathrm{d}t,\mathrm{d}x)+\int\limits_{\mathbb{R}^d}(\widetilde{Y}(t,x)-1)^2\,\nu^L(\mathrm{d}t,\mathrm{d}x)\\
&=0.
\end{aligned}
$$

Thus η is the stochastic exponential of a local $\mathbf{Q}^*$-martingale, and hence it is a local $\mathbf{Q}^*$-martingale itself, see Jacod and Shiryaev (1987, I.4.64b).

To state the next theorem we need some more notation from Chapter III in Jacod and Shiryaev (1987). $M^{\mathbf{Q}^*}_{\mu^L}$ is the positive measure on $(\widetilde{\Omega}\times\mathbb{R}_+\times\mathbb{R}^d,\mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d))$ defined by $M^{\mathbf{Q}^*}_{\mu^L}(U)=\mathbb{E}_{\mathbf{Q}^*}[(U*\mu^L)_{T^*}]$. The σ-field $\overline{\mathcal{P}}$ is given by $\overline{\mathcal{P}}=\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$, where $\mathcal{P}$ is the predictable σ-field on the enlarged space $\widetilde{\Omega}\times\mathbb{R}_+$. The "conditional expectation" $M^{\mathbf{Q}^*}_{\mu^L}(U\,|\,\widetilde{\mathcal{P}})$ is defined in Jacod and Shiryaev (1987, III.3.16). Note that if the jumps of L and U satisfy $\triangle L\triangle U=0$, then $M^{\mathbf{Q}^*}_{\mu^L}(\triangle U\,|\,\widetilde{\mathcal{P}})=0$.

Theorem 3.11 *If* $\mathbb{E}_{\mathbf{Q}^*}[\eta_{T^*}]=1$ *and* $M^{\mathbf{Q}^*}_{\mu^L}(\triangle R\,|\,\widetilde{\mathcal{P}})=0$, *then* $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{Q}^*}|_{\mathcal{G}_t}=\eta_t$ *defines an equivalent probability measure* $\mathbf{Q}$ *on the enlarged probability space* $\left(\widetilde{\Omega},\mathcal{G}\right)$ *such that* $\Pi_\Omega(\mathbf{Q})=\mathbf{P}$.

Proof: From the definition of $\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}}$ in Section 3.2 we conclude that $\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}}|_{\mathcal{F}_t}=\rho_t$. The density process ρ can be expressed as a Doléans-Dade exponential

$$\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}}|_{\mathcal{F}_t}=\rho_t=\mathcal{E}(N)_t,$$

where

$$N_t = \int_0^t \phi^\top(s)\, \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} (\widetilde{Y}(s,x) - 1)\, (\mu^L - \nu^L)(\mathrm{d}s, \mathrm{d}x).$$

Applying Itô's formula yields

$$\begin{aligned}\frac{1}{\rho_t} = \frac{1}{\rho_0} &- \int_0^t \frac{1}{\rho_{s-}^2}\, \mathrm{d}\rho_s \\ &+ \int_0^t \frac{1}{\rho_{s-}^3}\, \mathrm{d}\langle \rho^c, \rho^c\rangle_s + \int_0^t \int_{\mathbb{R}^d} \left(\frac{1}{\rho_{s-} + x} - \frac{1}{\rho_{s-}} + \frac{x}{\rho_{s-}^2} \right) \mu^\rho(\mathrm{d}s, \mathrm{d}x).\end{aligned}$$

Note that $\mathrm{d}\rho_s = \rho_{s-}\, \mathrm{d}N_s$, and $\rho_t^c = \int_0^t \rho_{s-}\phi^\top(s)\, \mathrm{d}W_s$ and hence $\langle \rho^c, \rho^c\rangle_t = \int_0^t \rho_{s-}^2 |\phi(s)|^2\, \mathrm{d}s$. If $f : [0,T^*] \times \mathbb{R} \to \mathbb{R}_+$ is a measurable function, then $\int_0^t \int_{\mathbb{R}^d} f(s,x)\, \mu^\rho(\mathrm{d}s, \mathrm{d}x) = \int_0^t \int_{\mathbb{R}^d} f(s, \rho_{s-}(\widetilde{Y}(s,x) - 1))\, \mu^L(\mathrm{d}s, \mathrm{d}x)$. Then it follows

$$\begin{aligned}\frac{1}{\rho_t} &= \frac{1}{\rho_0} - \int_0^t \frac{1}{\rho_{s-}}\, \mathrm{d}N_s + \int_0^t \frac{1}{\rho_{s-}} |\phi(s)|^2\, \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \left(\frac{1}{\rho_{s-} + \rho_{s-}(\widetilde{Y}(s,x) - 1)} - \frac{1}{\rho_{s-}} + \frac{\rho_{s-}(\widetilde{Y}(s,x) - 1)}{\rho_{s-}^2} \right) \mu^L(\mathrm{d}s, \mathrm{d}x) \\ &= \frac{1}{\rho_0} - \int_0^t \frac{1}{\rho_{s-}} \phi^\top(s)\, \mathrm{d}\widetilde{W}_s - \int_0^t \int_{\mathbb{R}^d} \left(\frac{1}{\rho_{s-}} (\widetilde{Y}(s,x) - 1) \right) \left(\mu^L - \nu^L\right)(\mathrm{d}s, \mathrm{d}x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \left(\frac{1}{\rho_{s-}\widetilde{Y}(s,x)} \left(\widetilde{Y}(s,x) - 1\right)^2 \right) \mu^L(\mathrm{d}s, \mathrm{d}x).\end{aligned}$$

The process $\frac{1}{\rho_t}$ can also be seen as a process on the enlarged probability space $(\widetilde{\Omega}, \mathcal{G})$. We define an auxiliary measure $\widetilde{\mathbf{Q}}$ by setting $\frac{\mathrm{d}\widetilde{\mathbf{Q}}}{\mathrm{d}\mathbf{Q}^*}|_{\mathcal{G}_t} = \frac{1}{\rho_t}$. Due to the construction of the measure $\widetilde{\mathbf{Q}}$, it satisfies $\Pi_\Omega(\widetilde{\mathbf{Q}}) = \mathbf{P}$.

The semimartingale characteristics under a new (equivalent) measure are determined by the semimartingale characteristics under the original measure and the functions ϕ and $\widetilde{Y}$, see Girsanov's theorem, Jacod and Shiryaev (1987, III.3.24). The density process η given in equation (3.60) is defined in such a way that ϕ and $\widetilde{Y}$ are the same as for $\frac{1}{\rho}$, see Jacod and Shiryaev (1987, III.5.17). Now, we replace $\frac{1}{\rho}$ by η and define the measure $\mathbf{Q}$ on $(\widetilde{\Omega}, \mathcal{G})$ by $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{Q}^*}|_{\mathcal{G}_t} = \eta_t$.

The semimartingale characteristics of the Lévy process L and of the bond price process $B(.,T)$ are the same under $\mathbf{Q}$ and under $\widetilde{\mathbf{Q}}$. Thus, the characteristics are the same under $\Pi_\Omega(\mathbf{Q})$ and $\Pi_\Omega(\widetilde{\mathbf{Q}}) = \mathbf{P}$. Remember that in contrast to $(\mathcal{G}_t)_{0\leq t\leq T^*}$ the filtration $(\mathcal{F}_t)_{0\leq t\leq T^*}$ is generated by the Lévy process L. By Theorem III.2.34 in Jacod and Shiryaev (1987) we conclude that the martingale problem has a unique solution on $(\Omega, \mathcal{F})$ and hence $\Pi_\Omega(\mathbf{Q}) = \mathbf{P}$. $\square$

In Bielecki and Rutkowski (1999) it is shown that under $\mathbf{Q}$ the migration process C^1 is still a conditional Markov process, and it has under $\mathbf{Q}$ the infinitesimal generator $\overline{\Lambda}_t$

$$\overline{\Lambda}_t = \begin{pmatrix} \overline{\lambda}_{1,1}(t) & \overline{\lambda}_{1,2}(t) & \cdots & \overline{\lambda}_{1,K-1}(t) & \overline{\lambda}_{1,K}(t) \\ \overline{\lambda}_{2,1}(t) & \overline{\lambda}_{2,2}(t) & \cdots & \overline{\lambda}_{2,K-1}(t) & \overline{\lambda}_{2,K}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{\lambda}_{K-1,1}(t) & \overline{\lambda}_{K-1,2}(t) & \cdots & \overline{\lambda}_{K-1,K-1}(t) & \overline{\lambda}_{K-1,K}(t) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{3.61}$$

with entries $\overline{\lambda}_{i,j}(t) = \phi_{i,j}(t)\lambda_{i,j}(t)$ for every $i,j \in \mathcal{K}$ with $i \neq j$ and $\overline{\lambda}_{i,i}(t) = -\sum_{j=1,j\neq i}^{K} \overline{\lambda}_{i,j}(t)$ for $i \in \mathcal{K}$.

In the Gaussian HJM-setting studied in Bielecki and Rutkowski (1999, 2000), the process ϕ is denoted as the market price of interest rate risk. Since we discuss here discontinuous processes, where the change of measure is characterized by the tuple $(\phi, \widetilde{Y})$, we propose to denote the tuple $(\phi, \widetilde{Y})$ as the market price of interest rate risk. Because the risk of default is fully characterized by the credit migration process which itself is given by its infinitesimal generator, it makes sense to denote $(\phi_{i,j}(t))_{i,j\in\mathcal{K}}$ as the market price of credit risk.

3.8 Outline of the implementation

The approach that we have presented here has to be seen as a framework for a credit risk model based on the Lévy HJM-framework. We outline how to implement this approach. A detailed description is beyond the scope of this thesis. The detailed empirical investigation is subject to future research, but the outline that we present here looks most encouraging.

In Raible (2000) it is shown that the empirical distributions of log-returns of zero-coupon bond prices deviate substantially in the tails and in the center from what one would expect in the case of the Gaussian HJM-model. The observed empirical distributions can be fitted with a high degree of accuracy by generalized hyperbolic distributions.

Let L be the generalized hyperbolic Lévy motion with parameters $(\lambda, \alpha, \beta, \delta, \mu)$, which is generated by a generalized hyperbolic distribution. The

Lebesgue density of L_1 is then given by

$$\begin{aligned}\mathrm{ghyp}(x;\lambda,\alpha,\beta,\delta,\mu) &= a(\lambda,\alpha,\beta,\delta)(\delta^2+(x-\mu)^2)^{0.5(\lambda-0.5)}\\ &\times \mathrm{K}_{\lambda-0.5}(\alpha\sqrt{\delta^2+(x-\mu)^2})\exp(\beta(x-\mu)),\quad x\in\mathbb{R},\end{aligned}$$

where $a(\lambda,\alpha,\beta,\delta)=(\alpha^2-\beta^2)^{\lambda/2}/(\sqrt{2\pi}\alpha^{\lambda-0.5}\delta^\lambda\mathrm{K}_\lambda(\delta\sqrt{\alpha^2-\beta^2}))$, and where K_λ is a modified Bessel function of the third kind with index λ. The range of the five parameters is given by

$$\begin{aligned}&\lambda,\mu\in\mathbb{R},\\ &\lambda<0 \quad\Rightarrow\quad \delta>0,|\beta|\le\alpha,\\ &\lambda=0 \quad\Rightarrow\quad \delta>0,|\beta|<\alpha,\\ &\lambda>0 \quad\Rightarrow\quad \delta\ge0,|\beta|<\alpha.\end{aligned}$$

Since a generalized hyperbolic Lévy motion is purely discontinuous, the Gaussian part is zero, i.e. c in equation (3.4) equals zero. For more details see e.g. Eberlein (2001) or Eberlein and Prause (2002).

It can be shown that there is a locally equivalent martingale measure $\mathbf{P}^*$ such that L_1 is generalized hyperbolic under $\mathbf{P}^*$ with parameters $(\widetilde{\lambda},\widetilde{\alpha},\widetilde{\beta},\widetilde{\delta},\widetilde{\mu})$ if and only if $\widetilde{\mu}=\mu$, $\widetilde{\delta}=\delta$ and the drift $A(.,T)$ coincides with the logarithm of the moment generating function of L_1 under $\mathbf{P}^*$ evaluated at $\Sigma(.,T)$, i.e.

$$A(t,T)=\mu\Sigma(t,T)+\ln\left(\frac{\left(\widetilde{\alpha}^2-\widetilde{\beta}^2\right)^{\widetilde{\lambda}/2}\mathrm{K}_{\widetilde{\lambda}}\left(\delta\sqrt{\widetilde{\alpha}^2-(\widetilde{\beta}+\Sigma(t,T))^2}\right)}{\left(\widetilde{\alpha}^2-(\widetilde{\beta}+\Sigma(t,T))^2\right)^{\widetilde{\lambda}/2}\mathrm{K}_{\widetilde{\lambda}}\left(\delta\sqrt{\widetilde{\alpha}^2-\widetilde{\beta}^2}\right)}\right).$$

Thus, the estimation of the drift leads to an estimation for the remaining parameters $\widetilde{\lambda},\widetilde{\alpha}$, and $\widetilde{\beta}$. Note that there is only one martingale measure such that L_1 is generalized hyperbolic with parameters $(\widetilde{\lambda},\widetilde{\alpha},\widetilde{\beta},\widetilde{\delta},\widetilde{\mu})$, see Raible (2000, Section 6.5).

Depending on the empirical data one has to choose a suitable volatility structure for the forward rate dynamics. Eberlein and Raible (1999) show that the Markov property of the short rate implies that Σ has either the Ho-Lee or the Vasiček volatility structure. Figure 3.4 shows the empirical density of the increments of the driving process $(L_t)_{0\le t\le T^*}$ in (3.6), where the underlying data are daily prices of bonds with 5 years to maturity and where the volatility structure has been assumed to be of Ho-Lee type. The two lines represent the fitted normal and the fitted generalized hyperbolic density.

Alternatively, if prices of options on zero coupon bonds are available, one can determine the risk-neutral parameters $(\widetilde{\lambda},\widetilde{\alpha},\widetilde{\beta},\widetilde{\delta},\widetilde{\mu})$ from secondary market data.

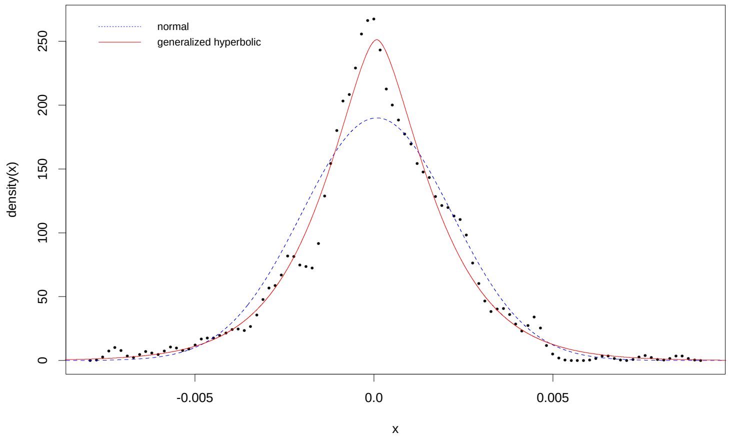


Figure 3.4: Empirical density derived from returns on 5-year bond prices (Sep-17-1999 to Sep-17-2001) and the fitted normal as well as the fitted generalized hyperbolic density.

The real-world parameters of the migration process can be fitted by Nelson-Aalen type estimators, see e.g. Andersen, Borgan, and Keiding (1993) for the mathematical background or Lando and Skødeberg (2002) for applications in rating migration processes. The Markov chain parameters under $\mathbf{Q}^*$ can be determined by using Condition ($\mathbb{M}2$).

3.9 Pure credit derivatives

A credit derivative is a financial asset where the underlying variable is credit risk, i.e. its payoff is determined by credit risk. The relevancy of credit derivatives is that the money lender or owner of a defaultable security can freely trade credit risk. This is similar to traditional insurance contracts where one wants to ensure losses from default, but in case of credit derivatives one does not need to own a defaultable bond to trade credit derivatives written on this defaultable bond. In addition, newer credit instruments have more flexible payoff patterns. For example, the payoff may be determined by the credit rating at maturity date.

Currently most of the credit derivatives are issued and traded over the counter. Therefore credit derivatives are subject to credit risk themselves. This risk will be ignored in this section.

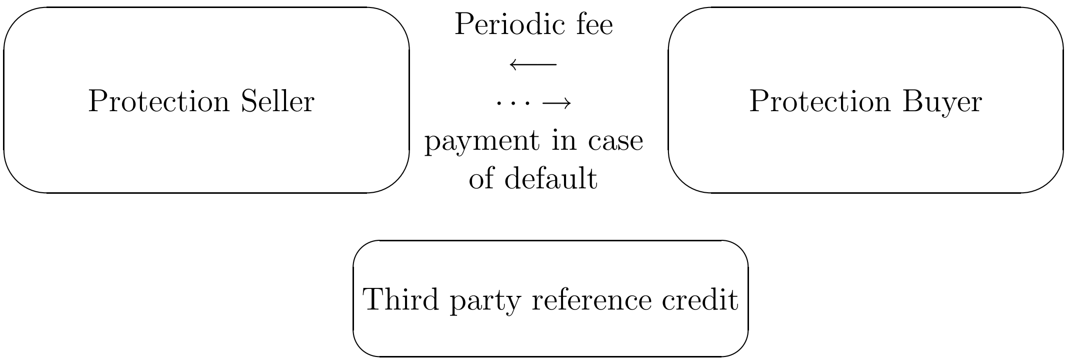
If the payoff of a credit derivative is determined only by credit risk, we say that it is a pure credit derivative. In practice the payoff of a credit derivative will be determined by at least two sources of risk: credit and market risk.

We can divide credit derivatives in two types. The payoff of a credit derivative of the first type is only influenced by the bankruptcy at maturity date. The payoff of a credit derivative of the second type is determined by the default probability of the underlying asset, therefore the maturity of a derivative of the second type is in general earlier than the maturity of the underlying, see Ammann (1999).

Examples for derivatives of the first type are for example credit risk or default swaps, default put options, and total return swaps. We will briefly characterize these three credit derivatives.

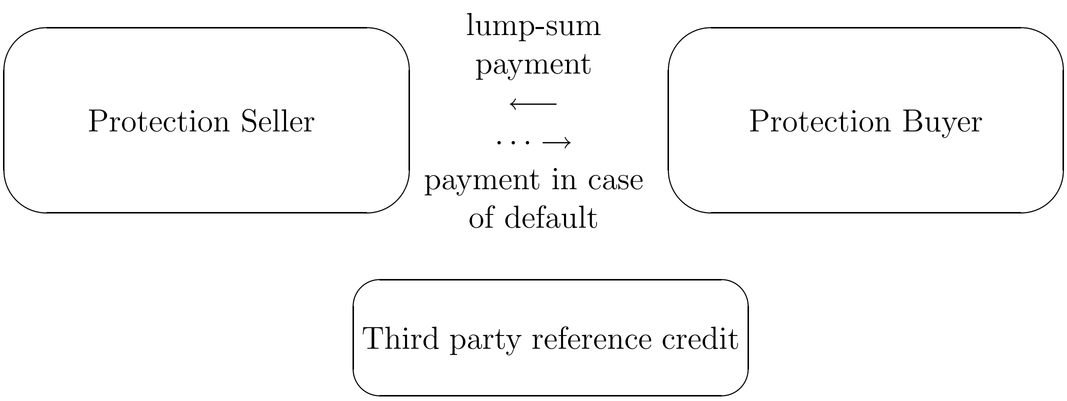
A survey on credit derivatives can be found in Ammann (1999), Nelken (1999), Schönbucher (2000a), Das (2000), and many other publications.

Credit risk or default swaps



The protection buyer is the insurance taking party, who has to pay a periodic fee until maturity of the bond or until default event, in general a fixed amount per year to the protection buyer, who in return has to make a contingent payment in case that the third party reference credit defaults. The payment that has to be made in case of default and the precise definition of default has to be specified in the contract. Possible definitions are e.g. bankruptcy, insolvency, or the failure to meet the payment obligation at maturity. Note that a credit risk swap is similar to an insurance policy, but here the protection buyer does not have to own the object (i.e. the defaultable bond) to buy the insurance (i.e. the default swap).

Default put option

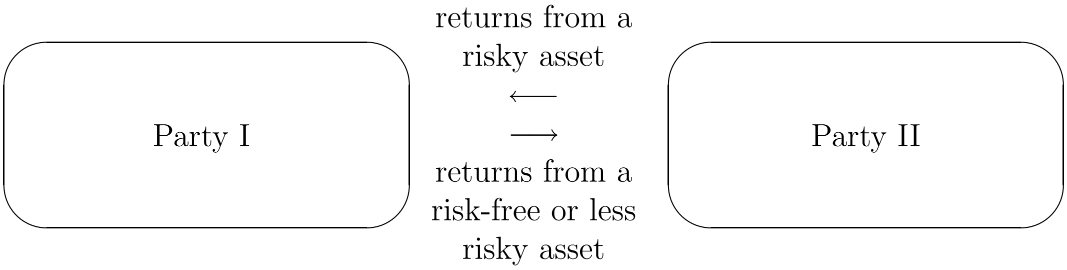


The structure of a default put option is very similar to the structure of a default swap. The difference is that the protection seller recieves the premium at the beginning of the contract as a lump-sum payment.

The payment in case of default is either the difference between the recovery and the face value of the defaulted bond, or a fixed payout which is in case of so-called digital or binary credit derivatives.

If the contingent payment is the loss that the bond holder suffers in case of default, the protection buyer has de facto a long position in a put option on the defaultable bond, whereas the protection buyer has a short position in the put option.

Total return swap



A total return swap is a contract between two parties. Both parties agree to change the returns from a risky and a less risky or risk-free asset.

Valuation

Default insurance products and credit derivatives of the first type can be easily valuated by any model in an arbitrage-free setting which can price the underlying defaultable bond. The price of a plain-vanilla credit derivative is simply the price spread between the defaultable and the default-free bond. The reason for this simple valuation relation is that a standard default swap or default put option insures all the credit risk inherent in the risky bond. This can be seen also from no-arbitrage arguments. Figure 3.5, page 91, shows the payoffs of a default-free bond, a defaultable bond, and a default put option. Let τ be the time of default, $P(t,T)$ the price at time t of a risk-free bond with maturity date T, and let $P^d(t,T)$ be the respective price of the defaultable bond.

The recovery rate δ can be an arbitrary random variable with values in $[0,1]$, and let $V^d(t,T)$ denote the time t price of the default put option with maturity date T. It can be easily seen, that a portfolio consisting of a long position in the default-free bond, and a short position in the defaultable bond, duplicates the credit derivative. At maturity date T, we have the following relationship: $P(T,T) = P^d(T,T) + V^d(T,T)$. For arbitrage reasons, we have that $V^d(t,T) = P(t,T) - P^d(t,T) \geq 0$.

This arbitrage argument can be generalized to more general credit swaps, see Duffie (1999), which include among other things coupons, or transaction costs. The credit swap is thereby duplicated similarly to the easier case by a portfolio consisting of a long position in the default free floating rate note and a short position in the defaultable corporate floating rate note. It is assumed that the defaultable floating rate note pays annual rate R_t+S, where R_t is the coupon rate of the default-free note and S is a fixed spread. For arbitrage reasons indicated above the annuity coupons of the credit swap, denoted by U, are equal to S.

If we consider the intensity based model presented in this Chapter, we can apply these arbitrage arguments. Following Bielecki and Rutkowski (2000) we consider a default swap triggered by the default event $\{C_t^1 = K\}$ which is settled at default time τ.

The payoff of the the default swap is

$$X = \Big(1 - \delta_{C_T^2} B(\tau, T)\Big) \mathbb{1}_{\{\tau \le T\}}.$$

We assume that the annuities are paid at fixed times $0 \le t_1 \le \ldots \le t_n \le T$. Then the annuity U has to satisfy the following equation

$$\pi_0(X) = U \mathbb{E}_{\mathbf{Q}^*} \left[\sum_{i=1}^n B_{t_i}^{-1} \mathbb{1}_{\{\tau > t_i\}} \right].$$

In case of a default put option the value at time 0 is

$$\pi_0(X) = \mathbb{E}_{\mathbf{Q}^*} \left[B_\tau^{-1} (1 - \delta_{C_T^2} B(\tau, T)) \mathbb{1}_{\{\tau \le T\}} \right].$$

The valuation formula for total rate of return swaps on coupon bearing bonds can be found in Bielecki and Rutkowski (2000, 2002).

3.10 A note on the Schönbucher approach

The idea of multiple defaults that we have used in Section 3.5 has been developed in Schönbucher (1998, 2000a). His model is closely related to Duffie and Singleton (1999a). In this section, we want to show how his basic approach can be generalized to Lévy processes. However, we will only state the starting point of the model for the one-dimensional case, without going into further details. This section can be seen as a further example, how models on Brownian motions can be pushed forward to models based on Lévy processes.

The Schönbucher model is formulated under a risk-neutral measure, where the instantaneous forward rates satisfy the assumptions of the Gaussian HJM framework. In Schönbucher (1998, 2000a) the HJM framework based on marked point process framework in Björk, Kabanov, and Runggaldier (1997) is also considered. We assume that the default-free term structure is of the same form as proposed in Eberlein and Raible (1999). This means that the default-free forward rate dynamics is given by the risk-neutral counterpart of equation (3.5), i.e.

$$\begin{aligned} \mathrm{d}f(t,T) = &\ \partial_2\Sigma(t,T)(\kappa'(\Sigma(t,T)) - b)\ \mathrm{d}t \\ &+ (-\sqrt{c}\partial_2\Sigma(t,T))\ \mathrm{d}W_t \\ &+ \int_{\mathbb{R}} (-x\partial_2\Sigma(t,T))\ \left(\mu^L(\mathrm{d}t,\mathrm{d}x) - \nu^L(\mathrm{d}t,\mathrm{d}x)\right), \end{aligned}$$

where Σ, W, μ^L and ν^L satisfy the conditions in Assumption ($\mathbb{A}1$) on page 40.

The default-free bond price is then given by $B(t,T) = \mathrm{e}^{-\int_t^T f(t,u)\ \mathrm{d}u}$.

The defaultable term structure of the instantaneous forward rates is assumed to satisfy

$$\begin{aligned} \mathrm{d}\overline{f}(t,T) = &\ \partial_2\overline{\Sigma}(t,T)(\overline{\kappa}'(\overline{\Sigma}(t,T)) - \overline{b})\ \mathrm{d}t \\ &+ (-\sqrt{\overline{c}}\,\partial_2\overline{\Sigma}(t,T))\ \mathrm{d}W_t \\ &+ \int_{\mathbb{R}} \left(-x\partial_2\overline{\Sigma}(t,T)\right)\ \left(\mu^L(\mathrm{d}t,\mathrm{d}x) - \nu^L(\mathrm{d}t,\mathrm{d}x)\right), \end{aligned} \tag{3.62}$$

where $\overline{\Sigma}$ satisfies the conditions of Assumption ($\mathbb{A}1$) and furthermore we assume $\overline{f}(t,T) > f(t,T)$ to ensure arbitrage-freeness.

Similar to Section 3.3, equation (3.35), a pseudo-defaultable bond is defined by $\widetilde{B}(t,T) := \tilde{B}(0,T)\exp(-\int_0^T \overline{f}(t,u)\ \mathrm{d}u)$.

Concerning the restructuring we assume Assumption ($\mathbb{A}3$), see Section 3.5, page 69. This means that at stopping time $\tau_i, i \geq 1$, the face value of the bond is changed to $1 - q_i$, where $0 \leq q_i < 1$. And we have defined $X_t = \sum_{i\geq 1} q_i \mathbb{1}_{\{\tau_i = t\}}$. We define the process $(Q_t)_{t\geq 0}$ by

$$Q_t := \prod_{0\leq s\leq t} (1 - X_s) = \prod_{\tau_i \leq t}(1 - q_i). \tag{3.63}$$

The defaultable bond price is given by

$$\overline{B}(t,T) = Q_t\widetilde{B}(t,T). \tag{3.64}$$

The defaultable bank account is given by

$$c(t) := Q_t \exp\left(\int_0^t \overline{r}_s\ \mathrm{d}s\right), \tag{3.65}$$

where $\overline{r}_t := \overline{f}(t,t), \quad t \geq 0$.

Remark: The payoff of a defaultable bond is $Q(T) := \prod_{\tau_i \le T}(1-q_i)$. In Proposition 3.10 we calculated that $\mathcal{L}(Q)_t = -\int_0^t \int_0^1 q\, \mu^X(\mathrm{d}s, \mathrm{d}q) = -N_t$.

The next proposition determines the dynamics of the defaultable bond price $\overline{B}(t,T)$ for fixed time horizon date T. Schönbucher (1998, 2000a) assumes the structure of the dynamics of $\overline{B}(t,T)$ and notes then that $\overline{B}(t,T) = Q_t\widetilde{B}(t,T)$. But we prefer the more intuitive way of defining $\overline{B}(t,T) = Q_t\widetilde{B}(t,T)$ and then deriving the dynamics of $\overline{B}(t,T)$.

Proposition 3.12 *The dynamics of the defaultable bond price* $\overline{B}(t,T) = Q_t\widetilde{B}(t,T)$ *for fixed* $T \in [0,T^*]$ *is given by*

$$
\begin{aligned}
\frac{1}{\overline{B}(t-,T)}\,\mathrm{d}\overline{B}(t,T) = &\left(\overline{r}_t + \overline{b}\,\overline{\Sigma}(t,T) - \overline{\kappa}(\overline{\Sigma}(t,T)) + \frac{1}{2}\overline{c}\,\overline{\Sigma}(t,T)^2\right)\mathrm{d}t \\
&+ \sqrt{\overline{c}}\,\overline{\Sigma}(t,T)\,\mathrm{d}W_t + \int_{\mathbb{R}} x\overline{\Sigma}(t,T)(\mu^L - \nu^L)(\mathrm{d}t,\mathrm{d}x) \\
&+ \int_{\mathbb{R}} \left(\mathrm{e}^{\overline{\Sigma}(t,T)x} - \overline{\Sigma}(t,T)x - 1\right)\mu^L(\mathrm{d}t,\mathrm{d}x) \\
&- \int_{\mathbb{R}^2} x_1 \mathrm{e}^{\overline{\Sigma}t,T)x_2}\,\mu^{(X,L)}(\mathrm{d}t,\mathrm{d}(x_1,x_2)).
\end{aligned}
\tag{3.66}
$$

Proof: Calculating the stochastic dynamics of $\frac{1}{\overline{B}(t-,T)}\,\mathrm{d}\overline{B}(t,T)$ is equivalent to calculating the stochastic logarithm $\mathcal{L}(\overline{B}(.,T))_t$. It is given by, see Lemma 2.4 in Kallsen and Shiryaev (2000),

$$
\begin{aligned}
\mathcal{L}(\overline{B}(.,T))_t &= \int_0^t \frac{1}{\overline{B}(s-,T)}\,\mathrm{d}\overline{B}(s,T) \\
&= \log\left(\frac{\overline{B}(t,T)}{\overline{B}(0,T)}\right) + \frac{1}{2}\int_0^t \frac{1}{\overline{B}(s-,T)^2}\,\mathrm{d}\langle \overline{B}^c(.,T), \overline{B}^c(.,T)\rangle_s \\
&\quad - \int_0^t \int_{\mathbb{R}} \left(\log\left(\left|1 + x\overline{B}(s-,T)^{-1}\right|\right) - x\overline{B}(s-,T)^{-1}\right)\mu^{\overline{B}(.,T)}(\mathrm{d}s,\mathrm{d}x).
\end{aligned}
$$

We calculate the three expressions on the right-hand side one by one.

The first expression is

$$\log\left(\frac{\overline{B}(t,T)}{\overline{B}(0,T)}\right) = \log\left(\frac{Q_t}{Q_0}\right) + \log\left(\frac{\widetilde{B}(t,T)}{\widetilde{B}(0,T)}\right)$$
$$= \log(Q_t) + \int_0^t \overline{r}_s + \overline{b}\,\overline{\Sigma}(s,T) - \overline{\kappa}\left(\overline{\Sigma}(s,T)\right)\,\mathrm{d}s$$
$$+ \sqrt{\overline{c}}\int_0^t \overline{\Sigma}(s,T)\,\mathrm{d}W_s$$
$$+ \int_0^t \int_{\mathbb{R}} x\overline{\Sigma}(s,T)\left(\mu^L - \nu^L\right)(\mathrm{d}s,\mathrm{d}x).$$

For the second expression note that since Q has paths of bounded variation and $\widetilde{B}(.,T)$ is a semimartingale, we have $\overline{B}(t,T) = Q_t\widetilde{B}(t,T) = \int_0^t Q_{s-}\,\mathrm{d}\widetilde{B}(s,T) + \int_0^t \widetilde{B}(s,T)\,\mathrm{d}Q_s$, see Jacod and Shiryaev (1987, I.4.49).
Therefore,

$$\overline{B}^c(t,T) = \left(\int_0^{\cdot} Q_{s-}\,\mathrm{d}\widetilde{B}(s,T)\right)_t^c$$
$$= \sqrt{\overline{c}}\int_0^t Q_{s-}\widetilde{B}(s-,T)\overline{\Sigma}(s,T)\,\mathrm{d}W_s$$
$$= \sqrt{\overline{c}}\int_0^t \overline{B}(s-,T)\overline{\Sigma}(s,T)\,\mathrm{d}W_s,$$

and therefore

$$\int_0^t \frac{1}{\overline{B}(s-,T)^2}\,\mathrm{d}\langle\overline{B}^c(.,T),\overline{B}^c(.,T)\rangle_s = \overline{c}\int_0^t \overline{\Sigma}(s,T)^2\,\mathrm{d}s.$$

The third expression deals with the jumps of $\overline{B}(.,T)$.

$$\int_0^t \int_{\mathbb{R}} \left(\log\left(\left|1+x\overline{B}(s-,T)^{-1}\right|\right) - x\overline{B}(s-,T)^{-1}\right) \mu^{\overline{B}(.,T)}(\mathrm{d}s,\mathrm{d}x)$$

$$= \sum_{0\le s\le t} \left(\log\left(\left|1+\frac{\triangle\overline{B}(s,T)}{\overline{B}(s-,T)}\right|\right) - \frac{\triangle\overline{B}(s,T)}{\overline{B}(s-,T)}\right)$$

$$= \sum_{0\le s\le t} \left(\log\left(\frac{\overline{B}(s,T)}{\overline{B}(s-,T)}\right) - \frac{\overline{B}(s,T)}{\overline{B}(s-,T)} + 1\right)$$

$$= \sum_{0\le s\le t} \Big(\log(1-X_s) + X_s \mathrm{e}^{\overline{\Sigma}(s,T)\triangle L_s}$$

$$+ \overline{\Sigma}(s,T)\triangle L_s - \mathrm{e}^{\overline{\Sigma}(s,T)\triangle L_s} + 1\Big)$$

$$= \log Q_t + \sum_{0\le s\le t} X_s \mathrm{e}^{\overline{\Sigma}(s,T)\triangle L_s}$$

$$+ \int_0^t \int_{\mathbb{R}} \left(\overline{\Sigma}(s,T)x - \mathrm{e}^{\overline{\Sigma}(s,T)x} + 1\right) \mu^L(\mathrm{d}s,\mathrm{d}x).$$

Thus,

$$\mathcal{L}(\overline{B}(.,T))_t = \int_0^t \left(\overline{r}_s + \overline{b}\,\overline{\Sigma}(s,T) - \overline{\kappa}\left(\overline{\Sigma}(s,T)\right) + \frac{1}{2}\overline{c}\,\overline{\Sigma}(s,T)^2\right) \mathrm{d}s$$

$$+ \sqrt{\overline{c}} \int_0^t \overline{\Sigma}(s,T)\ \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} x\overline{\Sigma}(s,T)\left(\mu^L - \nu^L\right)(\mathrm{d}s,\mathrm{d}x)$$

$$- \int_0^t \int_{\mathbb{R}} \left(\overline{\Sigma}(s,T)x - \mathrm{e}^{\overline{\Sigma}(s,T)x} + 1\right) \mu^L(\mathrm{d}s,\mathrm{d}x)$$

$$- \sum_{0\le s\le t} X_s \mathrm{e}^{\overline{\Sigma}(s,T)\triangle L_s}. \qquad \text{[Pr3.12.a]}$$

This proves our statement. □

Corollary 3.13 *The stochastic logarithm of the defaultable bond price process is given by*

$$\mathcal{L}(\overline{B}(.,T))_t = \mathcal{L}(\tilde{B}(.,T))_t - \int_0^t \int_{\mathbb{R}^2} x_1 \mathrm{e}^{\overline{\Sigma}(s,T)x_2} \, \mu^{(X,L)}(\mathrm{d}s, \mathrm{d}(x_1, x_2)). \tag{3.67}$$

Proof: Equation (3.67) can be easily derived from equation [Pr3.12.a] and Proposition 3.2, where the proposition has to be applied to $\widetilde{B}(t,T)$ instead of $B(t,T)$.
□

Remark: Equation (3.67) can be simplified, if we exclude the possibility that X and L jump simultaneously, which is e.g. the case when L has no jumps at all. In the Gaussian HJM framework in Schönbucher (1998) equation (3.67) has the following form

$$\mathcal{L}(\overline{B}(.,T))_t = \mathcal{L}(\tilde{B}(.,T))_t - N_t. \tag{3.68}$$

The dynamics of the defaultable bond is the main input for the fixing of a martingale measure. The approach here can be developed further in the same line as in the previous sections.

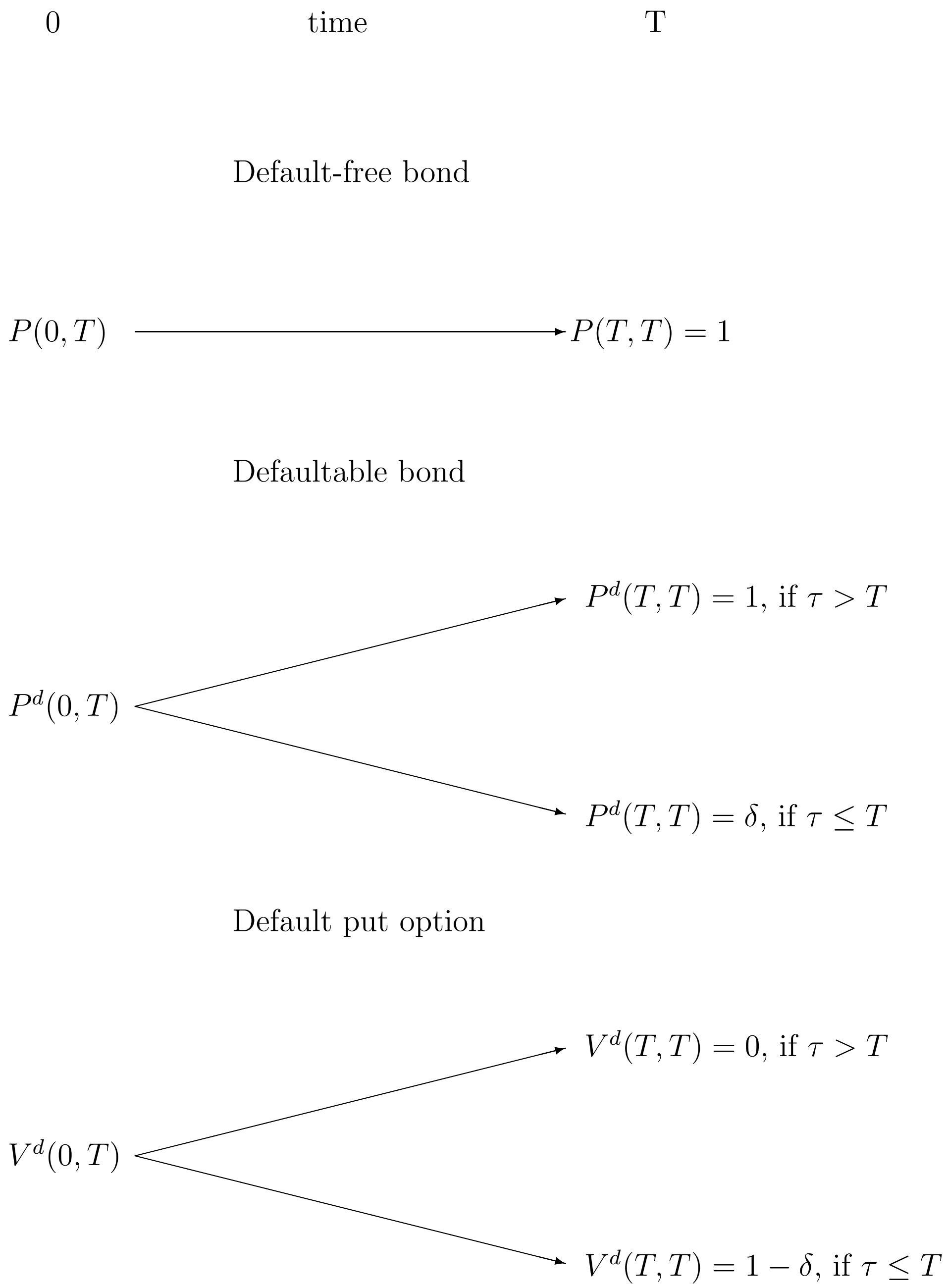


Figure 3.5: Payoffs: default-free bond, defaultable bond, and default put option.

Chapter 4

Market Models

In this chapter we show how a Libor market model based on general Lévy processes as driving processes can be constructed. Libor models are suited for practical modeling of interest rate derivatives such as swaptions, floors and caps.

A main feature of market or Libor models is that they are not necessarily based on a continuum of data, as in the case of instantaneous forward rates, see Chapter 3. Libor models have been developed in Brace, Gatarek, and Musiela (1997), Miltersen, Sandmann, and Sondermann (1997), and Jamshidian (1997) to construct an arbitrage-free term structure model which is consistent with the market practice of pricing caps and floors by Black's formula. Black's formula for caps is motivated by the option formula of Black and Scholes (1973) and implies that the Libor rates follow a geometric Brownian motion. We propose a more general model where the driving process of the Libor rates is a general Lévy process instead of a Brownian motion. An introduction to Libor models can be found in several textbooks and articles, e.g. in Musiela and Rutkowski (1998), Brigo and Mercurio (2001), or Rutkowski (2001).

There are different approaches to Libor market models. In the first approaches the Libor rates were modeled as a secondary process, i.e. the starting point for the modeling was different from the Libor rate itself. The most popular starting point is the HJM-approach. More recent models take the Libor rate as a primary process. We present both views in the following sections.

This chapter is related to Glasserman and Kou (2001) and to Jamshidian (1999). In Glasserman and Kou (2001) the term structure of Libor rates is driven by a jump diffusion process. In this case the purely discontinuous part is of bounded variation. In Jamshidian (1999) the Libor rate process is driven by a general semimartingale, but the pricing of caps and floors is not considered. This chapter can be seen in parts as a special case of the Jamshidian approach, but we specify the driving Lévy process and show how the Libor rates can be represented as an exponential of a stochastic integral driven by a Lévy process. Furthermore we show how the pricing of caps and floors can be done using Laplace transforms. The latter is based on an idea proposed in Raible (2000).

Throughout this chapter we restrict ourselves to the case where the stochastic integral can be written as the sum of the stochastic integrals with respect to the coordinates, although this is not the most general setting for stochastic integrals with respect to multi-dimensional semimartingales, see Jacod and Shiryaev (1987, III.4.§4a).

4.1 Modeling instantaneous forward rates

Libor rates can be derived from the bond prices which are determined by the instantaneous forward rates. The advantage of this approach is that Libor rates can be embedded in an existing HJM-framework, see Brace, Gatarek, and Musiela (1997). We follow this approach in the general setting of Björk, Di Masi, Kabanov, and Runggaldier (1997) and in the Lévy setting of Eberlein and Raible (1999).

4.1.1 The semimartingale setting

We assume a complete stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}^*, (\mathcal{F}_t)_{0\leq t\leq T^*})$, where the filtration satisfies the usual conditions. The predictable σ-field is given by $\mathcal{P}$.

For our purposes it is sufficient to consider the risk-neutral setting. This means that concerning the dynamics of the instantaneous forward rates we make the following assumption, see Björk, Di Masi, Kabanov, and Runggaldier (1997, Assumption 5.1).

Assumption ($\mathbb{S}$): The dynamics of the forward rates for $T \in [0, T^*]$ is

$$\mathrm{d}f(t,T) = \alpha(t,T)\ \mathrm{d}t + \sigma(t,T)^\top\ \mathrm{d}W_t^* + \int_{\mathbb{R}^r} \delta(x,t,T)(\mu - \nu^*)(\mathrm{d}t, \mathrm{d}x), \qquad (4.1)$$

where W^* is a standard Brownian motion in $\mathbb{R}^d$, μ is the jump measure of a semimartingale with continuous compensator ν^*, i.e. there exists $\lambda^* : \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^r) \to \mathbb{R}_+$ such that $\nu^*(\mathrm{d}t, \mathrm{d}x) = \lambda^*(t, \mathrm{d}x)\ \mathrm{d}t$. The coefficients are continuous in the second variable, where $\alpha : \Omega \times [0, T^*] \times [0, T^*] \to \mathbb{R}$ and $\sigma : \Omega \times [0, T^*] \times [0, T^*] \to \mathbb{R}_+$ are $\mathcal{P}\otimes\mathcal{B}([0, T^*])$-measurable and $\delta : \Omega\times\mathbb{R}^r\times[0, T^*]\times[0, T^*]$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^r)\otimes\mathcal{B}([0, T^*])$-measurable. The coefficients satisfy the following conditions for all finite $(t, T) \in \Delta := \{(s, u) \in \mathbb{R}_+ \times \mathbb{R}_+ \,|\, 0 \leq s \leq u \leq T^*\}$

$$\alpha(t,T) = \delta(x,t,T) = 0, \text{ if } (t,T) \notin \Delta$$
$$\sigma(t,T) = (0,\ldots,0)^\top, \quad \text{ if } (t,T) \notin \Delta$$

and

$$\int_0^T \int_t^T |\alpha(s,u)| \, \mathrm{d}u \, \mathrm{d}s < \infty, \qquad \int_0^T \int_t^T |\sigma(s,u)|^2 \, \mathrm{d}u \, \mathrm{d}s < \infty,$$

$$\int_0^T \int_{\mathbb{R}^r} \int_t^T |\delta(s,x,u)|^2 \, \mathrm{d}u \, \nu^*(\mathrm{d}s, \mathrm{d}x) < \infty.$$

We put

$$A(t,T) := -\int_t^T \alpha(t,u) \, \mathrm{d}u, \qquad S(t,T) := -\int_t^T \sigma(t,u) \, \mathrm{d}u,$$

$$\text{and } D(t,x,T) := -\int_t^T \delta(t,x,u) \, \mathrm{d}u.$$

Then the model is arbitrage-free if

$$\int_0^t \int_{\mathbb{R}^r} \left(\mathrm{e}^{D(t,x,T)} - 1 - D(t,x,T)\right) \, \nu^*(\mathrm{d}t, \mathrm{d}x) < \infty$$

and

$$A(t,T) + \frac{1}{2}|S(t,T)|^2 + \int_{\mathbb{R}^r} \left(\mathrm{e}^{D(t,x,T)} - 1 - D(t,x,T)\right) \lambda^*(t, \mathrm{d}x) = 0 \tag{4.2}$$

for any $t \in [0, T^*]$, see Björk, Di Masi, Kabanov, and Runggaldier (1997, Proposition 5.3).

If the discontinuous part of f has sample paths of bounded variation, then the following equation is sufficient for equation (4.2)

$$\alpha(t,T) = -\sigma(t,T)^\top S(t,T) - \int_{\mathbb{R}^r} \delta(x,t,u) \left(\exp\left(-\int_t^T \delta(t,x,u) \, \mathrm{d}u \right) - 1 \right) \lambda^*(t, \mathrm{d}x). \tag{4.3}$$

Equation (4.3) corresponds to equation (38) in Glasserman and Kou (2001) and can be obtained from equation (4.2) by taking derivatives with respect to the second variable.

The price at time t of a bond with maturity T is then given by

$$B(t,T) = B(0,T)\exp\left(\int_0^t (f(s,s) + A(s,T))\,\mathrm{d}s + \int_0^t S(s,T)^\top\,\mathrm{d}W_s^* \right.$$
$$\left. + \int_0^t \int_{\mathbb{R}^r} D(s,x,T)\,(\mu - \nu^*)(\mathrm{d}s,\mathrm{d}x)\right). \tag{4.4}$$

Remember that the savings account is $B_t = \exp\left(\int_0^t f(s,s)\,\mathrm{d}s\right)$.
Equations (4.4) and (4.2) yield for the discounted bond price at time t

$$Z(t,T) := \frac{B(t,T)}{B_t}$$

the following expression

$$Z(t,T) = Z(0,T)\exp\left(-\frac{1}{2}\int_0^t |S(s,T)|^2\,\mathrm{d}s + \int_0^t S(s,T)^\top\,\mathrm{d}W_s^* \right.$$
$$- \int_0^t \int_{\mathbb{R}^r} \left(\mathrm{e}^{D(s,x,T)} - 1 - D(s,x,T)\right)\lambda^*(s,\mathrm{d}x)\,\mathrm{d}s \tag{4.5}$$
$$\left. + \int_0^t \int_{\mathbb{R}^r} D(s,x,T)(\mu - \nu^*)(\mathrm{d}s,\mathrm{d}x)\right).$$

The forward process is defined by

$$F_B(t,T,T^*) := \frac{B(t,T)}{B(t,T^*)}. \tag{4.6}$$

Let $\delta > 0$ with $\delta \le T^* - T$, then the δ-(forward)-Libor rates are defined by

$$L(t,T) = \frac{1}{\delta}(F_B(t,T,T+\delta) - 1) \tag{4.7}$$
$$= \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right). \tag{4.8}$$

The δ-(forward)-Libor rate coincides with the forward swap rate of a single-period swap settled in arrears, see e.g. Rutkowski (2001). When we speak of Libor rates, we will always mean forward Libor rates.

The next theorem states the dynamics of $L(.,T)$ under the risk-neutral measure $\mathbb{P}^*$.

Theorem 4.1 *If $f(.,T)$ satisfies (4.1) and $A(.,T), D(.,.,T)$, and $S(.,T)$ satisfy (4.2), then the risk-neutral dynamics of $L(.,T)$ is given by*

$$\begin{aligned}\frac{\delta}{1+\delta L(t-,T)}\,\mathrm{d}L(t,T) &= \left(S(t,T+\delta)-S(t,T)\right)^\top S(t,T+\delta)\,\mathrm{d}t \\ &\quad + \int_{\mathbb{R}^r} \Big(\mathrm{e}^{D(t,x,T)-D(t,x,T+\delta)} - 1 \\ &\qquad\qquad + \mathrm{e}^{D(t,x,T+\delta)} - \mathrm{e}^{D(t,x,T)}\Big)\,\nu^*(\mathrm{d}t,\mathrm{d}x) \\ &\quad + \left(S(t,T)-S(t,T+\delta)\right)^\top \mathrm{d}W_t^* \\ &\quad + \int_{\mathbb{R}^r} \Big(\mathrm{e}^{D(t,x,T)-D(t,x,T+\delta)} - 1\Big)\,(\mu-\nu^*)\,(\mathrm{d}t,\mathrm{d}x).\end{aligned} \tag{4.9}$$

Proof: We have got the following relationship between the forward process $F_B(.,T,T+\delta)$, the Libor rate $L(.,T)$ and the discounted bond prices

$$\begin{aligned}F_B(t,T,T+\delta) = 1+\delta L(t,T) &= \frac{Z(t,T)}{Z(t,T+\delta)} \\ &= \frac{Z(0,T)}{Z(0,T+\delta)}\exp\Bigg(\int_0^t \left(A(s,T)-A(s,T+\delta)\right)\,\mathrm{d}s \\ &\qquad + \frac{1}{2}\int_0^t \left(S(s,T)-S(s,T+\delta)\right)^\top \mathrm{d}W_s^* \\ &\qquad + \int_0^t\int_{\mathbb{R}^r} \left(D(s,x,T)-D(s,x,T+\delta)\right)(\mu-\nu^*)(\mathrm{d}s,\mathrm{d}x)\Bigg).\end{aligned}$$

The connection between the forward process and the Libor process yields that the dynamics of the Libor rate is related to the dynamics of the forward process in the following way

$$\frac{\delta}{1+\delta L(t-,T)}\,\mathrm{d}L(t,T) = \frac{1}{F_B(t-,T,T+\delta)}\,\mathrm{d}F_B(t,T,T+\delta).$$

Since $F_B(.,T,T+\delta)$ is a positive semimartingale it can be expressed as a stochastic exponential of a process $\mathcal{L}(F_B(.,T,T+\delta))$; in fact we have $F_B(.,T,T+\delta) > 1$.

This process is given by, see Kallsen and Shiryaev (2000),

$$\mathcal{L}(F_B(.,T,T+\delta))_t = \ln\left(\frac{F_B(t,T,T+\delta)}{F_B(0,T,T+\delta)}\right) + \int_0^t \frac{1}{2F_B^2(s-,T,T+\delta)}\,\mathrm{d}\langle F_B^c(.,T,T+\delta), F_B^c(.,T,T+\delta)\rangle_s - \int_0^t\int_{\mathbb{R}^r}\left(\ln\left(|1+\frac{x}{F_B(s-,T,T+\delta)}|\right) + \frac{x}{F_B(s-,T,T+\delta)}\right)\mu^{F_B}(\mathrm{d}s,\mathrm{d}x). \qquad [Th4.1.a]$$

The first expression on the right-hand side is simply

$$\ln\left(\frac{F_B(t,T,T+\delta)}{F_B(0,T,T+\delta)}\right) = \int_0^t (A(s,T)-A(s,T+\delta))\,\mathrm{d}s + \int_0^t (S(s,T)-S(s,T+\delta))^\top\,\mathrm{d}W_s^* + \int_0^t\int_{\mathbb{R}^r}(D(s,x,T)-D(s,x,T+\delta))(\mu-\nu^*)(\mathrm{d}s,\mathrm{d}x). \qquad [Th4.1.b]$$

For the next expression on the right-hand side of equation [Th4.1.a] we have to calculate the quadratic covariation of $F_B(.,T,T+\delta)$. It can be derived from the quadratic covariation of $\ln F_B(.,T,T+\delta)$. Note that

$$\langle(\ln F_B(.,T,T+\delta))^c, (\ln F_B(.,T,T+\delta))^c\rangle_t = \int_0^t |S(s,T)-S(s,T+\delta)|^2\,\mathrm{d}s.$$

Therefore, the quadratic covariation of $F_B(.,T,T+\delta)$ is

$$\langle(F_B(.,T,T+\delta))^c, (F_B(.,T,T+\delta))^c\rangle_t = \int_0^t F_B^2(s-,T,T+\delta)|S(s,T)-S(s,T+\delta)|^2\,\mathrm{d}s.$$

Hence, the second expression on the right-hand side of [Th4.1.a] simplifies to

$$\int_0^t \frac{1}{2F_B^2(s-,T,T+\delta)} \, \mathrm{d}\langle F_B^c(.,T,T+\delta), F_B^c(.,T,T+\delta)\rangle_s$$

$$= \int_0^t \frac{1}{2}|S(s,T) - S(s,T+\delta)|^2 \, \mathrm{d}s. \quad \text{[Th4.1.c]}$$

For the remaining expression on the right-hand side of [Th4.1.a], we have to determine the jump measure of F_B, μ^{F_B}, by means of the random measure μ. The jumps of F_B and $\ln F_B$ are related by

$$\triangle F_B(t,T,T+\delta) = F_B(t-,T,T+\delta)\left(\mathrm{e}^{\triangle \ln F_B(t-,T,T+\delta)} - 1\right).$$

The jumps of $\ln F_B(.,T,T+\delta)$ are given by

$$\triangle \ln F_B(t,T,T+\delta) = \int_{\mathbb{R}^r} (D(t,x,T) - D(t,x,T+\delta))\, \mu(\{t\}, \mathrm{d}x).$$

Therefore for μ^{F_B}-integrable functions the following equation holds

$$\int_0^t \int_{\mathbb{R}^r} f(s,u)\, \mu^{F_B}(\mathrm{d}s,\mathrm{d}u) = \int_0^t \int_{\mathbb{R}^r} f(s, F_B(s-,T,T+\delta)(\mathrm{e}^x - 1))\, \mu^{\ln F_B}(\mathrm{d}s,\mathrm{d}x)$$

$$= \int_0^t \int_{\mathbb{R}^r} f(s, F_B(s-,T,T+\delta)(\exp(D(s,x,T) - D(s,x,T+\delta)) - 1))\, \mu(\mathrm{d}s,\mathrm{d}x).$$

Hence,

$$\int_0^t \int_{\mathbb{R}^r} \left(\ln\left(\left|1 + \frac{x}{F_B(s-,T,T+\delta)}\right|\right) - \frac{x}{F_B(s-,T+\delta)}\right) \mu^{F_B}(\mathrm{d}s,\mathrm{d}x)$$

$$= \int_0^t \int_{\mathbb{R}^r} \left((D(s,x,T) - D(s,x,T+\delta)) - \mathrm{e}^{D(s,x,T)-D(s,x,T+\delta)} + 1\right) \mu(\mathrm{d}s,\mathrm{d}x)$$

[Th4.1.d]

Now we put [Th4.1.b] to [Th4.1.d] in [Th4.1.a] and get the dynamics of

$F_B(.,T,T+\delta)$ under the measure $\mathbb{P}^*$

$$\begin{aligned}
\frac{\mathrm{d}F_B(t,T,T+\delta)}{F_B(t-,T,T+\delta)} &= (A(t,T)-A(t,T+\delta))\ \mathrm{d}t + (S(t,T)-S(t,T+\delta))^\top\ \mathrm{d}W_t^* \\
&\quad + \int\limits_{\mathbb{R}^r} (D(t,x,T)-D(t,x,T+\delta))\,(\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x) \\
&\quad + \frac{1}{2}(|S(t,T)-S(t,T+\delta)|^2)\ \mathrm{d}t \\
&\quad + \int\limits_{\mathbb{R}^r} \Big(\mathrm{e}^{D(t,x,T)-D(t,x,T+\delta)}-1 \\
&\qquad\qquad - (D(t,x,T)-D(t,x,T+\delta))\Big)\ \mu(\mathrm{d}t,\mathrm{d}x) \\
&= (A(t,T)-A(t,T+\delta)+\frac{1}{2}(|S(t,T)-S(t,T+\delta)|^2))\ \mathrm{d}t \\
&\quad + (S(t,T)-S(t,T+\delta))^\top\ \mathrm{d}W_t^* \\
&\quad + \int\limits_{\mathbb{R}^r} (D(t,x,T)-D(t,x,T+\delta))\,(\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x) \\
&\quad + \int\limits_{\mathbb{R}^r} \Big(\mathrm{e}^{D(t,x,T)-D(t,x,T+\delta)}-1 \\
&\qquad\qquad - (D(t,x,T)-D(t,x,T+\delta))\Big)\ (\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x) \\
&\quad + \int\limits_{\mathbb{R}^r} \Big(\mathrm{e}^{D(t,x,T)-D(t,x,T+\delta)}-1 \\
&\qquad\qquad - (D(t,x,T)-D(t,x,T+\delta))\Big)\ \nu^*(\mathrm{d}t,\mathrm{d}x).
\end{aligned}$$

Equation (4.2) yields

$$\begin{aligned}
\frac{\mathrm{d}F_B(t,T,T+\delta)}{F_B(t-,T,T+\delta)} &= \frac{1}{2}(|S(t,T+\delta)|^2-|S(t,T)|^2+|S(t,T)-S(t,T+\delta)|^2)\ \mathrm{d}t \\
&\quad + \int\limits_{\mathbb{R}^r} \Big(\mathrm{e}^{D(t,x,T)-D(t,x,T+\delta)}+\mathrm{e}^{D(t,x,T+\delta)}-\mathrm{e}^{D(t,x,T)}-1\Big)\ \nu^*(\mathrm{d}t,\mathrm{d}x) \\
&\quad + (S(t,T)-S(t,T+\delta))^\top\ \mathrm{d}W_t^* \\
&\quad + \int\limits_{\mathbb{R}^r} \Big(\mathrm{e}^{D(t,x,T)-D(t,x,T+\delta)}-1\Big)\ (\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x).
\end{aligned}$$

Note that

$$|S(t,T+\delta)|^2 - |S(t,T)|^2 + |S(t,T) - S(t,T+\delta)|^2 \\ = 2\left(S(t,T+\delta) - S(t,T)\right)^\top S(t,T+\delta).$$

This proves our theorem. □

The next theorem shows under which drift conditions one can embed a Libor-model within this framework. It is a generalization of Theorem 3.1 in Glasserman and Kou (2001) for the risk-neutral case. We consider a discrete-tenor structure $0 = T_0 < T_1 < T_2 < \ldots < T_M < T_{M+1} = T^*$ with $\delta = T_{i+1} - T_i$, $i = 1, \ldots, M$. We define the auxiliary function $\eta(t) := \inf\{k \in \{0, \ldots, M+1\} \mid T_k \geq t\}$.

Theorem 4.2 *For each $n = 1, 2, \ldots, M$ let $\gamma_n(.)$ be a bounded $\mathbb{R}^d$-valued function and $H_n : \mathbb{R}_+ \times \mathbb{R}^r \to (-1, \infty)$ be deterministic and in $G_{\mathrm{loc}}(\mu)$ and $\nu^*(\mathrm{d}t, \mathrm{d}x) = \lambda^*(t, \mathrm{d}x)\ \mathrm{d}t$. The model*

$$\mathrm{d}L(t,T_n) = L(t-,T_n)\left(\alpha_n(t)\ \mathrm{d}t + \gamma_n(t)^\top\ \mathrm{d}W_t^* + \int_{\mathbb{R}^r} H_n(t,x)\,(\mu - \nu^*)(\mathrm{d}t,\mathrm{d}x)\right), \tag{4.10}$$

is arbitrage-free if

$$\alpha_n(t) = \sum_{k=\eta(t)}^{n} \frac{\delta\gamma_n(t)^\top \gamma_k(t) L(t-,T_k)}{1+\delta L(t-,T_k)} \\ + \int_{\mathbb{R}^r} H_n(x,t)\left(1 - \prod_{k=\eta(t)}^{n} \frac{1+\delta L(t-,T_k)}{1+\delta L(t-,T_k)(1+H_k(x,t))}\right)\lambda^*(t,\mathrm{d}x). \tag{4.11}$$

Proof: We show that σ and δ can be chosen in equation (4.1), so that the dynamics of $L(.,T_n)$ in Theorem 4.1 simplifies to those in equation (4.10).

The risk-neutral dynamics of $L(.,T_n)$ under the measure $\mathbb{P}^*$ is given by

$$\begin{aligned} \mathrm{d}L(t,T_n) = \frac{1+\delta L(t-,T_n)}{\delta} \Bigg(& (S(t,T_{n+1}) - S(t,T_n))^\top S(t,T_{n+1})\ \mathrm{d}t \\ & + \int_{\mathbb{R}^r} \big(\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})} - 1 \\ & \qquad + \mathrm{e}^{D(t,x,T_{n+1})} - \mathrm{e}^{D(t,x,T_n)}\big)\ \nu^*(\mathrm{d}t,\mathrm{d}x) \\ & + (S(t,T_n) - S(t,T_{n+1}))^\top\ \mathrm{d}W_t^* \\ & + \int_{\mathbb{R}^r} \big(\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})} - 1\big)\ (\mu - \nu^*)(\mathrm{d}t,\mathrm{d}x) \Bigg). \end{aligned}$$

Let

$$S(t,T_n) - S(t,T_{n+1}) = \int_{T_n}^{T_{n+1}} \sigma(t,u)\ \mathrm{d}u := \frac{\delta\gamma_n(t)L(t-,T_n)}{1+\delta L(t-,T_n)} \qquad \text{[Th4.2.a]}$$

and

$$\begin{aligned} D(t,x,T_n) - D(t,x,T_{n+1}) &= \int_{T_n}^{T_{n+1}} \delta(t,x,u)\ \mathrm{d}u \\ &:= \ln\left(\frac{1+\delta L(t-,T_n)(1+H_n(t,x))}{1+\delta L(t-,T_n)}\right). \qquad \text{[Th4.2.b]} \end{aligned}$$

For simplicity, we assume that $\sigma(t,u) = \text{const}$, for $u \in [T_n, T_{n+1}]$ and $\sigma(t,u) = 0$, for $u \leq t$. Analogously, we assume piecewise constancy for each x and t for the function δ and $\delta(t,x,u) = 0$ for $u \leq t$. Note that we can also use piecewise affine linear or other smooth functions instead of constants.

A consequence of the definitions above is

$$S(t,T_{n+1}) = -\sum_{k=\eta(t)}^{n} \frac{\delta\gamma_k(t)L(t-,T_k)}{1+\delta L(t-,T_k)} \qquad \text{[Th4.2.c]}$$

and that

$$\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})} = \frac{1+\delta L(t-,T_n)(1+H_n(t,x))}{1+\delta L(t-,T_n)} = 1 + \frac{\delta H_n(t,x)L(t-,T_n)}{1+\delta L(t-,T_n)}. \qquad \text{[Th4.2.d]}$$

Furthermore,

$$D(t,x,T_{n+1}) = -\int_t^{T_{n+1}} \delta(t,x,u)\ \mathrm{d}u = -\sum_{k=\eta(t)}^{n} \int_{T_k}^{T_{k+1}} \delta(t,x,u)\ \mathrm{d}u$$

and hence

$$\mathrm{e}^{D(t,x,T_{n+1})} = \prod_{k=\eta(t)}^{n} \mathrm{e}^{-\int_{T_k}^{T_{k+1}} \delta(t,x,u)\ \mathrm{d}u} = \prod_{k=\eta(t)}^{n} \frac{1+\delta L(t-,T_k)}{1+\delta L(t-,T_k)(1+H_k(x,t))}. \qquad \text{[Th4.2.e]}$$

We calculate the integral with respect to the Brownian motion in $\mathrm{d}L(.,T_n)$. We have

$$\frac{1+\delta L(t-,T_n)}{\delta}(S(t,T_n)-S(t,T_{n+1}))^\top\ \mathrm{d}W_t^* = L(t-,T_n)\gamma_n(t)^\top\ \mathrm{d}W_t^*. \quad \text{[Th4.2.f]}$$

The corresponding drift part of $\mathrm{d}L(.,T_n)$ is

$$\begin{aligned}&\frac{1+\delta L(t-,T_n)}{\delta}(S(t,T_{n+1})-S(t,T_n))^\top S(t,T_{n+1})\ \mathrm{d}t\\ &\qquad = L(t-,T_n)\sum_{k=\eta(t)}^{n} \frac{\delta\gamma_n(t)^\top\gamma_k(t)L(t-,T_k)}{1+\delta L(t-,T_k)}\ \mathrm{d}t. \quad \text{[Th4.2.g]}\end{aligned}$$

The compensated jumps of $\mathrm{d}L(t,T_n)$ take the following form

$$\begin{aligned}&\frac{1+\delta L(t-,T_n)}{\delta}\int_{\mathbb{R}^r} \left(\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})}-1\right)\ (\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x)\\ &\qquad = \int_{\mathbb{R}^r} \frac{1+\delta L(t-,T_n)}{\delta}\frac{\delta H_n(t,x)L(t-,T_n)}{1+\delta L(t-,T_n)}\ (\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x)\\ &\qquad = \int_{\mathbb{R}^r} L(t-,T_n)H_n(t,x)\,(\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x). \quad \text{[Th4.2.h]}\end{aligned}$$

Now we simplify the integral with respect to the compensator

$$\frac{1+\delta L(t-,T_n)}{\delta}\int_{\mathbb{R}^r} \left(\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})}-1+\mathrm{e}^{D(t,x,T_{n+1})}-\mathrm{e}^{D(t,x,T_n)}\right)\ \nu^*(\mathrm{d}t,\mathrm{d}x).$$

For the first part of the integral we can use the calculation above. For the second part, we consider the expression

$$\begin{aligned}&\mathrm{e}^{D(t,x,T_{n+1})}-\mathrm{e}^{D(t,x,T_n)} = \left(1-\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})}\right)\mathrm{e}^{D(t,x,T_{n+1})}\\ &\qquad = -\frac{\delta}{1+\delta L(t-,T_n)}L(t-,T_n)H_n(t,x)\prod_{k=\eta(t)}^{n}\frac{1+\delta L(t-,T_k)}{1+\delta L(t-,T_k)(1+H_k(t,x))}. \quad \text{[Th4.2.i]}\end{aligned}$$

The last equation follows from [Th4.2.d] and [Th4.2.e].

Hence

$$\frac{1+\delta L(t-,T_n)}{\delta}\int\limits_{\mathbb{R}^r}\left(\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})}-1+\mathrm{e}^{D(t,x,T_{n+1})}-\mathrm{e}^{D(t,x,T_n)}\right)\nu^*(\mathrm{d}t,\mathrm{d}x)$$

$$=\frac{1+\delta L(t-,T_n)}{\delta}\int\limits_{\mathbb{R}^r}\frac{\delta}{1+\delta L(t-,T_n)}L(t-,T_n)H_n(t,x)$$

$$\times\left(1-\prod_{k=\eta(t)}^{n}\frac{1+\delta L(t-,T_k)}{1+\delta L(t-,T_k)(1+H_k(t,x))}\right)\nu^*(\mathrm{d}t,\mathrm{d}x)$$

$$=L(t-,T_n)\int\limits_{\mathbb{R}^r}H_n(t,x)\left(1-\prod_{k=\eta(t)}^{n}\frac{1+\delta L(t-,T_k)}{1+\delta L(t-,T_k)(1+H_k(t,x))}\right)\nu^*(\mathrm{d}t,\mathrm{d}x).$$

[Th4.2.j]

Now we only need to combine the equations [Th4.2.f], [Th4.2.g], [Th4.2.h] and [Th4.2.j] to get equation (4.10) where the drift is given by equation (4.11). □

4.1.2 The Lévy setting

We assume that the bond prices are given as in Eberlein and Raible (1999), i.e. the instantaneous forward rate process is modeled under the martingale measure $\mathbb{P}^*$ by

Assumption ($\mathbb{L}$): For any fixed maturity $T \le T^*$ the default-free instantaneous forward rate $f(t,T)$ satisfies

$$\mathrm{d}f(t,T)=\kappa'(\Sigma(t,T))\partial_2\Sigma(t,T)\,\mathrm{d}t-\partial_2\Sigma(t,T)\,\mathrm{d}L_t, \tag{4.12}$$

where ∂_2 denotes the derivation operator with respect to the second argument; κ is the logarithm of the moment generating function of L_1 and Σ is defined on $\Delta=\{(s,T)\in\mathbb{R}_+\times\mathbb{R}_+\,|\,0\le s\le T\le T^*\}$ and Σ is twice continuously differentiable in s and in T. In addition, we assume that $\Sigma(s,T)>0$ for all $(s,T)\in\Delta$ with $s\ne T$, and $\Sigma(s,T)=0$ if $s=T$.

For simplicity of notation we concentrate here on the one-dimensional case, but the results can be obtained analogously for the multi-dimensional case.

An equivalent representation of $\mathrm{d}f(t,T)$ can be obtained by using the canonical decomposition of L (see e.g. Jacod and Shiryaev (1987, II.2.38)), namely

$$L_t=bt+\sqrt{c}W_t+\int\limits_0^t\int\limits_{\mathbb{R}}x\,(\mu^L-\nu^L)(\mathrm{d}s,\mathrm{d}x), \tag{4.13}$$

where $b \in \mathbb{R}, c > 0, W$ is a standard Brownian motion and μ^L is the random measure of jumps, and ν^L is its compensator, which can be written as $\nu^L(\mathrm{d}t, \mathrm{d}x) = \nu(\mathrm{d}x)\,\mathrm{d}t$, where ν is the Lévy measure of the law of L_1. Note that we assume implicitly that ν satisfies the following integrability condition $\int_{\{|x|>1\}} \exp(ux)\,\nu(\mathrm{d}x) < \infty$, for $-M \leq u \leq M$, where M is a positive constant.

The short term interest rate is given by $r(t) = f(t,t), t \in [0, T^*]$.

The bond price is then given by

$$\begin{aligned} B(t,T) &= B(0,T)\exp\left(\int_0^t (r(s) - \kappa(\Sigma(s,T)))\ \mathrm{d}s + \int_0^t \Sigma(s,T)\ \mathrm{d}L_s\right) \\ &= B_t B(0,T)\exp\left(-\int_0^t \kappa(\Sigma(s,T))\ \mathrm{d}s + \int_0^t \Sigma(s,T)\ \mathrm{d}L_s\right). \end{aligned}$$

The forward process has the following form

$$\begin{aligned} F_B(t,T,T^*) = \frac{B(0,T)}{B(0,T^*)}\exp\Bigg(&\int_0^t (\kappa(\Sigma(s,T^*)) - \kappa(\Sigma(s,T)))\ \mathrm{d}s \\ &+ \int_0^t (\Sigma(s,T) - \Sigma(s,T^*))\ \mathrm{d}L_s\Bigg) \end{aligned} \tag{4.14}$$

and for $T \leq T^* - \delta$ we have

$$F_B(t,T,T+\delta) = \frac{B(t,T)}{B(t,T+\delta)}. \tag{4.15}$$

Note that $F_B(t,T,T+\delta)^{-1}$ is the forward price at time t of the $T+\delta$-maturity zero-coupon bond for the settlement date T.

One of the basic ideas of the Libor modeling approach is to construct a measure $\mathbb{P}_{T+\delta}$ such that $(L(t,T))_{0\leq t\leq T}$ follows a $\mathbb{P}_{T+\delta}$-martingale. This measure is denoted as the forward (martingale) measure associated with the date $T+\delta$.

By equation (4.7) $L(.,T)$ is a martingale under the measure $\mathbb{P}_{T+\delta}$ if and only if $F_B(.,T,T+\delta)$ is a $\mathbb{P}_{T+\delta}$-martingale. The next theorem gives the explicit construction of the forward measure associated with the maturity T^*.

Theorem 4.3 *The forward measure associated with the date T^* is given by*

$$\frac{\mathrm{d}\mathbb{P}_{T^*}}{\mathrm{d}\mathbb{P}^*} = \exp\left(\int_0^{T^*} \Sigma(s,T^*)\ \mathrm{d}L_s - \int_0^{T^*} \kappa(\Sigma(s,T^*))\ \mathrm{d}s\right). \tag{4.16}$$

Proof: We set $\eta_t := \frac{\mathrm{d}\mathbb{P}_{T^*}}{\mathrm{d}\mathbb{P}^*}|_{\mathcal{F}_t} = \exp\left(\int\limits_0^t \Sigma(s,T^*)\ \mathrm{d}L_s - \int\limits_0^t \kappa(\Sigma(s,T^*))\ \mathrm{d}s\right)$.

Then

$$
\begin{aligned}
F_B(t,T,T^*) &= \frac{B(0,T)}{B(0,T^*)} \exp\left(-\int\limits_0^t \kappa(\Sigma(s,T))\ \mathrm{d}s + \int\limits_0^t \Sigma(s,T)\ \mathrm{d}L_s\right) \\
&\qquad\qquad \times \exp\left(\int\limits_0^t \kappa(\Sigma(s,T^*))\ \mathrm{d}s - \int\limits_0^t \Sigma(s,T^*)\ \mathrm{d}L_s\right) \\
&= \frac{B(0,T)}{B(0,T^*)} \exp\left(-\int\limits_0^t \kappa(\Sigma(s,T))\ \mathrm{d}s + \int\limits_0^t \Sigma(s,T)\ \mathrm{d}L_s\right)\frac{1}{\eta_t} \\
&= \frac{1}{B(0,T^*)}\frac{B(t,T)}{B_t}\frac{1}{\eta_t}.
\end{aligned}
$$

Since $\left(\frac{B(t,T)}{B(0,T^*)}B_t^{-1}\right)_{0\le t\le T^*}$ is a $\mathbb{P}^*$-martingale it follows by Proposition III.3.8 in Jacod and Shiryaev (1987) that $F_B(.,T,T^*)$ is a $\mathbb{P}_{T^*}$-martingale. □

Note that

$$
\frac{\mathrm{d}\mathbb{P}_{T^*}}{\mathrm{d}\mathbb{P}^*} = \frac{1}{B_{T^*}B(0,T^*)} = \eta_{T^*} \text{ and } \frac{\mathrm{d}\mathbb{P}_{T^*}}{\mathrm{d}\mathbb{P}^*}|_{\mathcal{F}_t} = \frac{B(t,T^*)}{B_tB(0,T^*)} \qquad \mathbb{P}^*\text{-a.s.} \tag{4.17}
$$

The density process $\eta_t = \frac{\mathrm{d}\mathbb{P}_{T^*}}{\mathrm{d}\mathbb{P}^*}|_{\mathcal{F}_t}$ can be presented in the usual form

$$
\begin{aligned}
\eta_t = \exp\Bigg(&\int\limits_0^t \phi(s)\ \mathrm{d}W_s - \frac{1}{2}\int\limits_0^t \phi^2(s)\ \mathrm{d}s + \int\limits_0^t\int\limits_{\mathbb{R}} \left(\widetilde{Y}(s,x)-1\right)\ \left(\mu^L-\nu^L\right)(\mathrm{d}s,\mathrm{d}x) \\
&- \int\limits_0^t\int\limits_{\mathbb{R}} \left(\widetilde{Y}(s,x)-1-\ln(\widetilde{Y}(s,x))\right)\mu^L(\mathrm{d}s,\mathrm{d}x)\Bigg),
\end{aligned} \tag{4.18}
$$

where $\phi(s) = \sqrt{c}\Sigma(s,T^*)$ and $\widetilde{Y}(s,x) = \exp(\Sigma(s,T^*)x)$.

In the following we will say that a change of measure is determined by two functions (or predictable processes) ϕ and $\widetilde{Y}$ if the density process has the same representation as in equation (4.18).

Note that whenever $\widetilde{Y}$ depends on $s \in \mathbb{R}_+$ the $\mathbb{P}^*$-Lévy process L is not a Lévy process under the forward martingale measure $\mathbb{P}_{T^*}$. But due to the fact that ϕ and $\widetilde{Y}$ are deterministic, the semimartingale characteristics with respect to $\mathbb{P}_{T^*}$ remain deterministic, and thus L has independent increments under the forward martingale measure. This is an immediate consequence of Girsanov's theorem for semimartingales, see Jacod and Shiryaev (1987, III.3.24 and II.4.15).

As usual $W_t^{T^*} := W_t - \int_0^t \phi(s)\,\mathrm{d}s$ is a standard Brownian motion under $\mathbb{P}_{T^*}$ and $\nu^{T^*,L} := \widetilde{Y}\nu^L$ is the $\mathbb{P}_{T^*}$-compensator of μ^L.

The martingale property of the forward process under the forward martingale measure can also be seen in the following proposition.

Proposition 4.4 *The $\mathbb{P}_{T^*}$-dynamics of $F_B(.,T,T^*)$ is given by*

$$\frac{1}{F_B(t-,T,T^*)}\,\mathrm{d}F_B(t,T,T^*) = \sqrt{c}\Sigma(t,T,T^*)\,\mathrm{d}W_t^{T^*} + \int_{\mathbb{R}} \left(\mathrm{e}^{x\Sigma(t,T,T^*)} - 1\right)\left(\mu^L - \nu^{T^*,L}\right)(\mathrm{d}t,\mathrm{d}x),$$

where $\Sigma(t,T,T^*) := \Sigma(t,T) - \Sigma(t,T^*)$.

Proof: The relationship between the functions A, D, and S of Section 4.1.1 and the coefficients of the Lévy setting is given by the following equations $A(t,T) = \Sigma(t,T)b - \kappa(\Sigma(t,T))$, $D(t,x,T) = \Sigma(t,T)x$, and $S(t,T) = \Sigma(t,T)\sqrt{c}$, see the Proof of Lemma 6.1 in Raible (2000).

Using Theorem 4.1 we can derive the dynamics of the forward process under $\mathbb{P}^*$:

$$\begin{aligned}\frac{1}{F_B(t-,T,T^*)}\,\mathrm{d}F_B(t,T,T^*) =& -\sqrt{c}\Sigma(t,T,T^*)(\sqrt{c}\Sigma(t,T^*))\,\mathrm{d}t + \sqrt{c}\Sigma(t,T,T^*)\,\mathrm{d}W_t\\ &+ \int_{\mathbb{R}} \left(\mathrm{e}^{x\Sigma(t,T,T^*)} - 1\right)\left(\mu^L - \nu^L\right)(\mathrm{d}t,\mathrm{d}x)\\ &+ \int_{\mathbb{R}} \left(\mathrm{e}^{x\Sigma(t,T,T^*)} + \mathrm{e}^{x\Sigma(t,T^*)} - \mathrm{e}^{x\Sigma(t,T)} - 1\right)\nu^L(\mathrm{d}t,\mathrm{d}x).\end{aligned}$$

The first two expressions on the right-hand side of the last equation can be written as

$$\begin{aligned}&-\sqrt{c}\Sigma(t,T,T^*)(\sqrt{c}\Sigma(t,T^*))\,\mathrm{d}t + \sqrt{c}\Sigma(t,T,T^*)\,\mathrm{d}W_t\\ &\quad = \sqrt{c}\Sigma(t,T,T^*)\left(\mathrm{d}W_t - \sqrt{c}\Sigma(t,T^*)\,\mathrm{d}t\right) = \sqrt{c}\Sigma(t,T,T^*)\,\mathrm{d}W_t^{T^*}.\end{aligned}\quad \text{[Pr4.4.a]}$$

Now let us have a look at the terms related to the jumps of the forward process:

$$\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}-1\right)\left(\mu^L-\nu^L\right)(\mathrm{d}t,\mathrm{d}x)+\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}+\mathrm{e}^{x\Sigma(t,T^*)}-\mathrm{e}^{x\Sigma(t,T)}-1\right)\nu^L(\mathrm{d}t,\mathrm{d}x)$$
$$=\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}-1\right)\left(\mu^L-\nu^L\right)(\mathrm{d}t,\mathrm{d}x)+\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}-1\right)\left(1-\mathrm{e}^{x\Sigma(t,T^*)}\right)\nu^L(\mathrm{d}t,\mathrm{d}x)$$
$$=\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}-1\right)\left(\mu^L-\nu^L\right)(\mathrm{d}t,\mathrm{d}x)+\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}-1\right)\left(1-\widetilde{Y}(t,x)\right)\nu^L(\mathrm{d}t,\mathrm{d}x)$$
$$=\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}-1\right)\left(\mu^L-\nu^L\right)(\mathrm{d}t,\mathrm{d}x)+\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}-1\right)\left(\nu^L-\nu^{T^*,L}\right)(\mathrm{d}t,\mathrm{d}x)$$
$$=\int_{\mathbb{R}}\left(\mathrm{e}^{x\Sigma(t,T,T^*)}-1\right)\left(\mu^L-\nu^{T^*,L}\right)(\mathrm{d}t,\mathrm{d}x). \qquad \text{[Pr4.4.b]}$$

Combining [Pr4.4.a] and [Pr4.4.b] proves our assertion. □

4.1.3 Relationship to the Jamshidian approach

In this section we show how the Lévy term structure approach to Libor markets can be embedded in the general semimartingale approach of Jamshidian (1999). Furthermore we will derive the dynamics of the Libor rates under the forward martingale measure associated with T^*.

Note that the approach of Jamshidian (1999) is more general than our approach in the following two points. The discrete bond market is given by strictly positive semimartingales but is not specified any further. Even negative Libor rates are allowed. Furthermore, it is even possible that there exists no risk-neutral measure $\mathbb{P}^*$ within the bond market. It exists if a specific process, the state-price density, is a special semimartingale. For details we refer to Section 4 in Jamshidian (1999). Our interest in this chapter is to push further the market practice to price caps and floors by Black's formula to a formula which allows for more flexibility and our aim is not to develop a Libor market theory which is as general as possible.

We assume that the discrete-tenor setting is given by $0 < T_0 < T_1 < \ldots < T_{n+1} = T^*$, where $\delta = T_{i+1} - T_i$. According to this we consider the Libor rates $L(t,T_i) = \frac{1}{\delta}(F_B(t,T_i,T_{i+1})-1)$. With the help of the $\mathbb{P}^*$-dynamics of $F_B(.,T,T^*)$ which is given in the proof of Proposition 4.4 we can rewrite the Libor rate $L(t,T_i)$ as follows

$$
\begin{aligned}
L(t,T_i) = {} & \frac{1}{\delta}\left(F_B(0,T_i,T_{i+1})-1\right) \\
& + \frac{1}{\delta}\int_0^t cF_B(s-,T_i,T_{i+1})(\Sigma^2(s,T_{i+1})-\Sigma(s,T_i)\Sigma(s,T_{i+1}))\,\mathrm{d}s \\
& + \frac{1}{\delta}\int_0^t\int_{\mathbb{R}} F_B(s-,T_i,T_{i+1})\Big(\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})}+\mathrm{e}^{x\Sigma(s,T_{i+1})}-\mathrm{e}^{x\Sigma(s,T_i)}-1\Big)\nu(\mathrm{d}x)\,\mathrm{d}s \\
& + \int_0^t F_B(s-,T_i,T_{i+1})\frac{\sqrt{c}}{\delta}\Sigma(s,T_i,T_{i+1})\,\mathrm{d}W_s \\
& + \int_0^t\int_{\mathbb{R}} F_B(s-,T_i,T_{i+1})\frac{1}{\delta}\left(\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})}-1\right)(\mu^L-\nu^L)(\mathrm{d}s,\mathrm{d}x).
\end{aligned}
\tag{4.19}
$$

In order to improve the readability we will use the superscript ^{J} whenever we use the notation of Jamshidian (1999). For example in Jamshidian (1999) $L^J_{i,t}$ denotes the Libor rate $L(t,T_i)$. The connection to the semimartingale Libor approach is clarified by the next equations:

$$
\begin{aligned}
Q^J &= \mathbb{P}_{T^*}, \\
L_i^J(0) &= \frac{1}{\delta}\left(F_B(0,T_i,T_{i+1})-1\right), \\
A_i^J(t) &= \frac{1}{\delta}\int_0^t cF_B(s-,T_i,T_{i+1})(\Sigma^2(s,T_{i+1})-\Sigma(s,T_i)\Sigma(s,T_{i+1}))\,\mathrm{d}s \\
&\quad + \frac{1}{\delta}\int_0^t\int_{\mathbb{R}} F_B(s-,T_i,T_{i+1})\left(\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})}+\mathrm{e}^{x\Sigma(s,T_{i+1})}-\mathrm{e}^{x\Sigma(s,T_i)}-1\right)\nu(\mathrm{d}x)\,\mathrm{d}s, \\
\beta_i^J(t) &= F_B(t-,T_i,T_{i+1})\frac{\sqrt{c}}{\delta}\Sigma(t,T_i,T_{i+1}), \\
\phi_i^J(x,t) &= F_B(t-,T_i,T_{i+1})\frac{1}{\delta}\left(\mathrm{e}^{x\Sigma(t,T_i,T_{i+1})}-1\right) > -1, \\
\mu^J &= \mu^L, \qquad \nu^J = \nu^L.
\end{aligned}
$$

Therefore, the Libor rate $L_i^J(t) = L(t,T_i)$ has the following form

$$
L_i^J(t) = L_i^J(0) + A_i^J(t) + \int_0^t \beta_i^J(s)\,\mathrm{d}W_s + \int_0^t\int_{\mathbb{R}} \phi_i^J(x,s)\,(\mu^J-\nu^J)(\mathrm{d}s,\mathrm{d}x).
$$

The change of measure is described by

$$\alpha^J(s) = \phi(s) = \sqrt{c}\Sigma(s, T_{n+1})$$

$$\psi^J(s,x) = \widetilde{Y}(s,x) - 1 = \mathrm{e}^{x\Sigma(s,T_{n+1})} - 1 > -1$$

$$W_t^{Q^J} = W_t - \int_0^t \alpha^J(s)\,\mathrm{d}s = W_t^{T^*},$$

$$\nu^{Q^J}(\mathrm{d}s,\mathrm{d}x) = (1+\psi^J(s,x))\,\nu^J(\mathrm{d}s,\mathrm{d}x) = \widetilde{Y}(s,x)\nu^L(\mathrm{d}s,\mathrm{d}x) = \nu^{T^*,L}(\mathrm{d}s,\mathrm{d}x).$$

We want to mention that the conditions of Theorem 6 in Jamshidian (1999) are satisfied due to Assumption ($\mathbb{L}$) and that by applying equation (35) in Jamshidian (1999) we get the $\mathbb{P}_{T^*}$-dynamics of the forward process, cf. Proposition 4.4,

$$\begin{aligned}
\mathcal{L}\left(\frac{F_B(.,T_i,T^*)}{F_B(0,T_i,T^*)}\right)_t &= \int_0^t \sum_{j=i}^n \frac{\delta\beta_j^J(s)}{1+\delta L(s-,T_j)}\,\mathrm{d}W_s^{Q^J} \\
&\quad + \int_0^t\int_{\mathbb{R}} \left(\prod_{j=1}^n \left(1+\frac{\delta\phi_j^J(s,x)}{1+\delta L(s-,T_j)}\right) - 1\right)(\mu^J - \nu^{Q^J})(\mathrm{d}s,\mathrm{d}x) \\
&= \int_0^t \sqrt{c}\Sigma(s,T_i,T^*)\,\mathrm{d}W_s^{T^*} \\
&\quad + \int_0^t\int_{\mathbb{R}} \left(\mathrm{e}^{x\Sigma(s,T_i,T^*)} - 1\right)(\mu^L - \nu^{T^*,L})(\mathrm{d}s,\mathrm{d}x).
\end{aligned}$$

Proposition 4.5 *The $\mathbb{P}_{T^*}$-dynamics of $L(.,T_i)$ is given by*

$$\begin{aligned}
\mathrm{d}L(t,T_i) = &- \sum_{j=i+1}^n c\left(\frac{1}{\delta} + L(t-,T_j)\right)\Sigma^2(t,T_j,T_{j+1})\,\mathrm{d}t \\
&- \left(\frac{1}{\delta} + L(t-,T_i)\right)\int_{\mathbb{R}} \left(\mathrm{e}^{x\Sigma(t,T_i,T_{i+1})} - 1\right)\mathrm{e}^{x\Sigma(t,T_{i+1},T_{n+1})}\,\nu^{T^*,L}(\mathrm{d}t,\mathrm{d}x) \\
&+ \sqrt{c}\left(\frac{1}{\delta} + L(t-,T_i)\right)\Sigma(t,T_i,T_{i+1})\,\mathrm{d}W_t^{T^*} \\
&+ \left(\frac{1}{\delta} + L(t-,T_i)\right)\int_{\mathbb{R}} \left(\mathrm{e}^{x\Sigma(t,T_i,T_{i+1})} - 1\right)\left(\mu^L - \nu^{T^*,L}\right)(\mathrm{d}t,\mathrm{d}x).
\end{aligned}$$

Proof: Equation (32) in Jamshidian (1999) is given by

$$L^J_{i,t} = L^J_i(0) - \int_0^t \sum_{j=i+1}^n \frac{\delta_j \beta^J_{j,s} (\beta^J_{j,s})^\top}{1+\delta_j L^J_{j,s-}}\, \mathrm{d}t$$

$$- \int_0^t \int_{\mathbb{R}} \left(\phi^J_i(x,s) \left(\prod_{j=i+1}^n \left(1 + \frac{\delta_j \phi_j(x,s)}{1+\delta_j L^J_{j,s-}} \right) - 1 \right) \right) \nu^{J,Q^J}(\mathrm{d}s,\mathrm{d}x)$$

$$+ \int_0^t \beta^J_{i,s}\, \mathrm{d}W^{Q^J}_s + \int_0^t \int_{\mathbb{R}} \phi^J_i(s,x)\, (\mu^J - \nu^{J,Q^J})(\mathrm{d}s,\mathrm{d}x).$$

Therefore in our notation

$$L(t,T_i) = L(0,T_i) - \int_0^t \sum_{j=i+1}^n \frac{\delta F_B^2(s-,T_j,T_{j+1}) \frac{c}{\delta^2} \Sigma^2(s,T_j,T_{j+1})}{1+\delta L(s-,T_j)}\, \mathrm{d}s$$

$$- \int_0^t \int_{\mathbb{R}} \Bigg(F_B(s-,T_i,T_{i+1}) \frac{1}{\delta} \left(\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})} \right)$$

$$\times \prod_{j=i+1}^n \left(1 + \frac{\delta F_B(s-,T_j,T_{j+1}) \frac{1}{\delta} \left(\mathrm{e}^{x\Sigma(s,T_j,T_{j+1})} - 1 \right)}{1+\delta L(s-,T_j)} \right) \Bigg) \nu^{T^*,L}(\mathrm{d}s,\mathrm{d}x)$$

$$+ \int_0^t F_B(s-,T_i,T_{i+1}) \frac{\sqrt{c}}{\delta} \Sigma(s,T_i,T_{i+1})\, \mathrm{d}W^{T^*}_s$$

$$+ \int_0^t \int_{\mathbb{R}} F_B(s-,T_i,T_{i+1}) \frac{1}{\delta} \left(\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})} - 1 \right) \left(\mu^L - \nu^{T^*,L} \right) (\mathrm{d}s,\mathrm{d}x).$$

Because

$$1 + \delta L(s-,T_j) = F_B(s-,T_j,T_{j+1})$$

and

$$\sum_{j=i+1}^n \Sigma(s,T_j,T_{j+1}) = \sum_{j=i+1}^n (\Sigma(s,T_j) - \Sigma(s,T_{j+1})) = \Sigma(s,T_{i+1}) - \Sigma(s,T_{n+1})$$

$$= \Sigma(s,T_{i+1},T_{n+1}),$$

we get

$$
\begin{aligned}
L(t,T_i) = L(0,T_i) &- \sum_{j=i+1}^{n} \frac{cF_B(s-,T_j,T_{j+1})}{\delta}\Sigma^2(s,T_j,T_{j+1})\ \mathrm{d}s \\
&- \int_0^t\int_{\mathbb{R}} \frac{F_B(s-,T_i,T_{i+1})}{\delta}\left(\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})}-1\right)\mathrm{e}^{x\Sigma(s,T_{i+1},T_{n+1})}\nu^{T^*,L}(\mathrm{d}s,\mathrm{d}x) \\
&+ \int_0^t \frac{F_B(s-,T_i,T_{i+1})\sqrt{c}}{\delta}\Sigma(s,T_i,T_{i+1})\ \mathrm{d}W_t^{T^*} \\
&+ \int_0^t\int_{\mathbb{R}} \frac{F_B(s-,T_i,T_{i+1})}{\delta}\left(\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})}-1\right)\left(\mu^L-\nu^{T^*,L}\right)(\mathrm{d}s,\mathrm{d}x).
\end{aligned}
$$

Since $\frac{F_B(t,T_i,T_{i+1})}{\delta} = \frac{1}{\delta} + L(t,T_i)$ our statement is proved. □

The non-negativity of the Libor rate $L(.,T_i)$ can also be derived from Jamshidian (1999, Section 5.1). It is sufficient to check that $\gamma_i^J(x,s) := \frac{\phi_i^J(x,s)}{L_{i,s}^J} > -1$. In our case we have

$$
\begin{aligned}
\gamma_i^J(x,s) &= \frac{\frac{1}{\delta}F_B(s-,T_i,T_{i+1})\left(\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})}-1\right)}{\frac{1}{\delta}\left(F_B(s-,T_i,T_{i+1})-1\right)} \\
&= \frac{\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})}-1}{1-\frac{1}{F_B(s-,T_i,T_{i+1})}} \quad > \quad -1,
\end{aligned}
$$

since $\mathrm{e}^{x\Sigma(s,T_i,T_{i+1})} > 0$ and $F_B(.,T_i,T_{i+1}) > 1$.

4.2 A forward price model

Instead of embedding forward rates (or Libor rates) in an existing bond price model one can model the forward rates (or Libor rates) directly. Before we start with the Lévy Libor model we present a Lévy model for the forward processes. As in the previous section we assume a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}_{T^*}, (\mathcal{F}_t)_{0\le t\le T^*})$ and a family $(B(t,T))_{(t,T)\in\Delta}$ to be given. The bond prices are assumed to be positive semimartingales, decreasing in the second variable and they satisfy $B(T,T)=1$ for all $T\in[0,T^*]$. The dynamics of $B(.,T)$ remains unspecified in this section.

Concerning the modeling of the forward processes we make the following assumption.

(FPM.1): For fixed maturity date $T \leq T^*$ the dynamics of the forward process is given by

$$\mathrm{d}F_B(t,T,T^*) = F_B(t-,T,T^*)\ \mathrm{d}L_t^{T,T^*},$$

with initial condition

$$F_B(0,T,T^*) = \frac{B(0,T)}{B(0,T^*)},$$

where

$$L_t^{T,T^*} = L_0^{T,T^*} + \int_0^t \alpha(s,T,T^*)\ \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}} \beta(x,s,T,T^*)\ \left(\mu^L - \nu^{T^*,L}\right)(\mathrm{d}u,\mathrm{d}x),$$

where α and $\beta > -1$ are deterministic functions such that the integrals are well-defined and with the additional condition that $\alpha(t,T^*,T^*) = \beta(x,t,T^*,T^*) = 0$ for any $t \in [0,T^*]$, and $x \in \mathbb{R}$. Under $\mathbb{P}_{T^*}$ W^{T^*} is a standard Brownian motion, μ^L is the random measure of jumps of a Lévy process with respect to $\mathbb{P}_{T^*}$ and $\nu^{T^*,L}(\mathrm{d}t,\mathrm{d}x) = \nu^{T^*}(\mathrm{d}x)\ \mathrm{d}t$ is the $\mathbb{P}_{T^*}$-compensator of μ^L which satisfies the following integrability condition $\int_{\{|x|>1\}} \exp(ux)\, \nu^{T^*}(\mathrm{d}x) < \infty$, for $-M \leq u \leq M$, for some $M > 0$.

(FPM.2): The relationship between the bond price and the forward process is given by

$$F_B(t,T,T^*) = \frac{B(t,T)}{B(t,T^*)}.$$

Assumption (FPM.1) is equivalent to $F_B(.,T,T^*) = F_B(0,T,T^*)\mathcal{E}(L^{T,T^*})$ and hence by Jacod and Shiryaev (1987, I.4.64)

$$\begin{aligned} F_B(t,T,T^*) = F_B(0,T,T^*) \exp \Bigg(& \int_0^t \alpha(s,T,T^*)\ \mathrm{d}W_s^{T^*} - \frac{1}{2}\int_0^t \alpha^2(s,T,T^*)\ \mathrm{d}s \\ & + \int_0^t\int_{\mathbb{R}} \beta(x,s,T,T^*)\,(\mu^L - \nu^{T^*,L})(\mathrm{d}s,\mathrm{d}x) \\ & + \int_0^t\int_{\mathbb{R}} \left(\ln(1+\beta(x,s,T,T^*)) - \beta(x,s,T,T^*)\right) \mu^L(\mathrm{d}s,\mathrm{d}x) \Bigg). \end{aligned}$$

By construction $F_B(.,T,T^*)$ is bounded by 1 and $F_B(.,T,T^*)$ is a local $\mathbb{P}_{T^*}$-martingale. For simplicity we can assume that it is a martingale. The martingale property of the forward process implies that Assumption (FPM.2) is equivalent to

$$B(t,T) = \mathbb{E}_{\mathbb{P}_{T^*}}\left[\frac{B(t,T^*)}{B(T,T^*)} \,|\, \mathcal{F}_t\right],$$

because

$$\frac{B(t,T)}{B(t,T^*)} = F_B(t,T,T^*) = \mathbb{E}_{\mathbb{P}_{T^*}}\left[\frac{1}{B(T,T^*)} \,|\, \mathcal{F}_t\right].$$

Note that the forward modeling approach can be related to the last section by setting, see Proposition 4.4,

$$\alpha(t,T,T^*) = \sqrt{c}\Sigma(t,T,T^*),$$
$$\beta(x,t,T,T^*) = \exp\left(\Sigma(t,T,T^*)x\right) - 1.$$

Definition 4.6 *The forward process $F_B(t,T,U)$, $t \in [0, T \wedge U], 0 \le T, U \le T^*$, is defined by*

$$F_B(t,T,U) = \frac{F_B(t,T,T^*)}{F_B(t,U,T^*)}.$$

The forward process $F_B(t,T,U)$ can be written as

$$\begin{aligned} F_B(t,T,U) = F_B(0,T,U)\exp\Bigg(& \int_0^t \alpha(s,T,U)\, \mathrm{d}W_s^{T^*} \\ & - \frac{1}{2}\int_0^t \left(\alpha^2(s,T,T^*) - \alpha^2(s,U,T^*)\right)\, \mathrm{d}s \\ & + \int_0^t\int_{\mathbb{R}} \beta(x,s,T,U)\,(\mu^L - \nu^{T^*,L})(\mathrm{d}s,\mathrm{d}x) \\ & + \int_0^t\int_{\mathbb{R}} \left(\ln(\beta'(x,s,T,U)) - \beta(x,s,T,U)\right)\, \mu^L(\mathrm{d}s,\mathrm{d}x)\Bigg), \end{aligned}$$

where for abbreviation purposes we have set

$$\alpha(s,T,U) := \alpha(s,T,T^*) - \alpha(s,U,T^*),$$

$$\beta(x,s,T,U) := \beta(x,s,T,T^*) - \beta(x,s,U,T^*),$$

$$\beta'(x,s,T,U) := \frac{1+\beta(x,s,T,T^*)}{1+\beta(x,s,U,T^*)}.$$

Note that $\beta'(x,s,T,T^*) = 1+\beta(x,s,T,T^*)$.

Remark: In case we are starting with the Lévy HJM framework of Section 4.1.2 we get

$$\alpha(s,T,U) = \sqrt{c}\Sigma(s,T,U),$$

$$\beta(x,s,T,U) = (\mathrm{e}^{x\Sigma(s,T,T^*)} - 1) - (\mathrm{e}^{x\Sigma(s,U,T^*)} - 1) = \mathrm{e}^{x\Sigma(s,T,T^*)} - \mathrm{e}^{x\Sigma(s,U,T^*)},$$

$$\beta'(x,s,T,U) = \mathrm{e}^{x\Sigma(s,T,U)},$$

where $\Sigma(s,T,U) := \Sigma(s,T,T^*) - \Sigma(s,U,T^*) = \Sigma(s,T) - \Sigma(s,U)$.

The $\mathbb{P}_{T^*}$-dynamics of $F_B(.,T,U)$ can again be derived by Itô's formula:

$$\mathrm{d}F_B(t,T,U) = F_B(t-,T,U)\Bigg(\alpha(t,T,U)(\mathrm{d}W_t^{T^*} - \alpha(t,U,T^*)\,\mathrm{d}t) + \int_{\mathbb{R}} \beta(x,t,T,U)\,(\mu^L - \nu^{T^*,L})(\mathrm{d}t,\mathrm{d}x) + \int_{\mathbb{R}} (\beta'(x,t,T,U) - \beta(x,t,T,U) - 1)\,\mu^L(\mathrm{d}t,\mathrm{d}x)\Bigg). \tag{4.20}$$

The functions $\phi^U(t) := \alpha(t,U,T^*)$ and $Y^U(t,x) := \beta(x,t,U,T^*) + 1 = \beta'(x,t,U,T^*)$ determine a measure $\mathbb{P}_U \sim \mathbb{P}_{T^*}$ with the property that

$$W_t^U := W_t^{T^*} - \int_0^t \phi^U(s)\,\mathrm{d}s$$

is a standard Brownian motion under $\mathbb{P}_U$ and

$$\nu^{U,L}(\mathrm{d}t,\mathrm{d}x) := Y^U(t,x)\nu^{T^*,L}(\mathrm{d}t,\mathrm{d}x)$$

is the $\mathbb{P}_U$-compensator of μ^L.

In an analogous way to Proposition 4.4 it can be shown that

$$\mathrm{d}F_B(t,T,U) = F_B(t-,T,U)\Bigg(\alpha(t,T,U)\,\mathrm{d}W_t^U + \int_{\mathbb{R}} (\beta'(x,t,T,U)-1)\,(\mu^L-\nu^{U,L})(\mathrm{d}t,\mathrm{d}x)\Bigg). \quad (4.21)$$

4.3 The discrete tenor Lévy Libor rate model

In Section 4.1 and 4.2 we have discussed two frameworks, where the Libor rates can be derived from the bond prices or the forward prices. In this section the Libor rates itself are subject to modeling.

We make the following assumptions concerning the dynamics of the Libor rates. Let $T \le T^* - \delta$ be fixed, where $\delta > 0$. Again we assume a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}_{T^*}, (\mathcal{F})_{0\le t\le T^*})$ and let μ^L be the jump measure of a Lévy process L. Furthermore the initial bond prices $B(0,T), 0 \le T \le T^*$, are given and strictly decreasing in the second variable.

(LR.1): There exists a function $\lambda^1(.;.,T) : \Omega \times [0,T] \to \mathbb{R}_+$, and a function $\lambda^2(.;.,.,T) : \Omega \times \mathbb{R} \times [0,T] \to \mathbb{R}_+$ both predictable with $\lambda^2 \in G_{\mathrm{loc}}(\mu^L)$, such that

$$\mathrm{d}L(t,T) = L(t-,T)\Bigg(\lambda^1(t,T)\,\mathrm{d}W_t^{T+\delta} + \int_{\mathbb{R}} \lambda^2(x,t,T)\,(\mu^L - \nu^{T+\delta,L})(\mathrm{d}t,\mathrm{d}x)\Bigg), \quad (4.22)$$

where $W^{T+\delta}$ is a $\mathbb{P}_{T+\delta}$-standard Brownian motion and $\nu^{T+\delta,L}(\mathrm{d}t,\mathrm{d}x) = \nu^{T+\delta}(\mathrm{d}x)\,\mathrm{d}t$ is the $\mathbb{P}_{T+\delta}$-compensator of μ^L. Furthermore we assume that $\nu^{T+\delta,L}$ satisfies the following integrability condition $\int_{\{|x|>1\}} \exp(ux)\,\nu^{T+\delta}(\mathrm{d}x) < \infty$, for $-M \le u \le M$, where M is a positive constant. To guarantee that the Libor rate is positive we assume that $\lambda^2(\triangle L_t, t, T)\mathbb{1}_{\{\triangle L_t \neq 0\}} > -1$.

(LR.2) $\lambda^1(.;.,T)$ and $\lambda^2(.;.,.,T)$ satisfy the following integrability conditions

$$\int_0^T (\lambda^1(t,T))^2 \, \mathrm{d}t < \infty, \quad \text{a.s.}$$

$$\int_0^T \int_{\mathbb{R}} (\sqrt{\lambda^2(x,t,T)+1} - 1)^2 \, \nu^{T+\delta,L}(\mathrm{d}t,\mathrm{d}x) < \infty, \quad \text{a.s.}$$

(LR.3): The initial condition in equation (4.22) is given by

$$L(0,T) = \frac{1}{\delta}\left(\frac{B(0,T)}{B(0,T+\delta)} - 1\right).$$

The motivation for Assumption (LR.1) is the dynamics of the Libor rates within the forward rate approach in Section 4.2. From equation (4.7) and (4.21) we deduce

$$\begin{aligned}
\mathrm{d}L(t,T) &= \frac{1}{\delta}\,\mathrm{d}F_B(t,T,T+\delta) \qquad (4.23)\\
&= \frac{1}{\delta} F_B(t-,T,T+\delta)\Bigg(\alpha(t,T,T+\delta)\,\mathrm{d}W_t^{T+\delta}\\
&\qquad + \int_{\mathbb{R}} (\beta'(x,t,T,T+\delta)-1)(\mu^L - \nu^{T+\delta,L})(\mathrm{d}t,\mathrm{d}x)\Bigg)\\
&= \frac{1}{\delta}\,(1+\delta L(t-,T))\Bigg(\alpha(t,T,T+\delta)\,\mathrm{d}W_t^{T+\delta}\\
&\qquad + \int_{\mathbb{R}} (\beta'(x,t,T,T+\delta)-1)(\mu^L - \nu^{T+\delta,L})(\mathrm{d}t,\mathrm{d}x)\Bigg)\\
&= L(t-,T)\left(\lambda^1(t,T)\,\mathrm{d}W_t^{T+\delta} + \int_{\mathbb{R}} \lambda^2(x,t,T)\,(\mu^L - \nu^{T+\delta,L})(\mathrm{d}t,\mathrm{d}x)\right),
\end{aligned}$$

where

$$\lambda^1(t,T) = \frac{1+\delta L(t-,T)}{\delta L(t-,T)}\,\alpha(t,T,T+\delta), \qquad (4.24)$$

$$\lambda^2(x,t,T) = \frac{1+\delta L(t-,T)}{\delta L(t-,T)}\,(\beta'(x,t,T,T+\delta)-1). \qquad (4.25)$$

4.3.1 Construction of the forward measures

In this section we follow Musiela and Rutkowski (1997, 1998) to construct an arbitrage-free bond market which is based on the Lévy Libor model.

For simplicity we assume that $T_0^* := T^*$ is a multiple of δ, i.e. there exists $M \in \mathbb{N}$ such that $T_0^* = M\delta$. For the ease of notation, we shall assume that the discrete-tenor structure is given by $T_i^* := T_0^* - i\delta, \quad i \in \{1, \dots, M-1\}$.

The construction is based on backward induction, and so we will start with the longest maturity T_1^*. According to (LR.1) we postulate that the dynamics of $L(., T_1^*)$ is given by

$$\mathrm{d}L(t, T_1^*) = L(t-, T_1^*) \left(\lambda^1(t, T_1^*)\, \mathrm{d}W_t^{T_0^*} + \int_{\mathbb{R}} \lambda^2(x, t, T_1^*)(\mu^L - \nu^{T_0^*, L})(\mathrm{d}t, \mathrm{d}x) \right),$$

with initial condition

$$L(0, T_1^*) = \frac{1}{\delta} \left(\frac{B(0, T_1^*)}{B(0, T_0^*)} - 1 \right).$$

Motivated by equation (4.24) and (4.25) we define

$$\begin{aligned} \alpha(t, T_1^*, T_0^*) &:= \frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)} \lambda^1(t, T_1^*), \\ \beta'(x, t, T_1^*, T_0^*) &:= \frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)} \lambda^2(x, t, T_1^*) + 1, \qquad \forall t \in [0, T_1^*]. \end{aligned}$$

Then under the measure $\mathbb{P}_{T_0^*}$ the forward rate $F_B(., T_1^*, T_0^*) = \delta L(., T_1^*) + 1$ has the following dynamics

$$\begin{aligned} \mathrm{d}F_B(t, T_1^*, T_0^*) = F_B(t-, T_1^*, T_0^*) \Bigg(& \alpha(t, T_1^*, T_0^*)\, \mathrm{d}W_t^{T_0^*} \\ & + \int_{\mathbb{R}} (\beta'(x, t, T_1^*, T_0^*) - 1) \left(\mu^L - \nu^{T_0^*, L}\right)(\mathrm{d}t, \mathrm{d}x) \Bigg), \end{aligned}$$

with initial condition $F_B(0, T_1^*, T_0^*) = B(0, T_1^*)/B(0, T_0^*)$. This can be seen by the same arguments as in the motivation for Assumption (LR.1), page 116.

Note that due to Assumption (LR.1) and (LR.2) $\alpha(., T_1^*, T_0^*)$ and $\beta'(., ., T_1^*, T_0^*)$ are predictable and satisfy the following integrability conditions

$$\int_0^{T_1^*} (\alpha(t, T_1^*, T_0^*))^2\, \mathrm{d}t \le \int_0^{T_1^*} (\lambda^1(t, T_1^*))^2\, \mathrm{d}t < \infty$$

and that

$$\int_0^{T_1^*}\int_{\mathbb{R}} \left(\sqrt{\beta'(x,t,T_1^*,T_0^*)}-1\right)^2 \nu^{T_0^*,L}(\mathrm{d}t,\mathrm{d}x)$$
$$\leq \int_0^{T_1^*}\int_{\mathbb{R}} \left(\sqrt{\lambda^2(x,t,T_1^*)+1}-1\right)^2 \nu^{T_0^*,L}(\mathrm{d}t,\mathrm{d}x)$$
$$< \infty.$$

Then $\alpha(.,T_1^*,T_0^*)$ and $\beta'(.,.,T_1^*,T_0^*)$ define a measure $\mathbb{P}_{T_1^*}$ by

$$\frac{\mathrm{d}\mathbb{P}_{T_1^*}}{\mathrm{d}\mathbb{P}_{T_0^*}} = \mathcal{E}\left(\int_0^{\cdot} \alpha(s,T_1^*,T_0^*)\,\mathrm{d}W_t^{T_0^*} + \int_0^{\cdot}\int_{\mathbb{R}} (\beta'(x,s,T_1^*,T_0^*)-1)\left(\mu^L-\nu^{T_0^*,L}\right)(\mathrm{d}s,\mathrm{d}x)\right)_{T_1^*},$$

so that $W^{T_1^*} := W_t^{T_0^*} - \int_0^t \alpha(s,T_1^*,T_0^*)\,\mathrm{d}s$ is a standard Brownian motion under $\mathbb{P}_{T_1^*}$ and $\nu^{T_1^*}(\mathrm{d}t,\mathrm{d}x) := \beta'(x,t,T_1^*,T_0^*)\,\nu^{T_0^*,L}(\mathrm{d}t,\mathrm{d}x)$ is the $\mathbb{P}_{T_1^*}$-compensator of μ^L.

Analogously to the date T_1^* we postulate that the dynamics of $L(.,T_2^*)$ is given by

$$\mathrm{d}L(t,T_2^*) = L(t-,T_2^*)\left(\lambda^1(t,T_2^*)\,\mathrm{d}W_t^{T_1^*} + \int_{\mathbb{R}}\lambda^2(x,t,T_2^*)(\mu^L-\nu^{T_1^*,L})(\mathrm{d}t,\mathrm{d}x)\right),$$

with initial condition $L(0,T_2^*) = \frac{1}{\delta}\left(B(0,T_2^*)/B(0,T_1^*)-1\right)$. We define

$$\alpha(t,T_2^*,T_1^*) = \frac{\delta L(t-,T_2^*)}{1+\delta L(t-,T_2^*)}\lambda^1(t,T_2^*)$$
$$\beta'(x,t,T_2^*,T_1^*) = \frac{\delta L(t-,T_2^*)}{1+\delta L(t-,T_2^*)}\lambda^2(x,t,T_2^*)+1, \qquad t\in[0,T_2^*].$$

Note that the forward rate process $F_B(.,T_2^*,T_1^*)$ satisfies under $\mathbb{P}_{T_1^*}$

$$\mathrm{d}F_B(t,T_2^*,T_1^*) = F_B(t-,T_2^*,T_1^*)\left(\alpha(t,T_2^*,T_1^*)\,\mathrm{d}W_t^{T_1^*} + \int_{\mathbb{R}}(\beta'(x,t,T_2^*,T_1^*)-1)\left(\mu^L-\nu^{T_1^*,L}\right)(\mathrm{d}t,\mathrm{d}x)\right),$$

with $F_B(0,T_2^*,T_1^*) = B(0,T_2^*)/B(0,T_1^*)$.

As in the case of T_1^* $\alpha(.,T_2^*,T_1^*)$ and $\beta'(.,.,T_2^*,T_1^*)$ define a measure $\mathbb{P}_{T_2^*}$ on $(\Omega, \mathcal{F}_{T_2^*})$ by its density

$$\frac{\mathrm{d}\mathbb{P}_{T_2^*}}{\mathrm{d}\mathbb{P}_{T_1^*}} = \mathcal{E}\left(\int_0^{\cdot} \alpha(s,T_2^*,T_1^*)\,\mathrm{d}W_s^{T_1^*} + \int_0^{\cdot}\int_{\mathbb{R}} (\beta'(x,t,T_2^*,T_1^*)-1)(\mu^L - \nu^{T_1^*,L})(\mathrm{d}t,\mathrm{d}x)\right)_{T_2^*}.$$

Then $W^{T_2^*} := W^{T_1^*} - \int_0^t \alpha(s,T_2^*,T_1^*)\,\mathrm{d}s$ is a standard Brownian motion under $\mathbb{P}_{T_1^*}$ and $\nu^{T_2^*,L}(\mathrm{d}t,\mathrm{d}x) := \beta'(x,t,T_2^*,T_1^*)\,\nu^{T_1^*,L}(\mathrm{d}t,\mathrm{d}x)$ is the $\mathbb{P}_{T_1^*}$-compensator of μ^L. Note that

$$\frac{\mathrm{d}\mathbb{P}_{T_2^*}}{\mathrm{d}\mathbb{P}_{T_1^*}} = \frac{1}{F_B(0,T_2^*,T_1^*)} F_B(T_2^*,T_2^*,T_1^*).$$

By setting

$$\frac{\mathrm{d}\mathbb{P}_{T_{j+1}^*}}{\mathrm{d}\mathbb{P}_{T_j^*}} = \frac{1}{F_B(0,T_{j+1}^*,T_j^*)} F_B(T_{j+1}^*,T_{j+1}^*,T_j^*) \tag{4.26}$$

we define the measure $\mathbb{P}_{T_{j+1}^*}$ on $(\Omega,\mathcal{F}_{T_{j+1}})$ for the remaining dates. The relationship between the forward measure associated with the date T_{j+1}^* and T_j^* in equation (4.26) does not depend on the specific model for the Libor rates, see Rutkowski (2001).

The existence and uniqueness of the implied savings account can be shown by the same arguments as in Musiela and Rutkowski (1997). With a view to completeness we summarize their argumentation.

In the discrete-tenor setting the savings account is a discrete-time process $(B_t)_{t=0,\delta,\ldots,M\delta}$. The value B_t can be interpreted as the amount of money accumulated up to time t by rolling over a sequence of a series of zero-coupon bonds with the shortest maturities available. The bond prices will have to satisfy the following relationship

$$F_B(t,T_j,T_{j+1}) = \frac{B(t,T_j)}{B(t,T_{j+1})}.$$

Thus

$$B(T_j,T_{j+1}) = F_B^{-1}(T_j,T_j,T_{j+1}). \tag{4.27}$$

Hence it makes sense to define

$$
\begin{aligned}
B_0 &:= 1 \\
B_{T_j} &:= \prod_{k=1}^{j} \frac{1}{B(T_{k-1}, T_k)} = \prod_{k=1}^{j} F_B(T_{k-1}, T_{k-1}, T_k).
\end{aligned}
$$

The process $(B_{T_j})_{j=0,\dots,M}$ is predictable and increasing because

$$
\begin{aligned}
B_{T_{j+1}} &= \prod_{k=1}^{j+1} F_B(T_{k-1}, T_{k-1}, T_k) \\
&= B_{T_j} F_B(T_j, T_j, T_{j+1}) = B_{T_j}(1 + \delta L(T_j, T_{j+1})) \\
&> B_{T_j}.
\end{aligned}
$$

We define the candidate for the spot martingale measure $\mathbb{P}^* \sim \mathbb{P}_{T^*}$ by

$$
\frac{d\mathbb{P}^*}{d\mathbb{P}_{T^*}} = B_{T^*} B(0, T^*), \tag{4.28}
$$

and define the time-T_ℓ bond price of a zero-coupon bond with maturity T_k, $k \geq \ell$, as

$$
B(T_\ell, T_k) := \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_{T_\ell}}{B_{T_k}} \,|\, \mathcal{F}_{T_\ell}\right].
$$

Then indeed equation (4.27) is satisfied.

$$
\begin{aligned}
B(T_\ell, T_{\ell+1}) &= \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_{T_\ell}}{B_{T_{\ell+1}}} \,|\, \mathcal{F}_{T_\ell}\right] \\
&= \mathbb{E}_{\mathbb{P}^*}\left[\frac{1}{F_B(T_\ell, T_\ell, T_{\ell+1})} \,|\, \mathcal{F}_{T_\ell}\right] \\
&= \frac{1}{F_B(T_\ell, T_\ell, T_{\ell+1})},
\end{aligned}
$$

since $F_B(T_\ell, T_\ell, T_{\ell+1})$ is $\mathcal{F}_{T_\ell}$-measurable.

4.3.2 Special cases

In the following we will concentrate on the case that λ^1 and λ^2 are deterministic functions. There are two important special cases, where we are interested in. If there exists a constant $c > 0$ and a function λ on $[0, T]$, such that λ^1 and λ^2 are related through

$$
\begin{aligned}
\lambda^1(t, T) &= \sqrt{c}\lambda(t, T), \\
\lambda^2(x, t, T) &= \lambda(t, T)x,
\end{aligned}
$$

then $L_t^{T+\delta} := \sqrt{c}W_t^{T+\delta} + \int_0^t \int_{\mathbb{R}} x\,(\mu^L - \nu^{T+\delta,L})(\mathrm{d}s, \mathrm{d}x)$ is a Lévy process under $\mathbb{P}_{T+\delta}$. In case that λ^2 is constant zero, we are in the classical Gaussian setting. The dynamics of the Libor rate is driven by the Lévy process $L^{T+\delta}$:

$$\mathrm{d}L(t,T) = L(t-,T)\,\lambda(t,T)\;\mathrm{d}L_t^{T+\delta},$$

or equivalently

$$L(t,T) = L(0,T)\,\mathcal{E}\left(\int_0^{\cdot} \lambda(s,T)\;\mathrm{d}L_s^{T+\delta}\right)_t .$$

Note that one has to assume that the jumps of $\int_0^{\cdot} \lambda(s,T)\,\mathrm{d}L_s^{T+\delta}$ are strictly larger than -1 to ensure that the Libor rates are positive.

This can be replaced by the following alternative assumption.

(LLR.1) For fixed $T \leq T^* - \delta$ the Libor rate has the following representation

$$L(t,T) = L(0,T)\exp\left(\int_0^t \lambda(s,T)\;\mathrm{d}L_s^{T+\delta}\right), \tag{4.29}$$

where λ is a positive deterministic function such that $\int_0^T \lambda(s,T)^2\;\mathrm{d}s < \infty$. We assume that there exists $c \geq 0$, and a continuously differentiable function $b : \mathbb{R}_+ \to \mathbb{R}$ such that

$$L_t^{T+\delta} - b(t) = \sqrt{c}W_t^{T+\delta} + \int_0^t \int_{\mathbb{R}} x\,(\mu^L - \nu^{T+\delta,L})(\mathrm{d}s, \mathrm{d}x) \tag{4.30}$$

is a Lévy process under $\mathbb{P}_{T+\delta}$. The Lévy measure of $L_1^{T+\delta,L} - b(1)$ is denoted by $\nu^{T+\delta}$. Furthermore, we assume that $\int_{\{|x|>1\}} \exp(ux)\,\nu^{T+\delta}(\mathrm{d}x) < \infty$, for $-M \leq u \leq M$, and $|\lambda| \leq M$ where M is a positive constant.

(LLR.2) Furthermore we assume that the next condition holds

$$\int_0^t \lambda(s,T)b'(s)\;\mathrm{d}s = -\left(\int_0^t \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1 - \lambda(s,T)x\right)\,\nu^{T+\delta,L}(\mathrm{d}s, \mathrm{d}x) + \frac{1}{2}c\int_0^t \lambda^2(s,T)\;\mathrm{d}s\right). \tag{4.31}$$

Condition (LLR.2) gets more restrictive if we want $L^{T+\delta}$ to be a $\mathbb{P}_{T+\delta}$-Lévy process:

$$b\int_0^t \lambda(s,T)\ \mathrm{d}s = -\int_0^t\int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1 - \lambda(s,T)x\right) \nu^{T+\delta,L}(\mathrm{d}s,\mathrm{d}x) - \frac{1}{2}c\int_0^t \lambda^2(s,T)\ \mathrm{d}s.$$

Lemma 4.7 *Under Assumption (LLR.1) and under the condition given in equation (4.31) the Libor rate process $L(.,T)$ is a $\mathbb{P}_{T+\delta}$-martingale.*

Proof: Let $0 \le s \le t \le T$. The process $\int_0^{\cdot} \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}$ is an adapted process with independent increments, see e.g. Jacod and Shiryaev (1987, II.4.15) combined with Jacod and Shiryaev (1987, IX.5.3) or Kallsen and Shiryaev (2001, Lemma 2.5). Hence $\int_s^t \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}$ is independent from $\mathcal{F}_s$ and hence $\exp\left(\int_s^t \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}\right)$ is independent from $\mathcal{F}_s$. Then

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{T+\delta}}\left[\exp\left(\int_0^t \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}\right) \mid \mathcal{F}_s\right] \\
&\quad = \exp\left(\int_0^s \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}\right) \mathbb{E}_{\mathbb{P}_{T+\delta}}\left[\exp\left(\int_s^t \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}\right) \mid \mathcal{F}_s\right] \\
&\quad = \exp\left(\int_0^s \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}\right) \mathbb{E}_{\mathbb{P}_{T+\delta}}\left[\exp\left(\int_s^t \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}\right)\right].
\end{aligned}$$

Now we have to show that the expectation on the right-hand side equals one. Due to equation (4.30) this expectation can be written as

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{T+\delta}}\left[\int_s^t \lambda(u,T)\ \mathrm{d}L_u^{T+\delta}\right] &= \exp\left(\int_s^t \lambda(u,T)b'(u)\ \mathrm{d}u\right) \\
&\quad \times \mathbb{E}_{\mathbb{P}_{T+\delta}}\left[\int_s^t \lambda(u,T)\ \mathrm{d}\widetilde{L}_u^{T+\delta}\right],
\end{aligned} \qquad \text{[L4.7.a]}$$

where $\widetilde{L}_u^{T+\delta} := \sqrt{c}\ \mathrm{d}W_u^{T+\delta} + \int_0^t\int_{\mathbb{R}} x\left(\mu^L - \nu^{T+\delta,L}\right)(\mathrm{d}u,\mathrm{d}x)$ is a $\mathbb{P}_{T+\delta}$-Lévy process with drift zero.

The expectation on the right-hand side of equation [L4.7.a] can be calculated with Lemma 3.1 in Eberlein and Raible (1999) as follows

$$\mathbb{E}_{\mathbb{P}_{T+\delta}}\left[\exp\left(\int_s^t \lambda(u,T)\ \mathrm{d}\widetilde{L}_u^{T+\delta}\right)\right] = \exp\left(\int_s^t \widetilde{\kappa}(\lambda(u,T))\ \mathrm{d}u\right),$$

where $\widetilde{\kappa}(.)$ is the logarithm of the moment generating function of $\widetilde{L}_1^{T+\delta}$, i.e. $\widetilde{\kappa}$ is given by

$$\widetilde{\kappa}(x) = \frac{1}{2}cx^2 + \int_{\mathbb{R}} (\mathrm{e}^{zx} - 1 - zx)\ \nu^{T+\delta}(\mathrm{d}x).$$

By equation (4.31) we get

$$\mathbb{E}_{\mathbb{P}_{T+\delta}}\left[\exp\left(\int_s^t \lambda(u,T)\ \mathrm{d}\widetilde{L}_u^{T+\delta}\right)\right] = \exp\left(-\int_s^t \lambda(u,T)b'(u)\ \mathrm{d}u\right).$$

Hence the right-hand side of equation [L4.7.a] equals one. □

The following corollary embeds Assumption (LLR.1) in Assumption (LR.1).

Corollary 4.8 *The modeling approach*

$$L(t,T) = L(0,T)\exp\left(\int_0^t \lambda(s,T)\ \mathrm{d}L_s^{T+\delta}\right)$$

is equivalent to

$$L(t,T) = L(0,T)\,\mathcal{E}\Bigg(\sqrt{c}\int_0^{\cdot} \lambda(s,T)\ \mathrm{d}W_s^{T+\delta} + \int_0^{\cdot}\int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1\right)\left(\mu^L - \nu^{T+\delta,L}\right)(\mathrm{d}s,\mathrm{d}x)\Bigg)_t. \quad (4.32)$$

Proof: We have to calculate the exponential transform of $X := \int_0^{\cdot} \lambda(s,T)\ \mathrm{d}L_s^{T+\delta}$. It is defined by $\widetilde{X} := \mathcal{L}(\exp(X))$. Since $X_0 = 0$ we have $\mathcal{E}(\widetilde{X}) = \exp(X)$, see Kallsen and Shiryaev (2000, Lemma 2.4). The exponential transform is given by the following formula, see Kallsen and Shiryaev (2000, Lemma 2.6),

$$\widetilde{X}_t = X_t + \frac{1}{2}\langle X^c, X^c\rangle_t + \int_0^t\int_{\mathbb{R}} (\mathrm{e}^x - 1 - x)\ \mu^X(\mathrm{d}s,\mathrm{d}x).$$

1. Calculation of $\langle X^c, X^c\rangle_t$:

$$X_t^c = \int_0^t \sqrt{c}\lambda(s,T)\ \mathrm{d}W_s^{T+\delta} \qquad \Longrightarrow \qquad \langle X^c, X^c\rangle_t = c\int_0^t \lambda^2(s,T)\ \mathrm{d}s.$$

2. Calculation of $\int_0^t \int_{\mathrm{I\!R}} (\mathrm{e}^x - 1 - x)\, \mu^X(\mathrm{d}s, \mathrm{d}x)$:

   By construction the jumps of X are $\triangle X_t = \lambda(t,T)\triangle L_t^{T+\delta} = \lambda(t,T)\triangle L_t$. Hence

$$\int_0^t \int_{\mathrm{I\!R}} (\mathrm{e}^x - 1 - x)\, \mu^X(\mathrm{d}s, \mathrm{d}x) \quad = \quad \int_0^t \int_{\mathrm{I\!R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1 - \lambda(s,T)x\right) \mu^L(\mathrm{d}s, \mathrm{d}x).$$

Therefore

$$\begin{aligned}
\widetilde{X}_t = & \int_0^t \lambda(s,T)\ \mathrm{d}L_s^{T+\delta} + \frac{1}{2}c \int_0^t \lambda^2(s,T)\ \mathrm{d}s \\
& + \int_0^t \int_{\mathrm{I\!R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1 - \lambda(s,T)x\right)\ \mu^L(\mathrm{d}s, \mathrm{d}x) \\
= & \int_0^t b'(s)\lambda(s,T)\ \mathrm{d}s + \sqrt{c} \int_0^t \lambda(s,T)\ \mathrm{d}W_s^{T+\delta} \\
& + \int_0^t \int_{\mathrm{I\!R}} \lambda(s,T)x\ \left(\mu^L - \nu^{T+\delta,L}\right)(\mathrm{d}s, \mathrm{d}x) + \frac{1}{2}c \int_0^t \lambda^2(s,T)\ \mathrm{d}s \\
& + \int_0^t \int_{\mathrm{I\!R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1 - \lambda(s,T)x\right)\ \nu^{T+\delta,L}(\mathrm{d}s, \mathrm{d}x) \\
& + \int_0^t \int_{\mathrm{I\!R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1 - \lambda(s,T)x\right)\ (\mu^L - \nu^{T+\delta,L})(\mathrm{d}s, \mathrm{d}x).
\end{aligned}$$

And finally, by condition (4.31),

$$\widetilde{X}_t = \sqrt{c} \int_0^t \lambda(s,T)\ \mathrm{d}W_s^{T+\delta} + \int_0^t \int_{\mathrm{I\!R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1\right)\ (\mu^L - \nu^{T+\delta,L})(\mathrm{d}s, \mathrm{d}x).$$

□

4.3.3 Forward measure modeling

In Section 4.3.1 we make no assumption about the bond price system. The discrete bond price system is constructed from the Libor rate model.

In this section we make the following restriction concerning the form of the Libor rates

$$L(t,T) = \frac{1}{\delta}\left(\exp\left(\int_T^{T+\delta} f(t,u)\,\mathrm{d}u\right) - 1\right), \tag{4.33}$$

where f is the instantaneous forward rate of Section 4.1.1. This section is closely related to Brace, Gatarek, and Musiela (1997) and Glasserman and Kou (2001) since it provides the natural generalization to Lévy processes.

The connection between the martingale measure $\mathbb{P}^*$ and the forward measure associated with the date $T+\delta$ is given by

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{P}_{T+\delta}}{\mathrm{d}\mathbb{P}^*}\Big|_{\mathcal{F}_t} &= \frac{B(t,T+\delta)}{B_t B(0,T+\delta)} \\
&= \frac{1}{B(0,T+\delta)} \exp\left(\int_0^t A(s,T+\delta)\,\mathrm{d}s + \int_0^t S(s,T)^\top\,\mathrm{d}W_s^* \right. \\
&\qquad \left. + \int_0^t \int_{\mathbb{R}^r} D(s,x,T+\delta)(\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x)\right) \\
&= \frac{1}{B(0,T+\delta)} \exp\left(\int_0^t \phi^{T+\delta}(s)^\top\,\mathrm{d}W_s^* - \frac{1}{2}\int_0^t |\phi^{T+\delta}(s)|^2\,\mathrm{d}s \right. \\
&\qquad + \int_0^t \int_{\mathbb{R}^r} \left(Y^{T+\delta}(s,x) - 1\right)(\mu-\nu^*)(\mathrm{d}s,\mathrm{d}x) \\
&\qquad \left. - \int_0^t \int_{\mathbb{R}^r} \left(Y^{T+\delta}(s,x) - 1 - \ln Y^{T+\delta}(s,x)\right)\mu(\mathrm{d}s,\mathrm{d}x)\right),
\end{aligned}$$

where

$$\phi^{T+\delta}(t) = S(t,T+\delta), \tag{4.34}$$

$$Y^{T+\delta}(t,x) = \mathrm{e}^{D(t,x,T+\delta)} = \mathrm{e}^{-\int_t^{T+\delta}\delta(t,x,u)\,\mathrm{d}u}. \tag{4.35}$$

Under $\mathbb{P}_{T+\delta}$ the compensator of μ is given by

$$\nu^{T+\delta}(\mathrm{d}t, \mathrm{d}x) = \mathrm{e}^{D(t,x,T+\delta)}\nu^*(\mathrm{d}t, \mathrm{d}x) = \exp\left(-\int_t^{T+\delta} \delta(t,x,u)\ \mathrm{d}u\right)\nu^*(\mathrm{d}t, \mathrm{d}x),$$

and the process

$$W_t^{T+\delta} = W_t^* - \int_0^t S(s, T+\delta)\ \mathrm{d}s$$

is a standard Brownian motion.

The next theorem states the Libor rate model under the final forward measure $\mathbb{P}_{T_{M+1}}$ and generalizes Theorem 3.2 in Glasserman and Kou (2001).

Theorem 4.9 *Let γ_n be a bounded deterministic function, $W^{T_{M+1}}$ a standard Brownian motion with respect to $\mathbb{P}_{T_{M+1}}$ and $\nu^{T_{M+1}}(\mathrm{d}t, \mathrm{d}x) = \lambda^{T_{M+1}}(t, \mathrm{d}x)\ \mathrm{d}t$ is the $\mathbb{P}_{T_{M+1}}$-compensator of μ. Let $H_n : \mathbb{R}^r \to \mathbb{R}$ be a deterministic function in $G_{\mathrm{loc}}(\mu)$.*

The dynamics of $L(., T_n), n = 1, \ldots, M$, is assumed to satisfy

$$\mathrm{d}L(t, T_n) = L(t-, T_n)\Bigg(\alpha_n(t)\ \mathrm{d}t + \gamma_n(t)^\top\ \mathrm{d}W_t^{T_{M+1}} + \int_{\mathbb{R}^r} H_n(t,x)(\mu - \nu^{T_{M+1}})(\mathrm{d}t, \mathrm{d}x)\Bigg). \quad (4.36)$$

Then this model is arbitrage-free if

$$\alpha_n(t) = -\sum_{k=n+1}^{M} \frac{\delta\gamma_n(t)^\top\gamma_k(t)L(t-, T_k)}{1+\delta L(t-, T_k)} + \int_{\mathbb{R}^r} H_n(t,x)\left(1 - \prod_{k=n+1}^{M}\frac{1+\delta L(t-, T_k)(1+H_k(t,x))}{1+\delta L(t-, T_k)}\right)\lambda^{T_{M+1}}(t, \mathrm{d}x). \quad (4.37)$$

Proof: We show the arbitrage-freeness of the model by embedding it in the HJM-framework by choosing suitable functions $\sigma(t,u)$ and $\delta(t,x,u)$, where we can use the same formulas as in the proof of Theorem 4.2:

$$S(t, T_n) - S(t, T_{n+1}) = \int_{T_n}^{T_{n+1}} \sigma(t,u)\ \mathrm{d}u := \frac{\delta\gamma_n(t)L(t-, T_n)}{1+\delta L(t-, T_n)}$$

and

$$D(t,x,T_n) - D(t,x,T_{n+1}) = \int_{T_n}^{T_{n+1}} \delta(t,x,u)\ \mathrm{d}u$$
$$:= \ln\left(\frac{1+\delta L(t-,T_n)(1+H_n(t,x))}{1+\delta L(t-,T_n)}\right).$$

The risk-neutral dynamics of $L(.,T_n)$ is given by, see equation (4.9),

$$\mathrm{d}L(t,T_n) = \frac{1+\delta L(t-,T_n)}{\delta}\Bigg((S(t,T_n)-S(t,T_{n+1}))^\top(\mathrm{d}W_t^* - S(t,T_{n+1})\ \mathrm{d}t)$$
$$+\int_{\mathbb{R}^r}\left(\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})} - 1 + \mathrm{e}^{D(t,x,T_{n+1})} - \mathrm{e}^{D(t,x,T_n)}\right)\ \nu^*(\mathrm{d}t,\mathrm{d}x)$$
$$+\int_{\mathbb{R}^r}\left(\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})} - 1\right)\ (\mu-\nu^*)(\mathrm{d}t,\mathrm{d}x)\Bigg).$$

[Th4.9.a]

First, we consider the integrator of the first expression on the right-hand side

$$\mathrm{d}W_t^* - S(t,T_{n+1})\ \mathrm{d}t = \mathrm{d}W_t^* - S(t,T_{M+1})\ \mathrm{d}t + (S(t,T_{M+1}) - S(t,T_{n+1}))\ \mathrm{d}t$$
$$= \mathrm{d}W_t^{T_{M+1}} - \sum_{k=n+1}^{M}\frac{\delta\gamma_k(t)L(t-,T_k)}{1+\delta L(t-,T_k)}\ \mathrm{d}t,$$

where we have used equation [Th4.2.c], page 101, from the proof of Theorem 4.2. Hence,

$$(S(t,T_n)-S(t,T_{n+1}))^\top(\mathrm{d}W_t^* - S(t,T_{n+1})\ \mathrm{d}t)$$
$$= \frac{\delta}{1+\delta L(t-,T_n)}\gamma_n(t)^\top L(t-,T_n)\left(\mathrm{d}W_t^{T_{M+1}} - \sum_{k=n+1}^{M}\frac{\delta\gamma_k(t)L(t-,T_k)}{1+\delta L(t-,T_k)}\ \mathrm{d}t\right).$$

[Th4.9.b]

Before we consider the terms in equation [Th4.9.a] which are related to the jumps of $L(.,T_n)$ we state the following useful equality.

$$\mathrm{e}^{D(t,x,T_{n+1})} - \mathrm{e}^{D(t,x,T_n)} = \left(1-\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})}\right)\mathrm{e}^{D(t,x,T_{n+1})}$$
$$= \left(1-\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})}\right)$$
$$\times\ \mathrm{e}^{D(t,x,T_{n+1})-D(t,x,T_{M+1})}\mathrm{e}^{D(t,x,T_{M+1})}.$$

Hence,

$$\begin{aligned}
\mathrm{e}^{D(t,x,T_{n+1})} - \mathrm{e}^{D(t,x,T_n)} &= \left(1 - \mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})}\right) \mathrm{e}^{D(t,x,T_{M+1})} \\
&\qquad \times \prod_{k=n+1}^{M} \mathrm{e}^{D(t,x,T_k)-D(t,x,T_{k+1})} \\
&= -\frac{\delta L(t-,T_n) H_n(t,x)}{1+\delta L(t-,T_n)} \mathrm{e}^{D(t,x,T_{M+1})} \\
&\qquad \times \prod_{k=n+1}^{M} \frac{1+\delta L(t-,T_k)(1+H_k(t,x))}{1+\delta L(t-,T_k)},
\end{aligned}$$

where in the last equation we have used equation [Th4.2.d] from the proof of Theorem 4.2. By using equation [Th4.2.d] again we get

$$\begin{aligned}
&\mathrm{e}^{D(t,x,T_{n+1})} - \mathrm{e}^{D(t,x,T_n)} + \left(\mathrm{e}^{D(t,x,T_n)-D(t,x,T_{n+1})} - 1\right) \mathrm{e}^{D(t,x,T_{M+1})} \\
&\quad = \left(1 - \prod_{k=n+1}^{M} \frac{1+\delta L(t-,T_k)(1+H_k(t,x))}{1+\delta L(t-,T_k)}\right) \\
&\qquad \times \frac{\delta}{1+\delta L(t-,T_n)} L(t-,T_n) H_n(t,x)\, \mathrm{e}^{D(t,x,T_{M+1})}. \qquad \text{[Th4.9.c]}
\end{aligned}$$

Now we consider the expressions in equation [Th4.9.a], that are related to the jumps of $L(.,T_n)$. For the purpose of lucidity we use the notation for integrals with respect to random measures from Jacod and Shiryaev (1987, II.1.5).

$$\begin{aligned}
&\left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})} - 1 + \mathrm{e}^{D(.,.,T_{n+1})} - \mathrm{e}^{D(.,.,T_n)}\right) * \nu_t^* \\
&\qquad + \left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})} - 1\right) * (\mu - \nu^*)_t \\
&= \left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})} - 1 + \mathrm{e}^{D(.,.,T_{n+1})} - \mathrm{e}^{D(.,.,T_n)}\right) * \nu_t^* \\
&\qquad + \left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})} - 1\right) \left(\mathrm{e}^{D(.,.,T_{M+1})} - 1\right) * \nu_t^* \\
&\qquad + \left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})} - 1\right) * \left(\mu - \nu^{T_{M+1}}\right)_t \\
&= \left(\mathrm{e}^{D(.,.,T_{n+1})} - \mathrm{e}^{D(.,.,T_n)} + \left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})} - 1\right) \mathrm{e}^{D(.,.,T_{M+1})}\right) * \nu_t^* \\
&\qquad + \left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})} - 1\right) * \left(\mu - \nu^{T_{M+1}}\right)_t .
\end{aligned}$$

By using equation [Th4.9.c] and [Th4.2.d] we get

$$\frac{1}{\delta}(1+\delta L(.,T_n))\left(\left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})}-1+\mathrm{e}^{D(.,.,T_{n+1})}-\mathrm{e}^{D(.,.,T_n)}\right)*\nu^*_t\right.$$
$$\left.+\left(\mathrm{e}^{D(.,.,T_n)-D(.,.,T_{n+1})}-1\right)*(\mu-\nu^*)_t\right)$$
$$=\left(L_-(.,T_n)H_n(.,.)\left(1-\prod_{k=n+1}^{M}\frac{1+\delta L_-(.,T_k)(1+H_k(.,.))}{1+\delta L_-(.,T_k)}\right)\right)*\nu_t^{T_{M+1}}$$
$$+(L_-(.,T_n)H_n(.,.))*\left(\mu-\nu^{T_{M+1}}\right)_t. \qquad \text{[Th4.9.d]}$$

By combining equation [Th4.9.b] and [Th4.9.d] we get equation (4.36) where the drift coefficient satisfies (4.37). □

The next theorem shows that the market practice of assuming that caps and floors can be priced such that the Libor rates are driven by a Wiener process can be pushed further to a model where the Libor rates are driven by Lévy processes under each forward measure. This means that in our discrete tenor model under each forward measure $\mathbb{P}_{T_{n+1}}$ the Libor rate $L(.,T_n)$ has the form in Assumption (LLR.1) on page 121.

We proof a slightly more general version.

Let $r \geq M, I_n \subset \{1,\dots,M\}$, and let for all $n \in \{1,\dots,M\}$ $H_n \in G_{\mathrm{loc}}(\mu)$ such that $H_n(t,0)=0$ and H_n depends only on the I_n-coordinates, i.e. let Π_{I_n} : $\mathbb{R}^r \to \mathbb{R}^{I_n}$ be the projection $(x_i)_{i\in\{1,\dots,r\}} \mapsto (x_i)_{i\in I_n}$, then for all $x,y \in \mathbb{R}^r$ with $\Pi_{I_n}(x)=\Pi_{I_n}(y)$ we have $H_n(t,x)=H_n(t,y)$. We define $\widetilde{H}_n : [0,T^*]\times\mathbb{R}^{I_n} \to \mathbb{R}$ by setting $\widetilde{H}_n(t,x) := H_n(t,z)$, where $z \in \Pi_{I_n}^{-1}(x)$ is arbitrary. We define the random jump measure $\widetilde{\mu}_n$ on $\Omega\times[0,T^*]\times\mathbb{R}^{I_n}$ by setting $\widetilde{\mu}_n(\omega;[0,t)\times A) := \mu(\omega;[0,t)\times(A\times\mathbb{R}^{\{1,\dots,r\}\setminus I_n}))$ and let $\widetilde{\nu}^{T_{n+1}}(\mathrm{d}t,\mathrm{d}x)=\widetilde{f}_n(t,\mathrm{d}x)\,\mathrm{d}t$ be its $\mathbb{P}_{T_{n+1}}$-compensator. Let f_n be the continuation of $\widetilde{f}_n$ on $\mathbb{R}^r$ which depends only on the I_n coordinates and $\nu^{T_{n+1}}(\mathrm{d}t,\mathrm{d}x) := f_n(t,\mathrm{d}x)\,\mathrm{d}t$.

In the exponential Lévy Libor setting in Assumption (LLR.1) we have $I_n = \{n\}$ and $H_n(t,x)=\mathrm{e}^{\lambda(t,T_n)x_n}-1$, cf. Corollary 4.8. For practical purposes it is sufficient to consider the case where the Lévy measure has a Lebesgue density, i.e. $\widetilde{\nu}^{T_{n+1}}(\mathrm{d}t,\mathrm{d}x)=\widetilde{f}_n(x)\,\mathrm{d}x\,\mathrm{d}t$. The next proposition generalizes Proposition 3.1 in Glasserman and Kou (2001).

Proposition 4.10 *For each $n=1,2,\dots,M$ let $\gamma_n(.)$ be a bounded $\mathbb{R}^d$-valued function. Let $H_n,\widetilde{H}_n,\widetilde{\mu}_n$, $\nu^{T_{n+1}},\widetilde{\nu}^{T_{n+1}}$, $f_n,\widetilde{f}_n,W^{T_{n+1}}$ be given as above.*

Consider the following model, where for each $n\in\{1,\dots,M\}$ the dynamics of

$L(.,T_n)$ under its associated forward measure $\mathbb{P}_{T_{n+1}}$ is given by

$$\mathrm{d}L(t,T_n) = L(t-,T_n)\left(\gamma_n(t)^\top\,\mathrm{d}W_t^{T_{n+1}} + \int\limits_{\mathbb{R}^{I_n}} \widetilde{H}_n(t,x)(\widetilde{\mu}_n - \widetilde{\nu}^{T_{n+1}})(\mathrm{d}t,\mathrm{d}x)\right), \tag{4.38}$$

where

$$\widetilde{\nu}^{T_{n+1}}(\mathrm{d}t,\mathrm{d}x) = \widetilde{f}_n(t,\mathrm{d}x)\,\mathrm{d}t.$$

This model is arbitrage-free if the risk-neutral intensity satisfies for all $t \in [0,T^*]$.

$$\begin{aligned}&\int\limits_0^t\int\limits_{\mathbb{R}^r} g(s,x)\,\nu^*(\mathrm{d}s,\mathrm{d}x)\\ &\quad= \int\limits_0^t\int\limits_{\mathbb{R}^r} g(s,x)\left(\prod_{k=\eta(s)}^{n}\frac{1+\delta L(s-,T_k)(1+H_k(s,x))}{1+\delta L(s-,T_k)}\right) f_n(s,\mathrm{d}x)\,\mathrm{d}s,\end{aligned} \tag{4.39}$$

for every integrable $g : [0,T^*]\times\mathbb{R}^r \to \mathbb{R}$ that depends only on the $\bigcup_{k\le n} I_k$-coordinates.

Proof: It is sufficient to proof the statement for purely discontinuous $L(.,T_n)$. If the Gaussian parts are nonzero they can be handled in exactly the same way as in the classical setting of Brace, Gatarek, and Musiela (1997) or Musiela and Rutkowski (1997).

We follow the same strategy as in the proofs of Theorem 4.2 and Theorem 4.9. Again we set $D(t,x,T_n) - D(t,x,T_{n+1})$ as in equation [Th4.2.b]:

$$\begin{aligned}D(t,x,T_n) - D(t,x,T_{n+1}) &= \int\limits_{T_n}^{T_{n+1}} \delta(t,x,u)\,\mathrm{d}u\\ &:= \ln\left(\frac{1+\delta L(t-,T_n)(1+H_n(t,x))}{1+\delta L(t-,T_n)}\right).\end{aligned}$$

Note that $D(t,x,T_n) - D(t,x,T_{n+1})$ depends only on the I_n-coordinates, since H_n depends only on the I_n-coordinates and that $D(t,x,T_{n+1})$ depends on the $\bigcup_{k\le n} I_k$-coordinates, see [Th4.2.e].

Let us now consider the terms related to the jumps of the risk-neutral dynamics of $L(.,T_n)$, see equation [Th4.9.a].

$$\frac{1+\delta L(t-,T_n)}{\delta}\left(\int_{\mathbb{R}^r}\left(e^{D(t,x,T_n)-D(t,x,T_{n+1})}-1\right)(\mu-\nu^*)(dt,dx)\right.$$

$$\left.+\int_{\mathbb{R}^r}\left(e^{D(t,x,T_n)-D(t,x,T_{n+1})}-1+e^{D(t,x,T_{n+1})}-e^{D(t,x,T_n)}\right)\nu^*(dt,dx)\right)$$

$$=\frac{1+\delta L(t-,T_n)}{\delta}\left(\int_{\mathbb{R}^r}\left(e^{D(t,x,T_n)-D(t,x,T_{n+1})}-1\right)(\mu-\nu_n^*)(dt,dx)\right.$$

$$+\int_{\mathbb{R}^r}\left(e^{D(t,x,T_n)-D(t,x,T_{n+1})}-1+e^{D(t,x,T_{n+1})}-e^{D(t,x,T_n)}\right)$$

$$\left.\times\left(\prod_{k=\eta(t)}^{n}\frac{1+\delta L(t-,T_k)(1+H_k(t,x))}{1+\delta L(t-,T_k)}\right)f_n(t,dx)\;dt\right).$$

Now we use equations [Th4.2.d] and [Th4.2.i]. Hence the above sum can be written as

$$\frac{1+\delta L(t-,T_n)}{\delta}\left(\int_{\mathbb{R}^r}\left(\frac{\delta}{1+\delta L(t-,T_n)}L(t-,T_n)H_n(t,x)\right)(\mu-\nu_n^*)(dt,dx)\right.$$

$$+\int_{\mathbb{R}^r}\frac{\delta L(t-,T_n)H_n(t,x)}{1+\delta L(t-,T_n)}\left(1-\prod_{k=\eta(t)}^{n}\frac{1+\delta L(t-,T_k)}{1+\delta L(t-,T_k)(1+H_k(t,x))}\right)$$

$$\left.\times\left(\prod_{k=\eta(t)}^{n}\frac{1+\delta L(t-,T_k)(1+H_k(t,x))}{1+\delta L(t-,T_k)}\right)f_n(t,dx)\;dt\right)$$

$$=\int_{\mathbb{R}^r}L(t-,T_n)H_n(t,x)\left(\mu-\nu^{T_{n+1}}\right)(dt,dx)$$

$$+\int_{\mathbb{R}^r}L(t-,T_n)H_n(t,x)\left(f_n(t,dx)\;dt-\nu_n^*(dt,dx)\right)$$

$$+\int_{\mathbb{R}^r}L(t-,T_n)H_n(t,x)\left(1-\prod_{k=\eta(t)}^{n}\frac{1+\delta L(t-,T_k)}{1+\delta L(t-,T_k)(1+H_k(t,x))}\right)$$

$$\times\left(\prod_{k=\eta(t)}^{n}\frac{1+\delta L(t-,T_k)(1+H_k(t,x))}{1+\delta L(t-,T_k)}\right)f_n(t,dx)\;dt\Bigg)$$

$$= \int_{\mathbb{R}^r} L(t-, T_n) H_n(t,x) \left(\mu - \nu^{T_{n+1}}\right)(\mathrm{d}t, \mathrm{d}x)$$

$$+ \int_{\mathbb{R}^r} L(t-, T_n) H_n(t,x) \left(1 - \prod_{k=\eta(t)}^{n} \frac{1 + \delta L(t-, T_k)(1 + H_k(t,x))}{1 + \delta L(t-, T_k)}\right) f_n(t, \mathrm{d}x)\ \mathrm{d}t$$

$$+ \int_{\mathbb{R}^r} L(t-, T_n) H_n(t,x) \left(\prod_{k=\eta(t)}^{n} \frac{1 + \delta L(t-, T_k)(1 + H_k(t,x))}{1 + \delta L(t-, T_k)} - 1\right) f_n(t, \mathrm{d}x)\ \mathrm{d}t$$

$$= \int_{\mathbb{R}^{I_n}} L(t-, T_n) \widetilde{H}_n(t,x) \left(\widetilde{\mu}_n - \widetilde{\nu}^{T_{n+1}}\right)(\mathrm{d}t, \mathrm{d}x).$$

□

4.3.4 Examples for the volatility structure

In Brigo and Mercurio (2001, Section 6.3.1 and Chapter 7) one can find a list of different specifications of the volatility structure as well as some aspects of the calibration in the Gaussian setting. The different types of volatility structures can be applied in our setting, too.

We define an auxiliary function $h : [0, T^*] \to \mathbb{N}_0$ by

$$h(t) := k \mathbb{1}_{(T_{k-1}, T_k]}.$$

The simplest example for λ is the constant volatility structure

$$\lambda(t, T_k) = s_k,$$

where each $s_k, k \in 1, \ldots, M-1$ is a positive constant. The number of parameters can be reduced if we assume that λ depends only on the time to maturity. In this case we have

$$\lambda(t, T_k) = \eta_{k-h(t)}.$$

This can be also generalized to a more complex but still piecewise constant volatility structures:

$$\lambda(t, T_k) = s_k \eta_{h(t)+1}$$

or

$$\lambda(t, T_k) = s_k \eta_{k-h(t)}.$$

Brigo and Mercurio (2001) propose also two different types of parametric forms for λ. For $a, b, c, d > 0$ the first one is given by

$$\lambda(t, T_k; a, b, c, d) = (a(T_k - t) + d)\mathrm{e}^{-b(T_k - t)} + c,$$

or more general

$$\lambda(t, T_k; \Phi_k, a, b, c, d) = \Phi_k \left((a(T_k - t) + d)\mathrm{e}^{-b(T_k - t)} + c \right), \qquad \Phi_k, a, b, c, d > 0.$$

Note that in case we have $a = 0, c = \frac{\widehat{\sigma}}{b}, d = \frac{\widehat{\sigma}}{b}$, for some $\widehat{\sigma} > 0$, we get the Vasiček volatility structure, which is known as a volatility structure of instantaneous forward rates. But note that it is not suitable in our case, because the Vasiček volatility structure guarantees that the asset which is considered has a prespecified value at maturity, namely 1 in case of a zero-coupon bond. However it would not make any sense to postulate that the volatility $\lambda(t, T)$ of the Libor rate tends to zero for $t \to T$.

Figure 4.1 shows the typical shape of the parametric volatility structure.

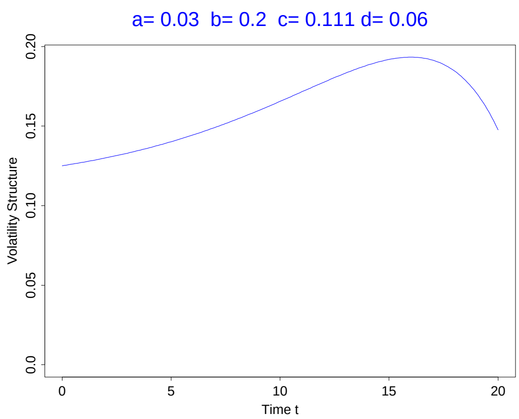


Figure 4.1: Example for the parametric volatility structure, $a = 0.03, b = 0.2, c = 0.111, d = 0.06$.

4.4 Pricing of caps and floors

An interest rate cap is a contract where the buyer of the cap places a maximum value, i.e. a strike rate, on the Libor rate index. He has to pay a premium to the seller of the cap. The premium will have to depend on the tenor structure $0 < T_0 < T_1 < \ldots < T_n$, the strike rate $\mathcal{K}$, the day-count convention, and the notional amount N. Similar to swaps an interest rate cap can be settled in advance or in arrears. The time-T_j payoff of a forward cap which starts at time T_0 and is settled in arrears at dates T_j is

$$N\delta(L(T_{j-1}, T_{j-1}) - \mathcal{K})^+.$$

The counterpart to interest rate caps are interest rate floors in the same sense as a call option is the counterpart of a put option. The payment of an interest rate floor settled in arrears at time T_j is

$$N\delta(\mathcal{K} - L(T_{j-1}, T_{j-1}))^+.$$

Without loss of generality we may assume that the notional amount N is one. The value at time $t \leq T_0$ is then given by

$$\mathrm{FC}_t = \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*}\left[B_t/B_{T_j}\delta(L(T_{j-1}, T_{j-1}) - \mathcal{K})^+ \mid \mathcal{F}_t\right].$$

The conditional expectation with respect to the measure $\mathbb{P}^*$ can be transformed to the conditional expectation with respect to the forward measure $\mathbb{P}_{T_j}$ by using the abstract Bayes rule, see e.g. Musiela and Rutkowski (1998, Lemma A.0.4). The density of $\mathbb{P}^*$ with respect to $\mathbb{P}_{T_j}$ is given by, see equation (4.17), page 105,

$$\frac{d\mathbb{P}^*}{d\mathbb{P}_{T_j}} = B_{T_j}B(0, T_j) \text{ and } \quad \frac{d\mathbb{P}^*}{d\mathbb{P}_{T_j}}|_{\mathcal{F}_t} = \frac{B_t B(0, T_j)}{B(t, T_j)}.$$

Hence

$$\begin{aligned}\mathrm{FC}_t &= \sum_{j=1}^{n} \frac{B(t, T_j)\ \mathbb{E}_{\mathbb{P}_{T_j}}\left[B_t/B_{T_j}\delta(L(T_{j-1}, T_{j-1}) - \mathcal{K})^+ B_{T_j}B(0, T_j) \mid \mathcal{F}_t\right]}{B_t B(0, T_j)} \\ &= \sum_{j=1}^{n} B(t, T_j)\,\mathbb{E}_{\mathbb{P}_{T_j}}\left[\delta(L(T_{j-1}, T_{j-1}) - \mathcal{K})^+ \mid \mathcal{F}_t\right].\end{aligned}$$

Remark: Each forward cap (floor) is a strip of caplets (floorlets). The j-th caplet with strike $\mathcal{K}$ and nominal value 1 is equivalent to a put option with maturity T_{j-1} and nominal value $\widetilde{\delta} := 1 + \mathcal{K}\delta$ on the bond with maturity T_j. The j-th caplet has positive value on the set $\{L(T_{j-1}, T_{j-1}) - \mathcal{K} > 0\}$. This is the same set where the put option described above is exercised, because

$$\begin{aligned}L(T_{j-1}, T_{j-1}) - \mathcal{K} > 0 \quad &\iff \quad 1 + L(T_{j-1}, T_{j-1})\delta = \frac{1}{B(T_{j-1}, T_j)} > \widetilde{\delta} = 1 + \mathcal{K}\delta \\ &\iff \quad B(T_{j-1}, T_j) < \frac{1}{\widetilde{\delta}}.\end{aligned}$$

In case that the payoff of the caplet discounted to the time T_j is positive it equals the payoff of the put option described above, since

$$B(T_{j-1},T_j)\delta(L(T_{j-1},T_{j-1})-\mathcal{K}) = 1-\widetilde{\delta}B(T_{j-1},T_j) \;=\; \widetilde{\delta}\left(\frac{1}{\widetilde{\delta}}-B(T_{j-1},T_j)\right).$$

In the remaining part of this section we show how caplets can be priced by using bilateral Laplace transforms, where we generalize slightly the approach in Raible (2000, Chapter 3). In view to applications using generalized hyperbolic Lévy processes, we concentrate on the purely discontinuous case. We assume that under the forward measure associated with the date T_j we have the dynamics as given in Proposition 4.10, i.e.

$$\mathrm{d}L(t,T_{j-1}) = L(t-,T_{j-1})\int_{\mathbb{R}}\left(\mathrm{e}^{\lambda(t,T_{j-1})x}-1\right)\left(\mu^L-\nu^{T_j,L}\right)(\mathrm{d}t,\mathrm{d}x),$$

where λ and $\nu^{T_j,L}$ satisfy the conditions given in (LLR.1) and (LLR.2).

From equation (4.29) we obtain

$$\begin{aligned}
L(T_{j-1},T_{j-1}) &= L(0,T_{j-1})\exp\left(\int_0^{T_{j-1}}\lambda(s,T_{j-1})\;\mathrm{d}L_s^{T_j}\right)\\
&= L(0,T_{j-1})\exp\Bigg(\int_0^{T_{j-1}} b_j'(s)\lambda(s,T_{j-1})\;\mathrm{d}s\\
&\qquad + \int_0^{T_{j-1}}\int_{\mathbb{R}}(x\lambda(s,T_{j-1}))\left(\mu^L-\nu^{T_j,L}\right)(\mathrm{d}s,\mathrm{d}x)\Bigg).
\end{aligned}$$

We define

$$X_t := \int_0^t \lambda(s,T_{j-1})\;\mathrm{d}L_s^{T_j},$$

so that

$$\begin{aligned}
X_{T_{j-1}} &= \ln\left(\frac{L(T_{j-1},T_{j-1})}{L(0,T_{j-1})}\right)\\
&= \int_0^{T_{j-1}} b_j'(s)\lambda(s,T_{j-1})\;\mathrm{d}s + \int_0^{T_{j-1}}\int_{\mathbb{R}} x\lambda(s,T_{j-1})\;\left(\mu^L-\nu^{T_j,L}\right)(\mathrm{d}s,\mathrm{d}x).
\end{aligned}$$

As a consequence of Lemma 3.1 in Eberlein and Raible (1999) we get the following form of the characteristic function χ of $X_{T_{j-1}}$

$$\begin{aligned}\chi(u) &:= \mathbb{E}_{\mathbb{P}_{T_j}}\left[\mathrm{e}^{iuX_{T_{j-1}}}\right]\\ &= \exp\Bigg(iu \int_0^{T_{j-1}} b_j'(s)\lambda(s,T_{j-1})\,\mathrm{d}s\\ &\qquad + \int_0^{T_{j-1}}\int_{\mathbb{R}} \left(\mathrm{e}^{iu\lambda(s,T_{j-1})x} - 1 - iu\lambda(s,T_{j-1})x\right)\,\nu^{T_j,L}(\mathrm{d}s,\mathrm{d}x)\Bigg).\end{aligned}$$

Equation (4.31) in Assumption (LLR.2) simplifies in the purely discontinuous case to

$$\int_0^t b'(s)\lambda(s,T)\,\mathrm{d}s = -\int_0^t\int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T)x} - 1 - \lambda(s,T)x\right)\,\nu^{T+\delta,L}(\mathrm{d}s,\mathrm{d}x).$$

Thus, the characteristic function of $X_{T_{j-1}}$ is given by

$$\begin{aligned}\chi(u) &= \exp\Bigg(-iu \int_0^{T_{j-1}}\int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T_{j-1})x} - 1 - \lambda(s,T_{j-1})x\right)\,\nu^{T_j,L}(\mathrm{d}s,\mathrm{d}x)\\ &\qquad + \int_0^{T_{j-1}}\int_{\mathbb{R}} \left(\mathrm{e}^{iu\lambda(s,T_{j-1})x} - 1 - iu\lambda(s,T_{j-1})x\right)\,\nu^{T_j,L}(\mathrm{d}s,\mathrm{d}x)\Bigg) \qquad (4.40)\\ &= \exp\left(\int_0^{T_{j-1}}\int_{\mathbb{R}} \left(\mathrm{e}^{iu\lambda(s,T_{j-1})x} - iu\,\mathrm{e}^{\lambda(s,T_{j-1})x} - (1-iu)\right)\,\nu^{T_j,L}(\mathrm{d}s,\mathrm{d}x)\right). \qquad (4.41)\end{aligned}$$

According to our model

$$(L(t,T_{j-1}))_{t\in[0,T_{j-1}]} = (L(0,T_{j-1})\exp(X_t))_{t\in[0,T_{j-1}]}$$

is a martingale with respect to the forward measure associated with the date T_j. Hence the moment generating function of $X_{T_{j-1}}$

$$\mathrm{mgf}(u) := \mathbb{E}_{\mathbb{P}_{T_j}}\left[\mathrm{e}^{uX_{T_{j-1}}}\right]$$

exists at least for $u \in [0,1]$ and we have

$$\mathrm{mgf}(1) = \mathbb{E}_{\mathbb{P}_{T_j}}\left[\mathrm{e}^{X_{T_{j-1}}}\right] = \mathbb{E}_{\mathbb{P}_{T_j}}\left[\frac{L(T_{j-1},T_{j-1})}{L(0,T_{j-1})}\right] = 1.$$

We make the following technical assumptions, cf. Raible (2000, Section 3.1 and 3.2). We assume that $\int_{-\infty}^{\infty} |\chi(u)|\,\mathrm{d}u < \infty$, so that the distribution of $X_{T_{j-1}}$ has a Lebesgue density, i.e. there exists a Lebesgue-measurable function ρ such that

$$\frac{\mathrm{d}\mathbb{P}_{T_j}^{X_{T_{j-1}}}}{\mathrm{dLeb}} = \rho, \quad \text{Leb-a.s.}$$

We assume that the extended characteristic function $\chi(z) := \mathbb{E}_{\mathbb{P}_{T_{j-1}}}\left[\mathrm{e}^{izX_{T_{j-1}}}\right]$ is defined for all $z \in \mathbb{R} - i[0,1] \subset \mathbb{C}$ with $\chi(-i) = 1$.

We define $w(x,\mathcal{K}) := (x-\mathcal{K})^+$ so that the payoff of the j-th caplet is $w(L(T_{j-1},T_{j-1}),\mathcal{K})$. The price of the j-th caplet at time 0 is given by $\mathbb{E}_{\mathbb{P}_{T_{j-1}}}\left[w(X_{T_{j-1}},\mathcal{K})\right]$. We introduce the modified payoff v by $v(x,\mathcal{K}) := w(\mathrm{e}^{-x},\mathcal{K})$.

Note that for $R < -1$ the function $x \mapsto \mathrm{e}^{-Rx}v(x,\mathcal{K})$ is bounded since

$$0 \le \mathrm{e}^{-Rx}\left(\mathrm{e}^{-x}-\mathcal{K}\right)^+ = \left(\mathrm{e}^{(|R|-1)x} - \mathrm{e}^{|R|x}\right)^+ \le \frac{1}{|R|}\left(\frac{|R|-1}{\mathcal{K}|R|}\right)^{|R|-1}.$$

The function $x \mapsto \mathrm{e}^{-Rx}v(x,\mathcal{K})$ is strictly greater than zero if and only if $x < -\ln(\mathcal{K})$. Hence $x \mapsto \mathrm{e}^{-Rx}v(x,\mathcal{K})$ is integrable:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-Rx}\left(\mathrm{e}^{-x}-\mathcal{K}\right)^+ \mathrm{d}x &= \int_{-\infty}^{-\ln(\mathcal{K})} \left(\mathrm{e}^{(|R|-1)x} - \mathcal{K}\mathrm{e}^{|R|x}\right)\mathrm{d}x \\ &= \left[\frac{1}{|R|-1}\mathrm{e}^{(|R|-1)x} - \frac{\mathcal{K}}{|R|}\mathrm{e}^{|R|x}\right]_{-\infty}^{-\ln(\mathcal{K})} \\ &= \frac{1}{|R|-1}\left(\frac{1}{\mathcal{K}}\right)^{|R|-1} - \frac{\mathcal{K}}{|R|}\left(\frac{1}{\mathcal{K}}\right)^{|R|} \\ &= \frac{\mathcal{K}^{1-|R|}}{|R|^2-|R|} < \infty.\end{aligned}$$

Theorem 4.11 *Let* $\zeta_j := -\ln(L(0,T_{j-1}))$. *Then* $L(T_{j-1},T_{j-1}) = \mathrm{e}^{-\zeta_j + X_{T_{j-1}}}$. *Assume that* $\mathrm{mgf}(-R) < \infty$. *Let* $V_j(\zeta_j,\mathcal{K})$ *be the time-0 price of the j-th caplet and let* $\mathrm{L}[v]$ *be the bilateral Laplace transform of* v, *i.e.*

$$\mathrm{L}[v](z) = \int_{-\infty}^{\infty} \mathrm{e}^{-zx}v(x)\,\mathrm{d}x, \qquad z = R + iu \in \mathbb{C}, u \in \mathbb{R}.$$

Then

$$V_j(\zeta_j, \mathcal{K}) = B(0, T_j) \frac{e^{\zeta_j R}}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} e^{iu\zeta_j} \, L[v](R+iu)\chi(iR-u) \; du \qquad (4.42)$$

whenever the right-hand side exists.

Proof: The price of the j-th caplet at time-0 can be written in the following way

$$\begin{aligned}
V_j(\zeta_j, \mathcal{K}) &= B(0, T_j)\mathbb{E}_{\mathbb{P}_{T_j}} \left[w(L(T_{j-1}, T_{j-1}), \mathcal{K})\right] \\
&= B(0, T_j)\mathbb{E}_{\mathbb{P}_{T_j}} \left[w(e^{-\zeta_j + X_{T_{j-1}}}, \mathcal{K})\right] \\
&= B(0, T_j)\mathbb{E}_{\mathbb{P}_{T_j}} \left[v(\zeta_j - X_{T_{j-1}}, \mathcal{K})\right] \\
&= B(0, T_j) \int_{\mathbb{R}} v(\zeta_j - X_{T_{j-1}}, \mathcal{K}) \, \mathbb{P}_{T_j}^{X_{T_{j-1}}}(dx) \\
&= B(0, T_j) \int_{\mathbb{R}} v(\zeta_j - x, \mathcal{K}_i)\rho(x) \; dx \\
&= B(0, T_j) \, (v * \rho) \, (\zeta_j, \mathcal{K}).
\end{aligned}$$

Thus,

$$L[V_j](R+iu) = B(0, T_j) \, L[v](R+iu) \, L[\rho](R+iu) \qquad \text{for} \quad u \in \mathbb{R}. \quad \text{[Th4.11.a]}$$

The integral that defines the bilateral Laplace transform $L[V_j](z)$ converges absolutely and $\zeta \mapsto V_j(\zeta, \mathcal{K})$ is a continuous function by Theorem B.2 in Raible (2000). Hence the payoff V_j can be obtained by inverting the Laplace transform, see Raible (2000, Theorem B.3).

$$\begin{aligned}
V_j(\zeta_j, \mathcal{K}) &= \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{\zeta_j z} \, L[V_j](z) \; dz \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\zeta_j (R+iu)} \, L[V_j](R+iu) \; du \\
&= \frac{e^{\zeta_j R}}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} e^{iu\zeta_j} \, L[V_j](R+iu) \; du.
\end{aligned}$$

By equation [Th4.11.a] we get

$$V_j(\zeta_j, \mathcal{K}) = B(0,T_j)\frac{\mathrm{e}^{\zeta_j R}}{2\pi}\lim_{M\to\infty}\int_{-M}^{M} \mathrm{e}^{iu\zeta_j}\,\mathrm{L}[v](R+iu)\chi(iR-u)\;\mathrm{d}u.$$

□

It is sufficient to consider the case there the strike price equals one because of Lemma 3.3 in Raible (2000),

$$V_j(\zeta_j, \mathcal{K}) = \mathcal{K}V_j(\zeta_j + \ln\mathcal{K}, 1).$$

The bilateral Laplace transform $\mathrm{L}[v]$ for $\mathcal{K} = 1$ is given by $\mathrm{L}[v](z) = (z(z+1))^{-1}$. Equation [Th4.11.a] remains true for more complicated payoff functions as long as the payoff depends only on $X_{T_{j-1}}$. Examples are given in Raible (2000, Table 3.1).

The characteristic function in (4.41) can be determined more precisely. Remember that $\nu^{T_j,L}(\mathrm{d}t, \mathrm{d}x) = \nu^{T_j}(\mathrm{d}x)\;\mathrm{d}t$. Let

$$\phi(u) := \exp\left(\int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux)\;\nu^{T_j}(\mathrm{d}x)\right)$$

be the characteristic function of L_1. Then it follows from equation (4.41)

$$\chi(u) = \exp\left(\int_0^{T_{j-1}} \left(-iu\ln\phi\left(-i\lambda(s,T_{j-1})\right) + \ln\phi(u\lambda(s,T_{j-1}))\right)\;\mathrm{d}s\right).$$

Let L be a generalized hyperbolic Lévy motion. Thus the distribution of L_1 is generalized hyperbolic with parameters $\lambda, \alpha, \beta, \delta, \mu$ and the characteristic function of L_1 is given by equation (1.14), page 13. Hence,

$$\chi(u) = \exp\left(T_{j-1}(1-iu)\ln\left(\frac{(\alpha^2-\beta^2)^{\lambda/2}}{\mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right)\right.$$
$$+\int_0^{T_{j-1}}\left(iu\frac{\lambda}{2}\ln\left(\alpha^2-(\beta+\lambda(s,T_{j-1}))^2\right)\right.$$
$$-\frac{\lambda}{2}\ln\left(\alpha^2-(\beta+iu\lambda(s,T_{j-1}))^2\right)$$
$$+\ln\left(\mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iu\lambda(s,T_{j-1}))^2}\right)\right)$$
$$\left.\left.-iu\ln\left(\mathrm{K}_\lambda\left(\delta\sqrt{\alpha^2-(\beta+\lambda(s,T_{j-1}))^2}\right)\right)\right)\mathrm{d}s\right).$$

In case that L is a normal inverse Gaussian process, i.e. $\lambda = -\frac{1}{2}$, the characteristic function χ simplifies to

$$\chi(u) = \exp\left(T_{j-1}(1-iu)\delta\sqrt{\alpha^2-\beta^2}\right.$$
$$\left.+\int_0^{T_{j-1}}\delta\left(iu\sqrt{\alpha^2-(\beta+\lambda(s,T_{j-1}))^2}-\sqrt{\alpha^2-(\beta+iu\lambda(s,T_{j-1}))^2}\right)\mathrm{d}s\right).$$

Hence, equation (4.42) can be calculated numerically in an efficient way. Future research will deal with the fitting of the model to real-life data sets as well as the pricing of swaptions. Another topic of future research is the modeling of defaultable Libor rates, see Lotz and Schlögl (2000), Schönbucher (2000b), and Bielecki and Rutkowski (2002). In analogy to equation (4.8), page 95, the defaultable δ-(forward)-Libor rates are defined by

$$L^{\mathrm{d}}(t,T) := \frac{1}{\delta}\left(\frac{D_C(t,T)}{D_C(t,T+\delta)}-1\right),$$

where $D_C(t,T)$ denotes the time-t price of the defaultable bond with maturity T which can be modeled according to Chapter 3.

Appendix A

Construction of the Conditional Markov Process

The construction that we present in this appendix is a slight modification of the construction in Bielecki and Rutkowski (1999).

The general starting point is a probability space

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}) = (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$$

where the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. This probability space corresponds to the probability space $(\Omega, \mathcal{F}, \mathbf{P}^*, (\mathcal{F}_t)_{0\leq t\leq T^*})$ in Section 3.2.1. The aim is to enlarge this probability space so that a further process, i.e. a conditional Markov process that describes the credit ratings is measurable as well.

For $i, j \in \mathcal{K} = \{1, \ldots, K\}, i \neq j$, let $\lambda_{i,j} : \mathbb{R}_+ \times \Omega \to [0, \infty)$ be $(\mathcal{F}_t)_{t\geq 0}-$adapted Lebesgue-integrable functions and for $i \in \mathcal{K}, \lambda_{i,i} := -\sum\limits_{j\in\mathcal{K}\setminus\{i\}} \lambda_{i,j}$.

Furthermore, we have a Hilbert cube, given by

$$(\Omega^U, \mathcal{F}^U, \mathbb{P}^U) = \left([0,1]^{\mathbb{N}}, \bigotimes_{i\geq 1} \mathcal{B}([0,1]), \bigotimes_{i\geq 1} (\text{Leb}|_{[0,1]}) \right),$$

where Leb denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The elements $\omega^U \in \Omega^U$ are assumed to have the following form $\omega^U = \left(\omega^U_{1,1}, \omega^U_{2,1}, \omega^U_{1,2}, \omega^U_{2,2}, \omega^U_{1,3}, \ldots\right)$. For $i \in \{1, 2\}$ and $j \in \mathbb{N}$ we define the projection $U_{i,j} : \Omega \to [0,1]$ by $U_{i,j}(\omega^U) = \omega^U_{i,j}$. Thus, the projections $U_{i,j}$ are uniformly distributed on $[0,1]$. This probability space, together with some auxiliary functions which will be introduced below, controls the law of jumping times and the law of the jumps.

Moreover we define

$$\left(\overline{\Omega}, 2^{\overline{\Omega}}, \mu\right) = \left(\mathcal{K}, 2^{\mathcal{K}}, \mu\right),$$

where $2^{\overline{\Omega}}$ denotes the set of all subsets of $\overline{\Omega}$, and μ is a probability measure on $\overline{\Omega} = \mathcal{K}$.

The set $\overline{\Omega}$ is the state space of the Markov process and μ is the starting distribution. In the credit risk framework μ is a one-point measure on the rating class that can be observed at time 0.

The product space is given by

$$(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}) = \left(\Omega \times \Omega^U \times \overline{\Omega}, \mathcal{F} \otimes \mathcal{F}^U \otimes 2^{\overline{\Omega}}, \mathbb{P} \otimes \mathbb{P}^U \otimes \mu\right).$$

We define the functions $\mathbb{T} : \mathbb{R}_+ \times \Omega \times [0,1] \times \overline{\Omega} \to \mathbb{R}_+$ by

$$\mathbb{T}(t, \omega, u, \overline{\omega}) = \inf \left\{ s \geq 0 : \exp\left(\int_t^{t+s} \lambda_{\overline{\omega},\overline{\omega}}(v, w)\ \mathrm{d}v \right) \leq u \right\}.$$

and let $\mathbb{C} : \mathbb{R}_+ \times \Omega \times [0,1] \times \overline{\Omega} \to \mathbb{R}_+$ be a random function with the following property

$$\mathrm{Leb}|_{[0,1]} \left(\{u \in [0,1] : \mathbb{C}(t, \omega, u, \overline{\omega}) = j\}\right) = \begin{cases} \frac{\lambda_{\overline{\omega},j}(t,\omega)}{-\lambda_{\overline{\omega},\overline{\omega}}(t,\omega)}, & \text{if } \lambda_{\overline{\omega},\overline{\omega}}(t,\omega) < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Next, to simplify the notation in Bielecki and Rutkowski (1999) we define the auxiliary functions $\mathbb{T}_k : \mathbb{R}_+ \times \widetilde{\Omega} \to \mathbb{R}_+$ and $\mathbb{C}_k : \mathbb{R}_+ \times \widetilde{\Omega} \to \overline{\Omega}, k \in \mathbb{N}$, by

$$\mathbb{T}_k(t, \omega, \omega^U, \overline{\omega}) = \mathbb{T}\left(t, \omega, U_{1,k}\left(\omega^U\right), \overline{\omega}\right)$$

and

$$\mathbb{C}_k\left(t, \omega, \omega^U, \overline{\omega}\right) = \mathbb{C}_k\left(t, \omega, U_{2,k}\left(\omega^U\right), \overline{\omega}\right),$$

respectively. The functions $\mathbb{T}_k$ and $\mathbb{C}_k$ will control the waiting time between the $(k-1)$-th and the k-th jump, denoted by η_k, and the state, $\widetilde{C}_k$, of the conditional Markov process after the k-th jump, respectively.

We will define the sequence of waiting times, η_k, $k = 0, 1, 2, \ldots$, the jumping times, τ_k, $k = 0, 1, 2, \ldots$, and the state after the k-th jump recursively. We start with $t = 0$ and define

$$\eta_0 : \widetilde{\Omega} \to \mathbb{R}, \eta_0(\omega, \omega^U, \overline{\omega}) = 0, \tau_0 = \eta_0, \text{ and } \widetilde{C}_0\left(\omega, \omega^U, \overline{\omega}\right) = \overline{\omega}.$$

The waiting time to the first jump is $\eta_1 : \widetilde{\Omega} \to \mathbb{R}_+$:

$$\begin{aligned} \eta_1\left(\omega, \omega^U, \overline{\omega}\right) &= \mathbb{T}_1\left(\tau_0\left(\omega, \omega^U, \overline{\omega}\right), \omega, \omega^U, \widetilde{C}_0(\omega, \omega^U, \overline{\omega})\right) \\ &= \mathbb{T}\left(0, \omega, U_{1,1}(\omega^U), \overline{\omega}\right). \end{aligned}$$

The jumping time of the first jump is $\tau_1 := \tau_0 + \eta_1$ and the state at τ_1 is a random variable given by $\widetilde{C}_1 : \widetilde{\Omega} \to \overline{\Omega}$,

$$\begin{aligned}\widetilde{C}_1(\omega, \omega^U, \overline{\omega}) &= \mathbb{C}_1\left(\eta_1(\omega, \omega^U, \overline{\omega}), \omega, \omega^U, \widetilde{C}_0(\omega, \omega^U, \overline{\omega})\right)\\ &= \mathbb{C}(\eta_1(\omega, \omega^U, \overline{\omega}), \omega, U_{2,1}(\omega^U), \overline{\omega}).\end{aligned}$$

The waiting time between the $(k-1)$-th and the k-th jump, $\eta_k : \widetilde{\Omega} \to \mathbb{R}_+$, is recursively defined by

$$\begin{aligned}\eta_k(\omega, \omega^U, \overline{\omega}) &= \mathbb{T}_k\left(\tau_{k-1}(\omega, \omega^U, \overline{\omega}), \omega, \omega^U, \widetilde{C}_{k-1}(\omega, \omega^U, \overline{\omega})\right)\\ &= \mathbb{T}\left(\tau_{k-1}(\omega, \omega^U, \overline{\omega}), \omega, U_{1,k}(\omega^U), \widetilde{C}_{k-1}(\omega, \omega^U, \overline{\omega})\right).\end{aligned}$$

The k-th jump happens at $\tau_k := \tau_{k-1} + \eta_k$. Then, the state $\widetilde{C}_k : \widetilde{\Omega} \to \overline{\Omega}$ is

$$\begin{aligned}\widetilde{C}_k(\omega, \omega^U, \overline{\omega}) &= \mathbb{C}_k\left(\eta_k(\omega, \omega^U, \overline{\omega}), \omega, \omega^U, \widetilde{C}_{k-1}(\omega, \omega^U, \overline{\omega})\right)\\ &= \mathbb{C}\left(\eta_k(\omega, \omega^U, \overline{\omega}), \omega, U_{2,k}(\omega^U), \widetilde{C}_{k-1}(\omega, \omega^U, \overline{\omega})\right).\end{aligned}$$

The conditional Markov process that we are interested in is $C = (C^1, C^2)$ and it is given by

$$\begin{aligned}C_t^1 &= \widetilde{C}_{k-1}, \qquad \text{for } t \in [\tau_{k-1}, \tau_k), k \geq 1,\\ C_t^2 &= \begin{cases}\widetilde{C}_{k-1}, & \text{for } t \in [\tau_k, \tau_{k+1}), k \geq 2,\\ \widetilde{C}_0, & \text{for } t \in [0, \tau_2).\end{cases}\end{aligned}$$

C_t^1 is the current rating at time t, while $C_t^2, t \geq \tau_2$, is the previous rating, which has an effect on the recovery rate in the default case.

Properties of η_k and $\widetilde{C}_k$

The following equations hold for every $k \geq 1$

$$\begin{aligned}\widetilde{\mathbb{P}}(\eta_k = 0) &= 0,\\ \widetilde{\mathbb{P}}(\lambda_{\widetilde{C}_k, \widetilde{C}_k}(\tau_{k+1}) < 0) &= 1.\end{aligned}$$

For further properties we need first some notation. If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are two measurable spaces and $\mathcal{X}_1$ is a sub-σ-algebra of $\mathcal{X}$ and $\mathcal{Z}$ is a sub-σ-algebra of $\mathcal{X} \otimes \mathcal{Y}$, then we define

$$\mathcal{X}_1 \vee \mathcal{Z} := \sigma\left((\mathcal{X}_1 \otimes \{\emptyset, Y\}) \cup \mathcal{Z}\right).$$

Let $\mathcal{H}_t^k := \sigma\left(1\!\!1_{\{\tau_k \le s\}} : 0 \le s \le t\right)$ and $\mathcal{G}_t^k := \mathcal{F}_t \vee \mathcal{H}_t^k \vee \sigma(\widetilde{C}_{k-1})$. Then it can be shown that, see Bielecki and Rutkowski (1999),

$$\widetilde{\mathbb{P}}\left(\eta_{k+1} > t \,|\, \mathcal{F}_{\tau_k+t} \vee \mathcal{H}_{\tau_k}^k \vee \sigma\left(\widetilde{C}_k\right)\right) = \exp\left(\int_{\tau_k}^{\tau_k+t} \lambda_{\widetilde{C}_k,\widetilde{C}_k}(v)\, \mathrm{d}v\right),$$

$$\widetilde{\mathbb{P}}(\eta_{k+1} > t) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\exp\left(\int_{\tau_k}^{\tau_k+t} \lambda_{\widetilde{C}_k,\widetilde{C}_k}(v)\, \mathrm{d}v\right)\right],$$

$$\widetilde{\mathbb{P}}\left(\widetilde{C}_{k+1} = j \,|\, \mathcal{G}_{\tau_{k+1}}^{k+1}\right) = \frac{\lambda_{\widetilde{C}_k,j}(\tau_{k+1})}{-\lambda_{\widetilde{C}_k,\widetilde{C}_k}(\tau_{k+1})}.$$

The enlarged probability space that we are interested in is the given filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}^*, (\mathcal{F}_t)_{0\le t\le T^*})$ enlarged by the information given by credit ratings. The enlarged probability space is

$$\left(\widetilde{\Omega}, \mathcal{G}, \mathbf{Q}^*, (\mathcal{G}_t)_{0\le t\le T^*}\right) = \left(\Omega \times \Omega^U \times \overline{\Omega}, \mathcal{F} \otimes \mathcal{F}^U \otimes 2^{\mathcal{K}}, \widetilde{\mathbb{P}}, \left(\mathcal{F}_t \vee \mathcal{F}_t^C\right)_{0\le t\le T^*}\right),$$

where $\mathcal{F}_t^C := \sigma\left(C_s : 0 \le s \le t\right)$. We denote the first time of a change in the credit rating at or after $t \ge 0$ by $\overline{\tau}(t) := \inf\left\{s \ge t : C_s \ne C_{s-}\right\}$. Then the conditional Markov property of C with respect to the filtration holds, i.e.

$$\begin{aligned}\mathbf{Q}^*(C_{\overline{\tau}(t)} = (j,i) \,|\, \mathcal{G}_t) &= \mathbf{Q}^*\left(C_{\overline{\tau}(t)} = (j,i) \,|\, \mathcal{F}_t \vee \mathcal{F}_t^C\right) \\ &= \mathbf{Q}^*\left(C_{\overline{\tau}(t)} = (j,i) \,|\, \mathcal{F}_t \vee \sigma\left(C_t\right)\right).\end{aligned}$$

It is worth noting that the conditions needed for the enlargement of the filtered probability space, i.e. the usual conditions, hold as well in the Bielecki and Rutkowski (1999) Gaussian HJM framework, as well as in the Lévy HJM framework.

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"*Ein Fatum waren aber auch die Naturgesetze, ehe man sie erforschte; nachdem dies geschehen war, ist es sogar gelungen, ihnen eine Technik überzuordnen.*"
Aus: Robert Musil, Der Mann ohne Eigenschaften, Bd. 2