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$L(Q)$ -preservation theorems

$L(Q)$ -PRESERVATION THEOREMS

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§0. Introduction. Much effort has been spent to prove that the reduced product operation preserves, and sometimes even strengthens the L -equivalence of structures, where L is some infinitary language. A similar result is suggested by the following well-known fact:

Assume D is a nonprincipal ultrafilter on ω and, for $n \in \omega$, C_n is a set. If the ultra-product $\prod_{\omega} C_n/D$ is infinite, it has a cardinality $\geq 2^{\aleph_0}$. Hence, by Łos' theorem, (i) if $\mathfrak{A}_n \equiv \mathfrak{B}_n$ and $\varphi(x)$ is a first-order formula, then $\prod_{\omega} \mathfrak{A}_n/D \models Qx\varphi(x)$ iff $\prod_{\omega} \mathfrak{B}_n/D \models Qx\varphi(x)$, where Q is the unary quantifier "there are $2^{\aleph_0}$ many."

We shall prove some generalizations of (i). In particular, we show

(ii) if D is a nonprincipal ultrafilter over $I = \omega$, and $\mathfrak{A}_i \equiv \mathfrak{B}_i$, then $\prod_I \mathfrak{A}_i/D \equiv_{L(Q)} \prod_I \mathfrak{B}_i/D$, where $L(Q)$ is the language obtained from the first-order language by adding the quantifier Q .

(ii) remains true, if D is an ω -regular or an atomless filter over a set I .

Lipner [7] proved that if $2^{\aleph_0}$ is regular, then the $L(Q)$ -equivalence is preserved under direct products. We show that the assumption " $2^{\aleph_0}$ is regular" is necessary. We determine the prefixes for which there is a sentence φ and there are structures $\mathfrak{A}_i$ and $\mathfrak{B}_i$, $i \in I$, such that $\mathfrak{A}_i \equiv_{L(Q)} \mathfrak{B}_i$, $\prod_I \mathfrak{A}_i \models \varphi$, but non $\prod_I \mathfrak{B}_i \models \varphi$.

The results of the present paper were announced in [4].

§1. Preliminaries. Let L be a first-order language. Assume, for simplicity, that L contains no function symbols and only *finitely* many predicate and constant symbols. For any cardinal $\lambda \geq 1$, let $L(Q_\lambda)$ be the language obtained from L by adding the unary quantifier "there exist at least λ many." $\mathfrak{A} \equiv_{Q_\lambda} \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are $L(Q_\lambda)$ -equivalent. In case $\lambda = 2^{\aleph_0}$ we omit the subscript λ , thus denoting by $L(Q)$ the logic $L(Q_{2^{\aleph_0}})$.

By $\prod_I \mathfrak{A}_i/D$ we denote the reduced product of a family $\langle \mathfrak{A}_i | i \in I \rangle$ of structures with respect to the filter D over the set I . For $f \in \prod_I A_i$ let f_D be the element of $\prod_I A_i/D$ containing f . A filter D over I is ω -regular if there is a family $\{X_n | n \in \omega\}$ such that $X_n \in D$ and $\bigcap_n X_n = \emptyset$. The 2-element Boolean algebra with universe $\{0, 1\}$ is denoted by **2**. Call a filter D over I *atomless* if the reduced power $\mathbf{2}^I/D$ is an atomless Boolean algebra. For a subset $X \subset I$ let X_D be the element of $\mathbf{2}^I/D$ containing the characteristic function of X . For $X_D \neq 0$ we denote by $D \upharpoonright X$ the filter $\{X \cap Y | Y \in D\}$.

We state two known results for future reference.

1.1. THEOREM. For any formula $\varphi(v_0, \dots, v_{n-1})$ of L there exist finitely many

formulas $\varphi_0(v_0, \dots, v_{n-1}), \dots, \varphi_m(v_0, \dots, v_{n-1})$, and a formula $\psi(x_0, \dots, x_m)$ of the language of Boolean algebras such that

(a) $\models \varphi_0 \vee \dots \vee \varphi_m$ and $\models \neg(\varphi_s \wedge \varphi_t)$ for $0 \leq s < t \leq m$,

(b) for any family $\langle \mathfrak{A}_i | i \in I \rangle$ of structures, any filter D over I and any $f_0, \dots, f_{n-1} \in \prod_I \mathfrak{A}_i$,

$$\prod_I \mathfrak{A}_i / D \models \varphi[f_{0D}, \dots, f_{n-1D}]$$

iff

$$2^I / D \models \psi \left[S \left(\prod_I \mathfrak{A}_i; \varphi_0; f_0, \dots, f_{n-1} \right), \dots, S \left(\prod_I \mathfrak{A}_i; \varphi_m; f_0, \dots, f_{n-1} \right) \right],$$

where $S(\prod_I \mathfrak{A}_i; \varphi_s; f_0, \dots, f_{n-1}) = \{i | \mathfrak{A}_i \models \varphi_s[f_0(i), \dots, f_{n-1}(i)]\}$ (cf. [3]).

We sometimes denote $f_0, \dots, f_{n-1}$ by f^n and $f_0(i), \dots, f_{n-1}(i)$ by $f^n(i)$.

1.2. THEOREM. Let $\mathfrak{A}$ and $\mathfrak{B}$ be atomless Boolean algebras. Assume that $a_0, \dots, a_m$ and $b_0, \dots, b_m$ are partitions of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then $(\mathfrak{A}, a_0, \dots, a_m) \equiv (\mathfrak{B}, b_0, \dots, b_m)$ iff $\{s | a_s \neq 0\} = \{s | b_s \neq 0\}$ (cf. [8]).

§2.

2.1. LEMMA. Let D be atomless over I . Suppose $X \subset I$ and $X_D \neq 0$. Then there are subsets $X_\xi \subset X$ for $\xi < 2^{\aleph_0}$ such that $X_{\xi_D} \neq 0$ and $X_{\xi_D} \neq X_{\eta_D}$ for $\xi \neq \eta$.

PROOF. Put $Y = I - X$. For $Y_D = 0$ the lemma is obvious. Suppose $Y_D \neq 0$. Then $2^I / D \simeq 2^Y / D \upharpoonright Y \times 2^X / D \upharpoonright X$. Hence $2^X / D \upharpoonright X$ is an atomless Boolean algebra, therefore $|2^X / D \upharpoonright X| \geq 2^{\aleph_0}$.

2.2. LEMMA. For any $\varphi(v_0, \dots, v_n) \in L$ there is a $n_\varphi \geq 1$ such that for any atomless D over I and any structures $\mathfrak{A}_i, i \in I$, $\prod_I \mathfrak{A}_i / D \models \forall v_0 \dots \forall v_{n-1} (Q_{n_\varphi} v_n \varphi \rightarrow Q v_n \varphi)$.

PROOF. Given φ choose $\varphi_0, \dots, \varphi_m$ and ψ such that 1.1 holds. Put $n_\varphi = 2^{m+1} + 1$. If, for some $f_0, \dots, f_{n-1} \in \prod_I \mathfrak{A}_i$, $\prod_I \mathfrak{A}_i / D \models Q_{n_\varphi} v_n \varphi[f_{0D}, \dots, f_{n-1D}]$, then there are $f, g \in \prod_I \mathfrak{A}_i$, $f_D \neq g_D$ such that, for all $s \leq m$,

$$F_{sD} = 0 \quad \text{iff} \quad G_{sD} = 0,$$

where $F_s = S(\prod_I \mathfrak{A}_i; \varphi_s; f^n, f)$ and $G_s = S(\prod_I \mathfrak{A}_i; \varphi_s; f^n, g)$ (for the definition of S see 1.1).

As $\bigcup_{s=0}^m F_s = \bigcup_{s=0}^m G_s = I$, we have $\bigcup_{s,t=0}^m F_s \cap G_t = I$.

Put $X_{f \neq g} = \{i | f(i) \neq g(i)\}$. As $f_D \neq g_D$ we get $s, t \leq m$ such that $X_D \neq 0$ holds for $X = F_s \cap G_t \cap X_{f \neq g}$. Choose X_ξ , $\xi < 2^{\aleph_0}$, as in Lemma 2.1. Define $h_\xi \in \prod_I \mathfrak{A}_i$ by

$$h_\xi(i) = \begin{cases} f(i) & \text{if } i \notin X_\xi, \\ g(i) & \text{if } i \in X_\xi. \end{cases}$$

Then $\{i | h_\xi(i) \neq h_\eta(i)\} \supset (X_\xi - X_\eta) \cup (X_\eta - X_\xi)$. Therefore, $h_{\xi_D} \neq h_{\eta_D}$ holds for $\xi \neq \eta$. For $\xi < 2^{\aleph_0}$ and $r \leq m$ let H_r^ξ be $S(\prod_I \mathfrak{A}_i; \varphi_r; f^n, h_\xi)$. We defined the h_ξ 's such that, for all $r \leq m$,

$$H_{rD}^\xi = 0 \quad \text{iff} \quad F_{rD} = 0.$$

Hence, by 1.1 and 1.2, $\prod_I \mathfrak{A}_i / D \models \varphi[f_{0D}, \dots, f_{n-1D}, h_{\xi_D}]$, and therefore $\prod_I \mathfrak{A}_i / D \models Q v_n \varphi[f_{0D}, \dots, f_{n-1D}]$.

2.3. COROLLARY. Suppose D and E are atomless filters over the sets I and J , respectively. If $\prod_I \mathfrak{A}_i / D \equiv \prod_J \mathfrak{B}_j / E$, then $\prod_I \mathfrak{A}_i / D \equiv_Q \prod_J \mathfrak{B}_j / E$.

PROOF (CF. [4, Lemma 2.7]). Replace by induction in any sentence ψ of $L(Q)$ each subformula of the form $Qv_n\varphi$ by $Q_{n_0}v_n\varphi$. Let ψ^* be the resulting formula of L . Now, by 2.2, for atomless D' over I' and $\langle \mathfrak{C}_i | i \in I' \rangle$,

$$\prod_{I'} \mathfrak{C}_i / D' \models \psi \quad \text{iff} \quad \prod_{I'} \mathfrak{C}_i / D' \models \psi^*.$$

Hence, if $\prod_I \mathfrak{A}_i / D \equiv \prod_I \mathfrak{B}_i / E$, then $\prod_I \mathfrak{A}_i / D \equiv_Q \prod_I \mathfrak{B}_i / E$.

2.4. COROLLARY. Suppose D is an atomless filter over I . If, for $i \in I$, $\mathfrak{A}_i \equiv \mathfrak{B}_i$, then $\prod_I \mathfrak{A}_i / D \equiv_Q \prod_I \mathfrak{B}_i / D$.

We shall assume that the reader is familiar with the r -move game G_r described by Ehrenfeucht in [2] and which may be played on any pair of structures $\mathfrak{A}$ and $\mathfrak{B}$. We write $\mathfrak{A} G_r \mathfrak{B}$ if the second player has a winning strategy in this game. Then,

2.5. THEOREM. (i) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of L with quantifier rank $\leq r$ iff $\mathfrak{A} G_r \mathfrak{B}$.

(ii) $\mathfrak{A} \equiv \mathfrak{B}$ iff, for all $r \in \omega$, $\mathfrak{A} G_r \mathfrak{B}$ (cf. [2]).

Lipner ([7], see also Vinner [9]) has introduced a game G_λ^λ to deal with the language $L(Q_\lambda)$. G_λ^λ is, as G_r , an r -move game between two players, I and II, and is played on a pair $(\mathfrak{A}, \mathfrak{B})$ of structures. There are two types of moves, $\exists$ -moves and Q_λ -moves; both end up with a choice of an element from $\mathfrak{A}$ and one from $\mathfrak{B}$. Player I on each move decides which type it is to be. The $\exists$ -moves are the same as moves in the Ehrenfeucht game, i.e. player I picks an element from $\mathfrak{A}$ or from $\mathfrak{B}$ and player II then picks an element from the other structure. Q_λ -moves take the following form. Player I chooses a subset X of λ elements either all from $\mathfrak{A}$ or all from $\mathfrak{B}$; player II then has to choose a set Y of λ elements from the other structure. Player I then selects y in Y and symmetrically, II picks x in X , thus completing this move. If after r moves the elements $a_0, \dots, a_{r-1}$ and $b_0, \dots, b_{r-1}$ have been successively chosen from $\mathfrak{A}$ and $\mathfrak{B}$, respectively (the chosen subsets are disregarded), then player II has won if the correspondence $a_i \mapsto b_i$ for $i < r$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

We write $\mathfrak{A} G_r^\lambda \mathfrak{B}$ if player II has a winning strategy in this game. Then

2.6. THEOREM. (i) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $L(Q_\lambda)$ of quantifier rank $\leq r$ iff $\mathfrak{A} G_r^\lambda \mathfrak{B}$.

(ii) $\mathfrak{A} \equiv_{Q_\lambda} \mathfrak{B}$ iff, for all $r \in \omega$, $\mathfrak{A} G_r^\lambda \mathfrak{B}$ (cf. [7]).

If $a_0, \dots, a_{n-1} \in A$ and $b_0, \dots, b_{n-1} \in B$ we write

$$a_0, \dots, a_{n-1} \sim_r^\lambda b_0, \dots, b_{n-1} \quad \text{for} \quad (\mathfrak{A}, a_0, \dots, a_{n-1}) G_r^\lambda (\mathfrak{B}, b_0, \dots, b_{n-1}),$$

i.e. by 2.6, if $a_0, \dots, a_{n-1}$ and $b_0, \dots, b_{n-1}$ satisfy the same formulas of $L(Q_\lambda)$ with free variables $v_0, \dots, v_{n-1}$ and quantifier rank $\leq r$. Notice that for $\lambda = 1$ the game G_r^λ is essentially Ehrenfeucht's game G_r . Therefore, we write $\sim_r$ for $\sim_r^1$, and in the case $\lambda = 2^{\aleph_0}$ we denote G_r^λ by G_r^Q and $\sim_r^\lambda$ by $\sim_r^Q$.

2.7. THEOREM. Suppose D is an ω -regular filter over I and, for $i \in I$, $\mathfrak{A}_i \equiv \mathfrak{B}_i$. Then $\prod_I \mathfrak{A}_i / D \equiv_Q \prod_I \mathfrak{B}_i / D$.

PROOF. By ω -regularity there are $X_k \in D$, $k \in \omega$, such that $I = X_0 \supset X_1 \supset \dots$ and $\bigcap_{k \in \omega} X_k = \emptyset$. For $i \in I$ let $k_i \in \omega$ be such that $i \in X_{k_i-1}$, $i \notin X_{k_i}$. Assume $r \in \omega$. For $f_0, \dots, f_{n-1} \in \prod_I A_i$, $g_0, \dots, g_{n-1} \in \prod_I B_i$ and $s \in \omega$, we write

$$f^n R_s g^n \quad \text{iff,} \quad \text{for all } i \in I, f^n(i) \sim_{s \cdot k_i} g^n(i).$$

Thus, if $f^n R_s g^n$, then the correspondence $f_{m_D} \mapsto g_{m_D}$, $m < n$, is a partial isomorphism from $\prod_I \mathcal{A}_i/D$ to $\prod_I \mathcal{B}_i/D$.

We prove that $\prod_I \mathcal{A}_i/D \equiv G^Q \prod_I \mathcal{B}_i/D$ by showing that there is a strategy for player II such that after n moves for the chosen elements from $\prod_I \mathcal{A}_i/D$ and $\prod_I \mathcal{B}_i/D$ there are representatives $f_0, \dots, f_{n-1}$ and $g_0, \dots, g_{n-1}$ such that $f^n R_{r-n} g^n$. The proof is by induction on n . Hence, assume $f^n R_{r-n} g^n$ and let us consider the $(n+1)$ th move: Suppose that player I decides to choose from $\mathcal{A}$ (by symmetry, we only need to consider this case).

$\exists$ -move. I chooses f_D in $\prod_I \mathcal{A}_i/D$. For $i \in I$ pick $g(i) \in B_i$ such that

$$f^n(i), f(i) \sim_{(r-n) \cdot k_i - 1} g^n(i), g(i).$$

In particular, $f^n(i), f(i) \sim_{(r-(n+1)) \cdot k_i} g^n(i), g(i)$, hence $f^n, f R_{r-(n+1)} g^n, g$.

Q -move. Player I chooses $F_D \subset \prod_I \mathcal{A}_i/D$ with $|F_D| = 2^{\aleph_0}$. Let F contain just one representative $\in \prod_I \mathcal{A}_i$ for each element in F_D . Thus $F_D = \{f_D | f \in F\}$. Define an equivalence relation R on F by

$$(1) \quad f R h \text{ iff } \{i | f^n(i), f(i) \sim_{(r-(n+1)) \cdot k_i} f^n(i), h(i)\} \in D.$$

Let $(F^k)_{k < \omega}$ be the R -equivalence classes. If F^k is finite, pick $f^k \in F^k$ and choose g^k as in the $\exists$ -move. Let G^k be $\{g^k\}$. Assume F^k is infinite and let $\bar{h}_0, \bar{h}_1, \dots$ be different elements from F^k . For $k \in \omega$ put

$$Y_k = \{i | f^n(i), \bar{h}_s(i) \sim_{(r-(n+1)) \cdot k_i} f^n(i), \bar{h}_t(i) \text{ for } s, t \leq k\}.$$

Then $Y_k \supset Y_{k+1}$ and, by (1), $Y_k \in D$. Define, for $s \in \omega$, $h_s \in \prod_I \mathcal{A}_i$ by

$$h_s(i) = \begin{cases} \bar{h}_s(i) & \text{if } i \in Y_s, \\ \bar{h}_0(i) & \text{if } i \notin Y_s. \end{cases}$$

It is easy to check that

$$(i) \quad h_{s_D} = \bar{h}_{s_D},$$

$$(ii) \quad \text{for all } i \in I \text{ and all } s, t \in \omega,$$

$$(2) \quad f^n(i), h_s(i) \sim_{(r-(n+1)) \cdot k_i} f^n(i), h_t(i).$$

By (i) we may assume $h_0, h_1, \dots \in F$. Let $J = \{i | \text{there are } s, t \in \omega \text{ such that } h_s(i) \neq h_t(i)\}$. Obviously, $J_D \neq \emptyset$.

Case 1. $2^J/D \upharpoonright J$ is infinite, and hence has cardinality $\geq 2^{\aleph_0}$. Assume $i \in J \cap X_1$. Choose $s, t \in \omega$ such that $h_s(i) \neq h_t(i)$. $i \in X_1$ implies $k_i \geq 2$. Thus, we find $g^0(i), g^1(i) \in B_i$ such that

$$(3) \quad f^n(i), h_s(i), h_t(i) \sim_{(r-(n+1)) \cdot k_i} g^n(i), g^0(i), g^1(i).$$

For $i \notin J \cap X_1$ choose $g(i) \in B_i$ with

$$(4) \quad f^n(i), h_0(i) \sim_{(r-(n+1)) \cdot k_i} g^n(i), g(i).$$

For $\eta \in 2^J$ define $g_\eta \in \prod_I B_i$ by

$$g_\eta(i) = \begin{cases} g(i) & \text{if } i \notin J \cap X_1, \\ g^{\eta(i)}(i) & \text{if } i \in J \cap X_1. \end{cases}$$

As $|2^J/D \upharpoonright J| \geq 2^{\aleph_0}$, we have $|\{g_{\eta_D} | \eta \in 2^J\}| \geq 2^{\aleph_0}$. Thus, there is a subset G^z of

$\{g_\eta | \eta \in 2^J\}$, $|G^\xi| = |F^\xi|$, and $g_D \neq \bar{g}_D$ for $g, \bar{g} \in G^\xi$. By (2), (3) and (4), for $s \in \omega$ and $g \in G^\xi$, $f^n, h_s R_{r-(n+1)} g^n, g$. Put $f^\xi = h_0$.

Case 2. $2^I/D \upharpoonright J$ is finite. Thus there is a finite partition $J_0, \dots, J_k$ of J such that $D_l = D \upharpoonright J_l$ is an ultrafilter over J_l ($l \leq k$). For $i \in I$ let H_i be $\{h_s(i) | s \in \omega\}$. Then

$$\{h_{s_D} | s \in \omega\} \subset \prod_I H_i/D \simeq \prod_{I-J} H_i/D \upharpoonright (I-J) \times \prod_{J_0} H_i/D_0 \times \dots \times \prod_{J_k} H_i/D_k.$$

As $\{h_{s_D} | s \in \omega\}$ is infinite and the first factor on the right side contains just one element, $\prod_{J_e} H_i/D_e$ is infinite for some $e \leq k$. By Łos' theorem, for $m \geq 0$,

$$J_e^m = \{i | |H_i| > m, i \in J_e\} \in D_e.$$

For $i \in J_e$ let m_i be such that $i \in J_e^{m_i-1} \cap X_{m_i-1}$, $i \notin J_e^{m_i} \cap X_{m_i}$. Pick m_i elements from H_i , say $h_{s_0}(i), \dots, h_{s_{m_i-1}}(i)$. As $m_i \leq k_i$ we find $g_0(i), \dots, g_{m_i-1}(i) \in B_i$ such that

$$(5) \quad f^n(i), h_{s_0}(i), \dots, h_{s_{m_i-1}}(i) \sim_{(r-(n+1)) \cdot k_i} f^n(i), g_0(i), \dots, g_{m_i-1}(i).$$

For $i \notin J_e$ choose $g(i) \in B_i$ such that

$$(6) \quad f^n(i), h_0(i) \sim_{(r-(n+1)) \cdot k_i} f^n(i), g(i).$$

D_e is ω -regular and, for $t \in \omega$, $\{i | m_i \geq t\} \supset J_e^t \cap X_t \in D_e$ holds. Thus, $|\prod_{J_e} m_i/D| \geq 2^{\aleph_0}$. Therefore, $|\{g_\eta | \eta \in 2^J\}| \geq 2^{\aleph_0}$. Hence, we find a subset G^ξ of $\{g_\eta | \eta \in \prod_{J_e} m_i\}$, $|G^\xi| = |F^\xi|$ such that $g_D \neq \bar{g}_D$ for $g, \bar{g} \in G^\xi$.

By (2), (5) and (6), for all $s \in \omega$ and $g \in G^\xi$,

$$f^n, h_s R_{r-(n+1)} g^n, g.$$

Put $f^\xi = h_0$. Let $G = \bigcup_{\xi < \lambda} G^\xi$. Player II chooses the subset $G_D = \{g_D | g \in G\}$ of $\prod_I B_i/D$. If now player I chooses $g_D \in G_D$, say $g \in G^\xi$, player II responds by choosing f^ξ .

We needed the ω -regularity of D only in Case 2, i.e. only if $2^I/D$ has an atom. Thus the above proof contains a proof of Corollary 2.4, but does not yield the stronger version 2.3.

In case $2^{\aleph_0} = \aleph_1$, 2.7 is contained in [1] and [6]. It should be clear to the reader familiar with [1] how to change the above proof to obtain the following stronger version of 2.7.

2.8. THEOREM. Let L be a countable language. Suppose D is an ω -regular filter over I and, for $i \in I$, $\mathfrak{A}_i \equiv \mathfrak{B}_i$. Then $\prod_I \mathfrak{A}_i/D \equiv_{L, \omega_1(Q)} \prod_I \mathfrak{B}_i/D$.

One consequence of Lipner's results in [7] is

$$(7) \quad \text{if } |I| \leq \aleph_0 \text{ and for } i \in I, \mathfrak{A}_i \equiv_Q \mathfrak{B}_i, \text{ then } \prod_I \mathfrak{A}_i \equiv_Q \prod_I \mathfrak{B}_i.$$

2.9. THEOREM. Assume $|I| \leq \aleph_0$ and let D be a filter over I . If, for $i \in I$, $\mathfrak{A}_i \equiv_Q$ then $\prod_I \mathfrak{A}_i/D \equiv_Q \prod_I \mathfrak{B}_i/D$.

PROOF. By (7) we may assume that $I = \omega$. For $n \in \omega$ choose $X_n \in D$ with $n \notin X_n$, if there is such an X_n ; otherwise put $X_n = I$. Let $X = \bigcap_n X_n$. Suppose $X \notin D$ and put $J = I - X$. Then $D \upharpoonright J$ is an ω -regular filter. Thus, by 2.7 and (7),

$$\prod_I \mathfrak{A}_i/D \simeq \prod_X \mathfrak{A}_i \times \prod_J \mathfrak{A}_i/D \upharpoonright J \equiv_Q \prod_X \mathfrak{B}_i \times \prod_J \mathfrak{B}_i/D \upharpoonright J \simeq \prod_I \mathfrak{B}_i/D.$$

Use (7) in case $X \in D$.

2.10. THEOREM. (a) Let D be an ultrafilter over I . If, for $i \in I$, $\mathfrak{A}_i \equiv_Q \mathfrak{B}_i$, then $\prod_I \mathfrak{A}_i / D \equiv_Q \prod_I \mathfrak{B}_i / D$.

(b) Assume $|I|$ is less than the first measurable cardinal and D is any nonprincipal ultrafilter over I . If, for $i \in I$, $\mathfrak{A}_i \equiv \mathfrak{B}_i$, then $\prod_I \mathfrak{A}_i / D \equiv_Q \prod_I \mathfrak{B}_i / D$.

PROOF. If D is a nonprincipal, ω -complete ultrafilter, then D is κ -complete and $|I| \geq \kappa$, where κ is the first measurable cardinal. Then D preserves $L_{\kappa\kappa}$ -equivalence, in particular $L(Q)$ -equivalence.

Is 2.10(a) true for filters? We prove in the next section that we get a negative answer, if $2^{\aleph_0}$ is singular. We do not know if 2.10(a) is true for filters in case $2^{\aleph_0}$ is regular.

§3. Throughout this section let λ be an infinite cardinal. Define the cardinals $j(\lambda)$ and $k(\lambda)$ by:

(a) $j(\lambda)$ is the least μ such that $2^\mu \geq \lambda$.

(b) $k(\lambda)$ is the least μ such that, for some family $\langle \lambda_\xi \mid \xi < \mu \rangle$ of cardinals $\lambda_\xi < \lambda$, $\prod_{\xi \in \mu} \lambda_\xi \geq \lambda$.

For any λ , we have $k(\lambda) \leq j(\lambda) \leq \lambda$. We say that $E(\lambda)$ holds, if $k(\lambda) = j(\lambda)$. If $E(\lambda)$ holds, then $2^{\text{cf}(\lambda)} \geq \lambda$. As $k(2^{\aleph_0}) = j(2^{\aleph_0}) = \aleph_0$, $E(2^{\aleph_0})$ holds.

Lipner proves that the $L(Q_\lambda)$ -equivalence is preserved by direct products, if λ is regular and $E(\lambda)$ holds. He shows that the assumption “ $E(\lambda)$ holds” is necessary but leaves open the question if the regularity of λ is necessary. Lipner’s proof and the proof in [10] are “global” as will be the proof of 3.4 below. We sketch a local proof of Lipner’s theorem, i.e. player II obtains a winning strategy for the product structure by playing in each component according to a winning strategy.

3.1. THEOREM. Assume $E(\lambda)$ holds and λ is regular. If, for $i \in I$, $\mathfrak{A}_i \equiv_{Q_\lambda} \mathfrak{B}_i$, then $\prod_I \mathfrak{A}_i \equiv_{Q_\lambda} \prod_I \mathfrak{B}_i$.

PROOF (SKETCH). For $f_0, \dots, f_{n-1} \in \prod_I A_i$, $g_0, \dots, g_{n-1} \in \prod_I B_i$ and $s \in \omega$ we write

$$f^n R_s g^n \quad \text{iff} \quad \text{for all } i \in I, f^n(i) \sim_{2_s}^{\lambda} g^n(i).$$

For $r \in \omega$ we prove $\prod_I \mathfrak{A}_i \equiv_{Q_\lambda} \prod_I \mathfrak{B}_i$ by showing that there is a strategy for player II such that if after n moves $f_0, \dots, f_{n-1}$ and $g_0, \dots, g_{n-1}$ have been chosen, then $f^n R_{r-n} g^n$. The proof is by induction on n and the $\exists$ -move is handled similar to the $\exists$ -move in 2.7. Assume that $f^n R_{r-n} g^n$ and that in the $(n+1)$ th move player I chooses $F \subset \prod_I A_i$ with $|F| = \lambda$. If there is an $i_0 \in I$ such that $|\{f(i_0) \mid f \in F\}| = \lambda$ we do a Q -move in the i_0 -component and an $\exists$ -move for $i \neq i_0$. Otherwise, for all $i \in I$,

$$(1) \quad |\{f(i) \mid f \in F\}| < \lambda.$$

Then, define an equivalence relation R on F by

$$f R h \quad \text{iff} \quad f^n, f R_{r-(n+1)} f^n, h.$$

If R has λ many equivalence classes, choose, for each $f \in F$, a $g_f \in \prod_I B_i$ as in the $\exists$ -move. Then, $|\{g_f \mid f \in F\}| = \lambda$ and player II may choose this subset. If R has less than λ equivalence classes, by regularity of λ there is a class F_0 with λ many ele-

ments. Let J be $\{i \mid \text{there are } f, \bar{f} \in F_0 \text{ with } f(i) \neq \bar{f}(i)\}$. By (1) and $E(\lambda)$, $|2^J| \geq \lambda$. Now we handle this situation similar to Case 1 of the Q -move in 2.7.

Lipner's global proof yields a stronger result, which we state in the terminology of this paper. Fix n : As there are only finitely many nonequivalent formulas of $L(Q_\lambda)$ with free variables $v_0, \dots, v_{n-1}$ and quantifier rank $\leq r$, the relation $\sim_r^\lambda$ has only finitely many equivalence classes, say $E_{n,r}^0, \dots, E_{n,r}^k$, where $k = k(n, r)$.

Then 3.1 is a special case of

3.2. THEOREM. *Assume λ is regular and $E(\lambda)$ holds. Suppose $f_0, \dots, f_{n-1} \in \prod_i \mathfrak{A}_i$, and $g_0, \dots, g_{n-1} \in \prod_j \mathfrak{B}_j$, and $r \in \omega$. For $s \leq k(n, 2r)$ put*

$$Y_s = \{i \in I \mid (f^n(i)) \in E_{n,2r}^s\} \quad \text{and} \quad Z_s = \{j \in J \mid (g^n(j)) \in E_{n,2r}^s\}.$$

Then $f^n \sim_r^\lambda g^n$, if the following conditions are satisfied:

- (a) $|Y_s| = |Z_s|$, if $|Y_s|$ is finite,
- (b) $\aleph_0 \leq |Y_s| < j(\lambda)$ iff $\aleph_0 \leq |Z_s| < j(\lambda)$,
- (c) $j(\lambda) \leq |Y_s| < \lambda$ iff $j(\lambda) \leq |Z_s| < \lambda$,
- (d) $\lambda \leq |Y_s|$ iff $\lambda \leq |Z_s|$.

The proof is similar to that of 3.4 below.

3.3. EXAMPLES. (a) Let λ be singular, say $\lambda = \bigcup_{\xi \in \mu} \mu_\xi$ where $\aleph_0 \leq \mu$, $\mu_\xi < \lambda$. Assume L has a binary predicate $\leq$ and an individual constant c . Let ψ be

$$\forall x c \leq x \wedge \forall x (x \leq x \wedge \forall y (x \leq y \rightarrow (x = c \vee x = y))).$$

Assume that, for $\xi \in \mu$, $\mathfrak{A}_\xi$ and $\mathfrak{B}_\xi$ are models of ψ , $|A_\xi| = \mu_\xi$ and $|B_\xi| = \aleph_0$. Then $\mathfrak{A}_\xi \equiv_{Q_\lambda} \mathfrak{B}_\xi$. Let $f \in \prod_\mu \mathfrak{A}_\xi$. If there is a ξ_0 such that $f(\xi) = c^{\mathfrak{A}_\xi}$ for $\xi \neq \xi_0$ and $f(\xi_0) \neq c^{\mathfrak{A}_{\xi_0}}$, then

$$\prod_\mu \mathfrak{A}_\xi \models \forall y ((y \leq x \wedge \neg y = x) \rightarrow y = c)[f].$$

Thus (recall $\lambda = \bigcup_{\xi \in \mu} \mu_\xi$), $\prod_\mu \mathfrak{A}_\xi \models \varphi$ where

$$\varphi = Q_\lambda x \forall y ((y \leq x \wedge \neg y = x) \rightarrow y = c).$$

Let $g \in \prod_\mu \mathfrak{B}_\xi$. If $\prod_\mu \mathfrak{B}_\xi \models \forall y ((y \leq x \wedge \neg y = x) \rightarrow y = c)[g]$, then there is a ξ_0 such that $g(\xi) = c^{\mathfrak{B}_\xi}$ for $\xi \neq \xi_0$ and $g(\xi_0) \neq c^{\mathfrak{B}_{\xi_0}}$. But there are only $\mu \cdot \aleph_0$, hence less than λ , such g 's. Thus $\prod_\mu \mathfrak{B}_\xi \models \neg \varphi$. In particular, non $\prod_\mu \mathfrak{A}_\xi \equiv_{Q_\lambda} \prod_\mu \mathfrak{B}_\xi$.

(b) Assume $2^{\aleph_0}$ is singular. Let c, d be two individual constants and $\leq$ a binary predicate symbol. It is not difficult to construct (similar to a) for $n \in \omega$ structures $\mathfrak{A}_n$ and $\mathfrak{B}_n$ such that $\mathfrak{A}_n \equiv_Q \mathfrak{B}_n$, $\prod_\omega \mathfrak{A}_n \models Qx \neg Qy y \leq x$, and non $\prod_\omega \mathfrak{B}_n \models Qx \neg Qy y \leq x$.

Thus not every formula whose prefix contains Q and $\forall$, or Q and $\neg Q$ is "product-stable". If $E(\lambda)$ holds, every formula only containing Q and $\exists$ is product-stable. More precisely:

Call φ a $(Q_\lambda, \exists)$ -formula if φ belongs to the smallest subset of $L(Q_\lambda)$ which contains all atomic formulas and negations of atomic formulas and is closed under $\wedge, \vee, Q_\lambda$ and $\exists$.

3.4. THEOREM. *Suppose $E(\lambda)$ holds. Let φ be any $(Q_\lambda, \exists)$ -sentence. If for $i \in I$ $\mathfrak{A}_i \equiv_{Q_\lambda} \mathfrak{B}_i$, then*

$$\prod_I \mathfrak{A}_i \models \varphi \quad \text{iff} \quad \prod_I \mathfrak{B}_i \models \varphi.$$

PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two structures. Let $\bar{G}_r^\lambda$ be the game obtained from G_r^λ by the following modification: In each move player I *has to pick from the structure* $\mathfrak{A}$ an element in an $\exists$ -move and a subset of power λ in a Q_λ -move. An analysis of the proof of 2.6 shows that

(2) every $(Q_\lambda, \exists)$ -sentence of quantifier rank $\leq r$ holding in $\mathfrak{A}$ holds in $\mathfrak{B}$ iff $\mathfrak{A} \bar{G}_r^\lambda \mathfrak{B}$.

By symmetry, it suffices to prove that, for all $r \in \omega$, $\prod_I \mathfrak{A}_i \bar{G}_r^\lambda \prod_I \mathfrak{B}_i$. We do this by showing there is a strategy for the second player with the following property:

Assume after n moves elements $f_0, \dots, f_{n-1} \in \prod_I A_i$ and $g_0, \dots, g_{n-1} \in \prod_I B_i$ have been chosen. For $s \leq k(n, 2r)$ (for notation see the remarks preceding 3.2) put

$$Y_s = \{i \in I \mid (f^n(i)) \in E_{n, 2(r-n)}^s\} \quad \text{and} \quad Z_s = \{i \in I \mid (g^n(i)) \in E_{n, 2(r-n)}^s\}.$$

Then

- (a) $|Y_s| \leq |Z_s|$.
- (b) $Y_s = \emptyset$ iff $Z_s = \emptyset$.

The proof is by induction on n . Thus, suppose $f_0, \dots, f_{n-1}, g_0, \dots, g_{n-1}$ satisfy (a) and (b), and in the $(n+1)$ th move player I does an $\exists$ -move and picks $f \in \prod_I A_i$. Suppose $s \leq k(n, 2(r-n))$ and $Y_s \neq \emptyset$; let $j: Y_s \rightarrow Z_s$ be injective. If $i \in Y_s$ and $(f^n(i), f(i)) \in E_{n+1, 2(r-(n+1))}^{e_i}$, choose $g(j(i)) \in B_{j(i)}$ such that $(g^n(j(i)), g(j(i))) \in E_{n+1, 2(r-(n+1))}^{e_i}$. Fix $i_0 \in Y_s$. If $i \in Z_s$, $i \notin j(Y_s)$ choose $g(i) \in B_i$ with $(g^n(i), g(i)) \in E_{n+1, 2(r-(n+1))}^{e_{i_0}}$. It is easy to see that f^n, f and g^n, g satisfy (a) and (b).

Next we consider the case when player I makes a Q_λ -move and chooses a subset $F \subset \prod_I A_i$ of power λ .

Case 1. There is an $i_0 \in I$ such that $|\{f(i_0) \mid f \in F\}| = \lambda$. Then, for some $e < k(n+1, 2(r-(n+1)))$ there are λ many $f \in F$ such that

$$(f^n(i_0), f(i_0)) \in E_{n+1, 2(r-(n+1))}^e.$$

If $i_0 \in Y_s$ pick any $j \in Z_s$ and choose λ many elements $g^z(j) \in B_j$ ($\xi < \lambda$) with

$$(g^n(j), g^z(j)) \in E_{n+1, 2(r-(n+1))}^e.$$

Now this case is finished similarly to the $\exists$ -move.

Case 2. For all $i \in I$, $|\{f(i) \mid f \in F\}| < \lambda$. Then, as $E(\lambda)$ holds, $2^{|J|} \geq \lambda$, where $J = \{i \mid |\{f(i) \mid f \in F\}| \geq 2\}$. Thus, $|J| \geq j(\lambda)$.

Case 2a. For some $s \leq k(n, 2(r-n))$, some $f \in F$ there are $i, j \in Y_s$ and $e, m \leq k(n+1, 2(r-(n+1)))$, $e \neq m$, such that

$$|J \cap Y_s| \geq j(\lambda), \quad (f^n(i), f(i)) \in E_{n+1, 2(r-(n+1))}^e$$

and

$$(f^n(j), f(j)) \in E_{n+1, 2(r-(n+1))}^m.$$

Choose a subset $Z \subset Z_s$ such that $|Z| \geq j(\lambda)$ and $|Z_s - Z| = |Z_s|$. For $j \in Z$ choose $g^0(j), g^1(j) \in B_j$ with $(g^n(j), g^0(j)) \in E_{n+1, 2(r-(n+1))}^e$, $(g^n(j), g^1(j)) \in E_{n+1, 2(r-(n+1))}^m$.

Now this case is finished by extending, similar to the $\exists$ -move, for $\eta \in 2^Z$, $\langle g^{\eta(j)}(j) \mid j \in Z \rangle$ to an element $g^\eta \in \prod_I B_i$.

Case 2b. Assume otherwise that for all $s \in \omega$ such that $|J \cap Y_s| \geq j(\lambda)$, all $f \in F$ there is an $e \leq k(n+1, 2(r-(n+1)))$ such that, for all $i \in Y_s$, $(f^n(i), f(i)) \in$

$E_{n+1, 2(r-(n+1))}^e$. Then, by a cardinality argument, for some s there are $i_0 \in Y_s$ and $f, h \in F$ such that $|J \cap Y_s| \geq j(\lambda)$, $f(i_0) \neq h(i_0)$ and

$$f^n(i_0), f(i_0) \sim_{2(r-(n+1))}^\lambda f^n(i_0), h(i_0).$$

For $j \in Z_s$ choose $g^0(j), g^1(j) \in B_j$ with

$$g^n(j), g^0(j), g^1(j) \sim_{2(r-(n+1))}^\lambda f^n(i_0), f(i_0), h(i_0).$$

Now finish this case by extending each $\langle g^{n(j)}(j) | j \in Z_s \rangle$, $\eta \in 2^{Z_s}$, to an element $g^n \in \prod_i B_i$.

As the proof shows, the hypothesis of 3.4 can be weakened considerably. The proof is global; thus we need only global assumptions as in 3.2. Secondly, whenever in components $(\mathfrak{A}_i, f^n(i))$ and $(\mathfrak{B}_j, g^n(j))$ was played, player I chose from $(\mathfrak{A}_i, f^n(i))$. Thus we do not need the full $L(Q_\lambda)$ -equivalence of $\mathfrak{A}_i$ and $\mathfrak{B}_i$.

REFERENCES

- [1] M. BENDA, *Reduced products and non-standard logics*, this JOURNAL, vol. 34 (1969), pp. 424–436.
- [2] A. EHRENFUCHT, *An application of games to the completeness problem for formalized theories*, *Fundamenta Mathematicae*, vol. 49 (1961), pp. 129–141.
- [3] S. FEFERMAN and R. VAUGHT, *The first-order properties of algebraic systems*, *Fundamenta Mathematicae*, vol. 47 (1959), pp. 57–103.
- [4] J. FLUM, *Generalized quantifiers and reduced products*, *Notices of the American Mathematical Society*, vol. 21 (1974).
- [5] ———, *On Horn theories*, *Mathematische Zeitschrift*, vol. 138 (1974), pp. 205–212.
- [6] H. J. KEISLER, *Formulas with linearly ordered quantifiers*, *The syntax and semantics of infinitary languages*, Lecture Notes in Mathematics, vol. 72 (1968), pp. 96–130.
- [7] L. D. LIPNER, *Some aspects of generalized quantifiers*, Thesis, University of California, Berkeley, 1970.
- [8] T. SKOLEM, *Untersuchungen über die Axiome des Klassenkalküls und über die "Productions und Summationsprobleme", welche gewisse Klassen von Aussagen betreffen*, *Skrifter utgit av Videnskapsselskapet i Kristiania, I. Klasse* no. 3, Oslo 1919.
- [9] S. VINNER, *A generalization of Ehrenfeucht's game and some applications*, *Israel Journal of Mathematics*, vol. 12 (1972), pp. 279–298.
- [10] A. WOJCIECHOWSKA, *Generalized products for Q -languages*, *Bulletin de l'Académie Polonaise des Sciences*, vol. 17 (1969), pp. 337–339.

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