The Harder-Narasimhan Filtrations and Rational Contractions

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Introduction

Special subsheaves of the tangent sheaf and Harder-Narasimhan Filtration

The study of special subsheaves E of the tangent sheaf $\mathcal{T}_X$ of a projective variety X reveals the geometry of underlying space. For example, Miyaoka use the existence of positive subsheaf E of the tangent sheaf $\mathcal{T}_X$ to deduce the uni-ruledness of X, i.e., there is rational curve passing through a general point of X. More precisely, he showed the following theorem:

Theorem 0.0.1 (cf. [MP97, Theorem 2.14] and references there)**.** *Let X be a normal, projective variety defined over the complex numbers $\mathbb{C}$. If there exist ample line bundles $H_1, \ldots, H_{\dim X-1}$ and a subsheaf $E \subset \mathcal{T}_X$ such that*

$$c_1(E) \cdot H_1 \cdot \ldots \cdot H_{\dim X-1} > 0,$$

then X is uni-ruled.

Here the existence of positive subsheaf with respect to $H_1, \ldots, H_{\dim X-1}$ can be rephrased as follows. Given ample line bundles $\{H_i\}$, we can use $H_1, \ldots, H_{\dim X-1}$ to filtrate tangent sheaf $\mathcal{T}_X$ into

$$\mathrm{HNF}_{\{H_i\}}(\mathcal{T}_X) : 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{T}_X,$$

which is called the Harder-Narasimhan filtration, by sorting out subsheaves of maximal *normalized degree* with respect to $\{H_i\}$ of tangent sheaf $\mathcal{T}_X$ repeatedly.

Definition-Theorem 0.0.2 (Harder-Narasimhan filtration,[HL97, Theorem 1.3.4])**.** *Let X be a normal, projective variety over the complex numbers, $\mathcal{T}_X$ the tangent sheaf, and $H_1, \ldots, H_{\dim X-1}$ ample line bundles. There exists a unique filtration of $\mathcal{T}_X$, the* Harder-Narasimhan filtration with respect to $\{H_i\}$*, depending on the chosen ample line bundles*

$$\mathrm{HNF}_{\{H_i\}}(\mathcal{F}) := 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{T}_X,$$

with the following properties:

(i) The sheaves $\mathcal{F}_i$ are saturated in $\mathcal{F}$, i.e., $\mathcal{F}/\mathcal{F}_i$ is torsion free.

(ii) The quotients $\mathcal{G}_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ are torsion-free and for any $0 \neq \mathcal{F} \subsetneq \mathcal{G}_i$ with $\operatorname{rank} F < \operatorname{rank} \mathcal{G}_i$ we have

$$\frac{c_1(\mathcal{F}).H_1.\ldots.H_{\dim X-1}}{\operatorname{rank} \mathcal{F}} \leq \frac{c_1(\mathcal{G}_i).H_1.\ldots.H_{\dim X-1}}{\operatorname{rank} \mathcal{G}_i}.$$

(iii) Furthermore, we have

$$\mu_{\{H_i\},\max}(\mathcal{T}_X) := \frac{c_1(\mathcal{G}_1).H_1.\dots.H_{\dim X-1}}{\operatorname{rank}\mathcal{G}_1} > \dots > \frac{c_1(\mathcal{G}_k).H_1.\dots.H_{\dim X-1}}{\operatorname{rank}\mathcal{G}_k}.$$

With this insight, we can rephrase Miyaoka's theorem as follows:

Theorem 0.0.3 (cf. [MP97, Theorem 2.14])**.** *Let X be a normal, projective variety over the complex numbers $\mathbb{C}$. If there exist ample line bundles $H_1, \dots, H_{\dim X-1}$, such that*

$$\mu_{\{H_i\},\max}(\mathcal{T}_X) > 0,$$

then X is uni-ruled.

Relation between Harder-Narasimhan filtration and rational maps

Problems

The terms of Harder-Narasimhan filtrations possess a unique property. If

$$c_1(\mathcal{F}_1).H_1.\dots.H_{\dim X-1} > 0,$$

then the sheaf $\mathcal{F}_1$ is closed under Lie bracket, i.e., an integrable subsheaf, cf. Proposition 1.3.32. The classic Frobenius theorem asserts that sheaf $\mathcal{F}_1$ is the relative tangent sheaf of some rational map $\phi : U \to Y$, $U \subset X$ an open set. It is natural to ask the following question.

Question 0.0.4 (Baby version)**.** *For a given relative tangent sheaf $\mathcal{T}_f$ of a rational map f, can we find ample line bundles $\{H_i\}$ such that $\mathcal{T}_f$ is a term of $\mathrm{HNF}_{\{H_i\}}(\mathcal{T}_X)$? Furthermore, if that is the case, what is the relation between rational map f and ample line bundles $\{H_i\}$?*

To make the second question more precise, we introduce the cone of movable curves.

Definition 0.0.5. *Let X be a smooth projective variety. The cone of movable curves $\overline{\mathrm{Mov}}_1(X)$ is the closure of following cone*

$$\mathrm{Mov}_1(X) := \{C \in N_1(X)_{\mathbb{R}} : C.E \geq 0, \textit{for all effective Cartier divisor } E\}$$

in $N_1(X)_{\mathbb{R}}$.

The cone of movable curves is naturally related to rational maps of fibre type, since it contains all general complete intersection curves in general fibre of such maps. Furthermore, the construction of Harder-Narasimhan filtration $\mathrm{HNF}_{\{H_i\}}(\mathcal{T}_X)$ with respect to ample line bundles $\{H_i\}$ can be generalized to Harder-Narasimhan filtration $\mathrm{HNF}_{\alpha}(\mathcal{T}_X)$ with respect to movable curve class α, cf. Theorem 1.3.20.

Question 0.0.6 (Main problem). *For a given tangent sheaf $\mathcal{T}_f$ of a rational map f, can we find a movable curve class α such that $\mathcal{T}_f$ is a term of $\mathrm{HNF}_\alpha(\mathcal{T}_X)$? If that is the case, what is relation between rational map f and the movable curves class α? Alternatively, what is the relation between rational map f and the geometry of the cone of movable curves?*

Known results

For various rational maps, the first half of Question 0.0.6 have an affirmative answer. Sola Conde and Toma considered the case when rational map f is the *Maximal rationally connected quotient* map of an algebraic variety X, an important rational map related whether any two points can by connected by a chain of rational curves. They showed that for a uni-ruled, projective manifold, there exists a movable curve class such that the relative tangent sheaf of the maximal rationally connected quotient is part of the Harder-Narasimhan filtration of the tangent sheaf with respect to such a curve class (cf. [SCT, Theorem 1.1]). Neumann considered the case when rational map f is composition of maps in the Minimal Model program. He showed that for a smooth Fano 3-fold each term of the Harder-Narasimhan filtration of the tangent sheaf, with respect to a given movable curve class, is the relative tangent sheaf of a (not necessarily elementary) Mori fibration (cf. [Neu10, Theorem 4.1]).

Rational contractions

We investigate a special kind of rational maps which fit into Question 0.0.6 quite well, the *Rational contractions.* Rational contractions are natural generalization of birational contractions to non-birational case. Intuitively, they are rational maps which are composition of small birational maps with a contraction morphism.

Definition 0.0.7 (Rational contraction, cf. Definition 1.4.1). *Let $f : X \dashrightarrow Y$ be a dominant rational map between normal projective varieties. We say that f is a* rational contraction*, if there exists a resolution of f*

$$\begin{tikzcd} X' \arrow[d, "\mu"] \arrow[dr, "f'"] & \\ X \arrow[r, dashed, "f"] & Y, \end{tikzcd}$$

where X' is smooth and projective, μ is birational, and for every μ-exceptional effective divisor E on X' we have

$$f'_*\mathcal{O}_{X'}(E) = \mathcal{O}_Y.$$

For a $\mathbb{Q}$-Cartier divisor $D \subset Y$, $f^*(D)$ is defined to be $\mu_*(f'^*(D))$.

Remark 0.0.8. *In our case, toric varieties, rational contractions are indeed composition of small birational maps followed by a contraction morphism(cf. Corollary 1.4.15).*

We note that for a rational contraction $f : X \dashrightarrow Y$, the Neron-Severi group of Y can be identified as a subgroup of the Neron-Severi group of X. Intuitively, this follows from that f is composition of small rational maps and a contraction morphism.

Results

Main Result

In this paper, we investigate both problems in Question 0.0.6 for rational contractions between tori varieties. We prove that the relative tangent sheaf of a rational contraction between toric varieties is a term of Harder-Narasimhan filtration of the tangent sheaf with respect to a specified movable curve C. In fact, we can choose C to be general complete intersection curve in a general fibre of such map. Furthermore, as a byproduct, the rational contraction and rational map defined by the foliation associated to the relative tangent sheaf coincide generically. Specifically, we prove the following theorem.

Theorem 0.0.9 (cf. Theorem 3.3.16, Corollary 3.3.17)**.** *Let $\phi : X \dashrightarrow Z$ be a rational contraction, where Z is normal, projective and $\dim Z < \dim X$. Then the relative tangent sheaf of ϕ is a term of the Harder-Narasimhan filtration of the tangent sheaf $\mathcal{T}_X$ with respect to a suitable movable curve class. Furthermore, the rational map ϕ and rational map*

$$\begin{array}{rccc} q: & X & \dashrightarrow & \mathrm{Chow}(X) \\ & x & \to & \text{‘}\mathcal{T}_{X/Z}\text{-leaf through } x\text{'} \end{array}$$

coincide on an open set of X.

The class of rational contractions contains a vast variety of interesting maps, including all the maps in the minimal model program. Since the important of the Minimal Model Program, we extract the following statement from Theorem 0.0.9. The following theorem shows that we can reconstruct the end result of the minimal model program via the study of foliations associated to terms of the Harder-Narasimhan filtration of the tangent sheaf and geometry of the cone of movable curves.

Theorem 0.0.10 (cf. Section 3.2)**.** *Let X be a $\mathbb{Q}$-factorial, projective, toric variety. Then for any K_X-minimal model with scaling $\phi : X \dashrightarrow X'$ and Mori fibre space $f : X' \to Z$*

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[dr, dashed, "\psi"'] & X' \arrow[d, "f"] \\ & Z, \end{tikzcd}$$

there is a movable curve C such that the rational map $\psi = f \circ \phi : X \dashrightarrow Z$ coincide generically with

$$\begin{array}{cccc} q: & X & \dashrightarrow & \mathrm{Chow}(X) \\ & x & \to & \mathcal{T}_{X/Z}\text{-leaf through } x\text{'} \end{array}$$

where $\mathcal{F}$ is a term of $\mathrm{HNF}_C(\mathcal{T}_X)$. In fact, we can choose C to be the numerical pullback of a curve in the general fibre of $f : X' \to Z$.

Further Results

The key ingredient for the proof of Theorem 0.0.9 is the following theorem, which allows us to calculate the Harder-Narasimhan filtration of the tangent sheaf with respect to a given movable curve. We will prove Theorem 0.0.11 in Section 2.2.

Theorem 0.0.11. *For a given projective, $\mathbb{Q}$-factorial, toric variety X and a movable curve class C, we give an algorithm to calculate the Harder-Narasimhan filtration of the tangent sheaf with respect to C.*

In the course of the proof of Theorem 0.0.9 in the case of Mori fibre spaces. We prove a Mehta-Ramanathan type theorem.

Theorem 0.0.12 (Mehta-Ramanathan type theorem)**.** *Let $\phi : \mathrm{X}_\Delta \to X_{\Delta_R}$ be a Mori fibre space of $\mathbb{Q}$-factorial, projective, complex, toric varieties contracting extremal ray $\mathcal{R}$. Let C be a complete intersection in a general fibre F of ϕ, $C \in \mathcal{R}$. Then we have*

$$\mathrm{HNF}_C(\mathcal{T}_{\mathrm{X}_\Delta})|_F = \mathrm{HNF}_C(\mathcal{T}_{\mathrm{X}_\Delta}|_F).$$

Outline of the dissertation

The dissertation is structure as follows: In Chapter 1 we set up notation and basic properties of toric varieties. Then we give a combinatorial description of the geometry of the cone of movable curves on a toric variety in Section 1.3.1. Chapter 2 is devoted to the statement and proof of the algorithm of calculation of Harder-Narasimhan filtration, and we give a simple way to determine the Harder-Narasimhan filtration of the tangent sheaf of weight projective spaces in the end of Chapter. Chapter 3 contains the proof of Theorem 0.0.9.

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1 Preliminaries

Throughout this paper all spaces are assumed to be algebraic varieties over complex number $\mathbb{C}$.

1.1 Convex geometry

We recall some basic definitions of convex geometry.

Definition 1.1.1. *Let N be a lattice such that $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. Let $\mathcal{C} \subset N_{\mathbb{R}}$ be a subset of a finite dimensional real vector space. We say that $\mathcal{C}$ is a* cone*(respectively* convex subset*) if whenever α and $\beta \in \mathcal{C}$ then*

$$\lambda\alpha + \mu\beta \in \mathcal{C} \text{ for all } \lambda \geq 0,\ \mu \geq 0,$$

(respectively such that $\lambda + \mu = 1$ with $\lambda \geq 0,\ \mu \geq 0$). We say that $\mathcal{C}$ is strictly convex *if $\mathcal{C}$ contains no positive dimensional linear subspaces. We say that $\mathcal{R} \subset \mathcal{C}$ is a ray of a cone $\mathcal{C}$ if $\mathcal{R} = \mathbb{R}_{\geq 0}\alpha$, for some non-zero vector $\alpha \in \mathcal{C}$. We say that $\mathcal{R}$ is an* extremal ray *if whenever $\beta + \gamma \in \mathcal{R}$, where β and $\gamma \in \mathcal{C}$, then β and $\gamma \in \mathcal{R}$; an element $C \in \mathcal{C}$ is called an* extremal element *if $\mathbb{R}_{\geq 0}C$ is an extremal ray. We say that a cone $\mathcal{C}$ is a* rational polyhedron *if $\mathcal{C}$ is a cone and if there exist finitely many $u_i \in N$ such that $\mathcal{C} = \{\sum a_i u_i : a_i \in \mathbb{R}_{\geq 0}\}$.*

Definition 1.1.2. *Given a finite subset S of $N_{\mathbb{R}}$, the cone generated by S is the set $\{\sum a_s s; a_s \in \mathbb{R}_{\geq 0},\ s \in S\}$. We denote it by* $\mathrm{Cone}(S)$.

Definition 1.1.3 (Dual cone)**.** *Let M be the dual lattice of N with a perfect pairing $\langle\ ,\ \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$. Given a polyhedral cone $\sigma \subset N_{\mathbb{R}}$, its* dual cone *is defined by*

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} : \langle m, u\rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Definition 1.1.4. *A* face *of a polyhedral cone σ is $\tau = \sigma \cap H_m$ for some $m \in \sigma^{\vee}$ and $H_m = \{u \in N_{\mathbb{R}} : \langle m, u\rangle = 0\} \subset N_{\mathbb{R}}$. We call $m \in \sigma^{\vee}$ a defining element of τ.*

1.2 Toric varieties

In this section, we recall the basic properties of toric varieties and fix the notation. For the proofs, see [CLS], [Ful].

Setting 1.2.1. *Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n and M its dual lattice. A* fan *Δ is a finite collection of rational, strongly convex cones $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, such that each face τ of a convex cone $\sigma \in \Delta$ again belongs to Δ and every intersection of two cones in Δ is a face of each. To each cone $\sigma \in \Delta$, we can associate an affine variety $U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$, where $\sigma^\vee$ denotes the dual cone of σ. Then the toric variety X_Δ associated to fan Δ is the normal variety obtained by gluing the U_σ. We write $T_N := \operatorname{Spec} \mathbb{C}[M] \cong N \otimes_{\mathbb{Z}} \mathbb{C}^* = (\mathbb{C}^*)^n$. The torus T_N may be regarded as an open subset of X_Δ and we call it a big torus of X_Δ. There is an action of T_N on X_Δ, which extends the action of T_N on itself.*

We use notation and assumptions in Setting 1.2.1 in the following subsection. We have Δ, which is a fan in a lattice N of rank n, and M will be the dual lattice of N. The variety X_Δ will always stand for the toric variety associated to the fan Δ.

Definition 1.2.2 (Dimension of cones)**.** *For $\sigma \in \Delta$ the* dimension $\dim \sigma$ *of σ is the dimension of the linear space $W_\sigma = \sigma + (-\sigma)$ spanned by σ. Let $\Delta(k)$ to be the set of cones in Δ of dimension k.*

For each cone $\tau \in \Delta(k)$ there is a torus invariant closed subvariety $V(\tau)$ associated to τ. We need the following definition in order to define $V(\tau)$.

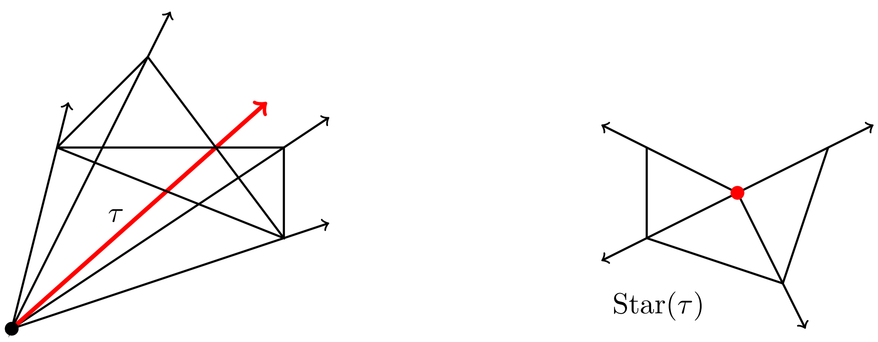
Definition 1.2.3 ($\mathrm{Star}(\tau)$)**.** *For $\tau \in \Delta(k)$ we set*

$$N_\tau := \tau \cap N + (-\tau \cap N), \text{ and } N(\tau) := N/N_\tau,$$

the sub-lattice generated by $\tau \cap N$ and the quotient lattice. Now for a cone σ which contains τ as a face, we can consider its image $\overline{\sigma}$ in $N(\tau)$ and set

$$\overline{\sigma} := (\sigma + (N_\tau)_{\mathbb{R}}/(N_\tau)_{\mathbb{R}}) \subset N(\tau)_{\mathbb{R}}.$$

The collection $\{\overline{\sigma} : \tau \text{ is a face of } \sigma\}$ forms a fan in $N(\tau)$ and we denote this fan by $\mathrm{Star}(\tau)$*. We note that $N(\tau)$ is of rank $n-k$.*



Definition 1.2.4 (Invariant subvarieties associated to cones)**.** *For a cone $\tau \in \Delta(k)$ we have a fan* $\mathrm{Star}(\tau)$ *in a rank $n-k$ lattice $N(\tau)$. Then we define $V(\tau) := X_{\mathrm{Star}(\tau)}$, which is of dimension $n-k$. The variety $V(\tau)$ can be regarded as a closed subvariety of X_Δ and it is invariant under the action of T_N. In case of* $\dim V(\tau) = 1$ *(resp. $n-1$) we call $V(\tau)$* a torus invariant curve *(resp.* a torus invariant divisor*).*

Definition 1.2.5 ($\mathbb{Q}$-divisors)**.** *A* $\mathbb{Q}$-divisor D *is a finite combination of Weil divisors with coefficient in* $\mathbb{Q}$.

Definition 1.2.6 ($\mathbb{Q}$-factoriality)**.** *A normal variety* X *is said to be* $\mathbb{Q}$-factorial *if every prime divisor* D *on* X *is* $\mathbb{Q}$*-Cartier, i.e., for each* D *there exists a positive integer* m_D *such that* $m_D D$ *is a Cartier divisor.*

Proposition 1.2.7 ($\mathbb{Q}$-factoriality criterion for toric varieties, cf. [**?**, Proposition 4.2.7])**.** *A toric variety* X_Δ *is* $\mathbb{Q}$*-factorial if and only if each cone* $\sigma \in \Delta$ *is simplicial, i.e., the primary generators of the one-skeleton of any cone are linearly independent.* □

Proposition 1.2.8 (cf. [Ful, Section 3.4])**.** *Let* D *be a Cartier (resp.* $\mathbb{Q}$*-Cartier) divisor on a toric variety. Then* D *is linearly (resp.* $\mathbb{Q}$*-linearly) equivalent to a sum of torus invariant divisors (resp.* $\mathbb{Q}$*-divisors).* □

Definition 1.2.9. *A toric variety* X_Δ *has a* torus factor *if it is equivariantly isomorphic to the product of a nontrivial torus and a toric variety of smaller dimension.*

Proposition 1.2.10 (cf. [CLS, Proposition 3.3.9])**.** *A toric variety* X_Δ *has no torus factor if and only if the set* $\{\mathrm{u}_\rho : \rho \in \Delta(1), \mathbb{Z}_{\geq 0}\,\mathrm{u}_\rho = \rho\}$ *spans the vector space* $N_\mathbb{R}$. □

Definition 1.2.11 (Tangent sheaf, cf. [Har77, Section II.8])**.** *Let* X *be an algebraic variety. The* tangent sheaf $\mathcal{T}_X$ *is the dual of the sheaf* Ω^1_X *of Kähler differentials. If* $f : X \to Y$ *is a morphism between algebraic varieties, then the* relative tangent sheaf $\mathcal{T}_{X/Y}$ *is the dual of sheaf of relative differential forms* $\Omega^1_{X/Y}$ *and canonical map between Kähler differentials*

$$f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0$$

defines an injective map $\mathcal{T}_{X/Y} \to \mathcal{T}_X$.

Definition 1.2.12 (Map of fans, cf. [Oda, Section 1.5])**.** *Let* $\Delta \subset N$ *and* $\Delta' \subset N'$ *be two fans. A map of fans* $\phi : (N', \Delta') \to (N, \Delta)$ *is a* $\mathbb{Z}$*-linear homomorphism* $\phi : N' \to N$ *whose scalar extension* $\phi : N'_\mathbb{R} \to N_\mathbb{R}$ *satisfies the following properties: For each* $\sigma' \in \Delta'$ *there exists* $\sigma \in \Delta$ *such that* $\phi(\sigma') \subset \sigma$.

Theorem 1.2.13 (cf. [Oda, Theorem 1.13.])**.** *A map of fans* $\phi : (N', \Delta') \to (N, \Delta)$ *gives rise to a homomorphic map*

$$\phi_* : X_{\Delta'} \to X_\Delta,$$

whose restriction to the open subset $T_{N'} \subset X_{\Delta'}$ *coincides with the homomorphism of algebraic tori*

$$\phi \otimes 1 : T_{N'} = N' \otimes \mathbb{C}^* \to T_N = N \otimes \mathbb{C}^*$$

arising from ϕ. Furthermore, the morphism ϕ_* is equivariant with respect to the actions of $T_{N'}$ and T_N on the toric varieties.

Conversely, suppose $f' : T_{N'} \to T_N$ is a homomorphism of algebraic tori and $f : X_{\Delta'} \to X_\Delta$ is a holomorphic map equivariant with respect to f'. Then there exists a unique map of fans $\phi : (N', \Delta') \to (N, \Delta)$ such that $f = \phi_*$. If a morphism $\phi : X_{\Delta'} \to X_\Delta$ satisfies above equivalent condition, we call it $\square$

Definition 1.2.14 (Toric morphism)**.** *Let $\Delta \subset N$ and $\Delta' \subset N'$ be two fans. We call a morphism $\varphi : X_{\Delta'} \to X_\Delta$ a toric morphism if $\varphi = \phi_*$ is the induced morphism for some map of fans $\phi : (N', \Delta') \to (N, \Delta)$.*

Proposition 1.2.15 (General fibres of Toric morphism, [HLY02, Theorem 2.1.4])**.** *Let $\phi : X_{\Delta'} \to \mathrm{X}_\Delta$ be a surjective, projective, toric morphism between projective, toric varieties associated with $\psi : (N', \Delta') \to (N, \Delta)$ with $\dim \mathrm{X}_\Delta < \dim X_{\Delta'}$. Then over the big torus T_N of X_Δ, we have $\phi^{-1}(T_N) \cong F \times T_N$, where F a projective, toric variety, and fan structure of F is $\Delta'_{\{0\}} := \{\sigma' \in \Delta' : \psi(\sigma') = \{0\}\} \subset \ker \psi$*

Lemma 1.2.16 (Relative tangent sheaf)**.** *Let $\phi : (N', \Delta') \to (N, \Delta)$ be a map of fans and $\phi_* : X_{\Delta'} \to X_\Delta$ the corresponding homomorphic morphism. We assume that $\operatorname{rank} \ker(\phi) > 0$, then the relative tangent sheaf $\mathcal{T}_{X_{\Delta'}/X_\Delta}$ is the saturation of $\ker(\phi) \otimes \mathcal{O}_{X_{\Delta'}}$ in $\mathcal{T}_{X_{\Delta'}}$.*

Proof. Since the restriction of ϕ_* on $\mathcal{T}_{N'}$ is $N' \otimes \mathbb{C}^* \to N \otimes \mathbb{C}^*$, we only need to check that the relative tangent sheaf of $\phi_* : T_{N'} \to T_N$ is $\ker(\phi) \otimes \mathcal{O}_{T_{N'}}$ which is obvious. $\square$

Proposition 1.2.17 (Exact sequence of divisors, [CLS, Theorem 4.1.3])**.** *Let X_Δ be a $\mathbb{Q}$-factorial, complete, toric variety. Then there exists an exact sequence*

$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{\alpha^*} \mathbb{R}^{\Delta(1)} \xrightarrow{\beta^*} \operatorname{Pic}(\mathrm{X}_\Delta)_{\mathbb{R}} \longrightarrow 0$$

where

$$\begin{aligned} \alpha^*(m) &= (\langle m, \mathrm{u}_\rho \rangle)_{\rho \in \Delta(1)} \\ \beta(e_\rho) &= [D_\rho] \qquad\qquad e_\rho \text{ a standard basis of } \mathbb{R}^{\Delta(1)} \end{aligned}$$

Theorem 1.2.18 (Euler sequence, cf. [CLS, Theorem 8.1.6])**.** *Let X_Δ be a $\mathbb{Q}$-factorial, toric variety without torus factor.*

Then there exists an exact sequence for the tangent sheaf.

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Cl}(\mathrm{X}_\Delta) \otimes \mathcal{O}_{\mathrm{X}_\Delta} & \overset{\alpha}{\to} & \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_{\mathrm{X}_\Delta}(D_\rho) & \overset{\beta}{\to} & \mathcal{T}_{X_\Delta} & \to & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \to & \mathrm{Cl}(\mathrm{X}_\Delta) \otimes \mathcal{O}_{\mathrm{X}_\Delta} & \to & \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_{\mathrm{X}_\Delta} & \overset{\beta}{\to} & N \otimes \mathcal{O}_{\mathrm{X}_\Delta} & \to & 0 \end{array} \tag{1.1}$$

Furthermore, if X_Δ is smooth, then we have

$$0 \to \operatorname{Pic}(\mathrm{X}_\Delta) \otimes \mathcal{O}_{\mathrm{X}_\Delta} \overset{\alpha}{\to} \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_{\mathrm{X}_\Delta}(D_\rho) \overset{\beta}{\to} \mathcal{T}_{X_\Delta} \to 0$$

$\square$

Definition 1.2.19 (Fake weighted projective space)**.** *A* fake weighted projective space *is a $\mathbb{Q}$-factorial, complete, toric variety with Picard number one but which is not a weighted projective space. In fact, a fake weighted projective space is always projective.*

Proposition 1.2.20 ([Mat, Corollary 14-2-2])**.** *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety and $\mathcal{R} \subset \overline{\mathrm{NE}}_1(\mathrm{X}_\Delta)$ an extremal ray. Let*

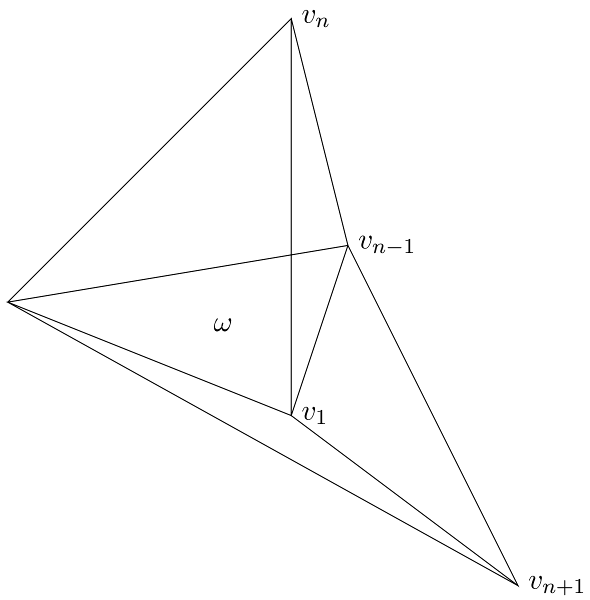
$$\phi_{\mathcal{R}} : \mathrm{X}_\Delta \to Y_\Sigma$$

be the contraction of $\mathcal{R}$ with $\dim Y_\Sigma < \dim \mathrm{X}_\Delta$. Then the general fibre of $\phi_{\mathcal{R}}$ is either a weighted projective space or a fake weighted projective space.

Proposition 1.2.21 (Intersection theory, [Ful, Section 5.1])**.** *Let X_Δ be a $\mathbb{Q}$-factorial toric variety, i.e., every cone in Δ is simplicial. We pick $\omega = \langle v_1, \ldots, v_{n-1} \rangle \in \Delta(n-1)$, the cone generated by $v_1, \ldots, v_{n-1}$. There are two unique primitive lattice points v_n, v_{n+1} such that $\langle v_1, \ldots, v_{n-1}, v_n \rangle$ and $\langle v_1, \ldots, v_{n-1}, v_{n+1} \rangle$ belong to $\Delta(n)$. If we write down the equation $\sum a_i v_i = 0$ with $a_{n+1} = 1$, then we have*

$$V(\langle v_i \rangle) \cdot V(\omega) = a_i V(\langle v_{n+1} \rangle) \cdot V(\omega)$$

for $i = 1, \ldots, n$. □



Theorem 1.2.22 (Small $\mathbb{Q}$-factorialization, [Fuj, Theorem 5.5])**.** *Let X be a projective, toric variety, then there exists a projective morphism $f : X' \to X$ such that X' is a $\mathbb{Q}$-factorial, projective, toric variety and the exceptional set of f is of codimension ≥ 2. We call X' a* small $\mathbb{Q}$-factorialization *of X.* □

1.2.1 Toric structure of weighted projective spaces and fake weighted projective spaces

(ref: [CLS], [Kas]) We need the following toric interpretation of weighted projective space and fake weighted projective spaces for the simple algorithm of the calculation of the Harder-Narasimhan filtration of the tangent sheaf in Subsection 2.3.

1.2.23 (Toric structure of weighted projective space)**.** *For any given positive integers $a_0, \ldots, a_n$ with $\gcd(a_0, \ldots, a_n) = 1$ the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ has a toric structure as follows. Let $\{u_0, \ldots, u_n\} \subset N \cong \mathbb{Z}^n$ be a set of $n+1$ primitive lattice points that fulfill the following two conditions*

(1) $a_0 u_0 + a_1 u_1 + \cdots + a_n u_u = 0.$

(2) the u_i generate the lattice N.

and the fan Δ made up of the cones generated by all the proper subsets of $\{u_0, \ldots, u_n\}$. Then the toric variety associated to this fan in the lattice N is isomorphic to $\mathbb{P}(a_0, \ldots, a_n)$. Equivalently, if we define the lattice

$$N = \mathbb{Z}^{n+1}/\mathbb{Z} \cdot (a_0, \ldots, a_n)$$

and let u_i for $i = 0, \ldots, n$ be the images in N of the standard basis vectors of $\mathbb{Z}^{n+1}$, and Δ the fan made up of the cones generated by all proper subsets of $\{u_0, \ldots, u_n\}$, then $\mathbb{P}(a_0, \ldots, a_n) \cong \mathrm{X}_\Delta$.

1.2.24 (Toric structure of fake weighted projective space)**.** *Consider any given positive integers $a_0, \ldots, a_n$ with $\gcd(a_0, \ldots, a_n) = 1$, any set $\{u_0, u_1, \ldots, u_n\}$ $\subset N$ of $n+1$ primitive lattice points in $N \cong \mathbb{Z}^n$ that fulfill the conditions*

(1) $a_0 u_0 + \cdots + a_n u_n = 0$

(2) The u_i generate a sublattice $N' \subset N$ of finite index

and the fan Δ made up of the cones generated by all proper subset of $\{u_0, \ldots, u_n\}$. Then X_Δ is a fake weighted projective space *and is denoted by $\mathbb{P}(u_0, \ldots, u_n)$. In fact, every fake weighted projective space is of this form.*

1.3 Harder-Narasimhan filtration and foliations

1.3.1 The cone of movable curves

We discuss the notion of movable curves, and provide a structure theorem of the cone of movable curves on a $\mathbb{Q}$-factorial, projective, toric variety.

Definition 1.3.1 (Movable curve, cf. [BDPP], [CP11], [GKP12, Appendix A])**.** *Let X be a normal, $\mathbb{Q}$-factorial, projective variety. We call a curve C movable if C lies in a covering family $(C_i)_{t \in S}$ with an irreducible and projective S and C_t is* *irreducible for general* $t \in S$. *We set* $\mathrm{Mov}_1(X)$ *to be the convex cone generated by all movable curves in* $N_1(X)_{\mathbb{R}}$ *and* $\overline{\mathrm{Mov}_1}(X)$ *to be its closure in* $N_1(X)_{\mathbb{R}}$ *and call it the cone of movable curves, where* $N_1(X)$ *is the free abelian group generated by irreducible curves on* X *modulo numerical equivalence and* $N_1(X)_{\mathbb{R}} := N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 1.3.2 (Duality, [BDPP, Theorem 2.2])**.** *Let* X *be a* $\mathbb{Q}$*-factorial, normal, projective variety, then* $\overline{\mathrm{Mov}_1}(X)$ *and* $\overline{\mathrm{Eff}}(X)$ *are dual to each other via the intersection of divisors and curves.*

Proof. In the original form, the theorem was prove for X a projective manifold. It is well-known that the theorem can be extended to the case X $\mathbb{Q}$-factorial, normal, projective variety. For the reader's convenience, we represent the proof here. It is clear that if C is a movable curve class then $C.E \geq 0$ for any effective divisor, therefore $C.E' \geq 0$ for $E' \in \overline{\mathrm{Eff}(X)}$. We only need to show that the converse: if a divisor E satisfies E is non-negative on $\overline{\mathrm{Mov}_1}(X)$, then E is pseudo-effective. Let $f : X' \to X$ be a resolution of singularity of X and X' a projective manifold. The main observation is that divisor E is pseudo-effective if and only if f^*E is pseudo-effective. If E is not pseudo-effective, then f^*E is also not pseudo-effective. Hence there exists a movable curve class $C' \in \overline{\mathrm{Mov}_1}(X')$ such that $f^*E.C' < 0$. We may assume C' is a movable curve which lies in a covering family $(C'_t)_{t \in S}$. Then the family of curves $(f_*C_t)_{t \in S}$ is a covering family on X and $E.f_*C' = f^*E.C' < 0$. □

Remark 1.3.3. *For the case of Mori dream space, there is a simple proof of this fact, cf. [KO12, Proposition 2.6].*

1.3.4. *The cone of movable curves* $\overline{\mathrm{Mov}_1}(X)$ *of a* $\mathbb{Q}$*-factorial, projective, toric variety* X *is equal to* $\mathrm{Mov}_1(X)$ *by the following Proposition 1.3.6 and Lemma 1.3.5. Hence we shall not distinguish the difference between* $\mathrm{Mov}_1(X)$ *and* $\overline{\mathrm{Mov}_1}(X)$.

Lemma 1.3.5 (cf. [Pay06, Proposition 2, Page 427])**.** *Let* X *be a* $\mathbb{Q}$*-factorial, projective, toric variety and let* $\{a_\rho\}_{\rho \in \Delta} \subset \mathbb{Z}_{\geq 0}$ *such that* $\sum a_\rho \mathrm{u}_\rho = 0$, *where* $\rho = \mathbb{Z}_{\geq 0} \mathrm{u}_\rho$. *Then there exists an irreducible movable curve* C *such that* $D_\rho.C = a_\rho$ *for all* $\rho \in \Delta$. □

The following proposition represents a curve by a vector in a vector spaces.

Proposition 1.3.6 (Exact sequence of curves, [CLS, Proposition 6.4.1])**.** *Let* X_Δ *be a* $\mathbb{Q}$*-factorial, projective, toric variety and* $N_1(\mathrm{X}_\Delta)$ *the free abelian group generated by irreducible curves on* X *module numerical equivalence and* $N_1(\mathrm{X}_\Delta)_{\mathbb{R}} := N_1(\mathrm{X}_\Delta) \otimes_{\mathbb{Z}} \mathbb{R}$. *Then we have an exact sequence*

$$0 \longrightarrow N_1(\mathrm{X}_\Delta)_{\mathbb{R}} \xrightarrow{\beta} \mathbb{R}^{\Delta(1)} \xrightarrow{\alpha} N_{\mathbb{R}} \longrightarrow 0$$

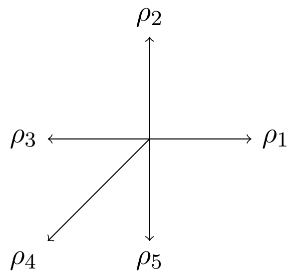
where

$$\begin{aligned} &\alpha(e_\rho) = u_\rho && e_\rho \text{ a standard basis of } \mathbb{R}^{\Delta(1)} \\ &\beta([C]) = (D_\rho.C)_{\rho \in \Delta(1)} && C \subset \mathrm{X}_\Delta \text{ an irreducible complete curve.} \end{aligned}$$

Thus $N_1(\mathrm{X}_\Delta)_\mathbb{R}$ can be interpreted as the space of linear relation among the minimal generators of Δ. Furthermore, a tuple $(a_\rho) \in \mathbb{R}^{\Delta(1)}$ represents a curve class C if and only if that $\sum a_\rho u_\rho = 0$. □

Definition 1.3.7. *For a movable curve C, we see that $\beta(C)$ lies in the first quadrant of the vector space $\mathbb{R}^{\Sigma(1)}$. We set $\Sigma_C := \{\rho \in \Sigma(1) : C.D_\rho > 0\}$. We note that if $C' \in \mathbb{R}_{>0}C$, then $\Sigma_{C'} = \Sigma_C$. We also use the notation $\Sigma_\mathcal{C}$, where $\mathcal{C} \subset \overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ is a cone, which can be defined to be $\cup_{C\in\mathcal{C}}\Sigma_C$. If $\{C_j\}$ generates the cone $\mathcal{C}$, then $\Sigma_\mathcal{C} = \cup\Sigma_{C_j}$.*

Example 1.3.8. *Let S be the toric surface associated to the following fan in N^2. Then S is the blowing up of Hirzebruch surface F_1 along a torus fixed point.*



There are 5 torus invariant divisors on S, and the intersection table is the following.

	D_{ρ_1}	D_{ρ_2}	D_{ρ_3}	D_{ρ_4}	D_{ρ_5}
D_{ρ_1}	0	1	0	0	1
D_{ρ_2}	1	0	1	0	0
D_{ρ_3}	0	1	-1	1	0
D_{ρ_4}	0	0	1	-1	1
D_{ρ_5}	1	0	0	1	-1

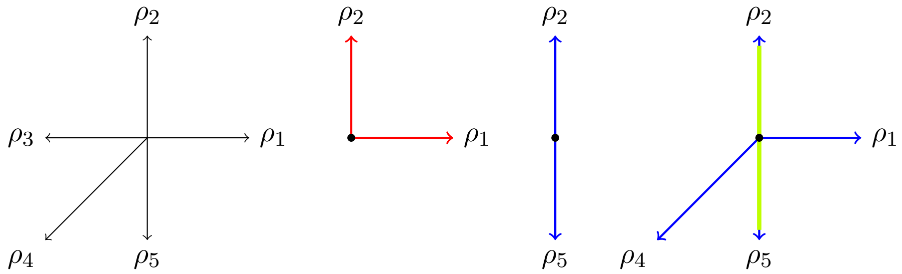
Since curves and divisors are the same on S, we can use intersection table to see $\Sigma_{D_{\rho_i}}$ and $\beta(D_{\rho_i})$ via the intersection table. From the intersection table, we see that D_{ρ_1}, D_{ρ_2} are movable curves, $\Sigma_{D_{\rho_1}} = \{\rho_2, \rho_4\}$, $\Sigma_{D_{\rho_2}} = \{\rho_1, \rho_3\}$.

Remark 1.3.9. *If V_{Σ_C} the minimal sub vector space of $N_\mathbb{R}$ containing all elements of Σ_C, then the vector space $V_{\Sigma_C} = \mathrm{Cone}(\{\mathrm{u}_\rho : \rho \in \Sigma_C,\ \rho = \mathbb{Z}_{\geq 0}\,\mathrm{u}_\rho\})$*

Remark 1.3.10. *The set Σ_C is also used to study the relation between the Mori cone of a toric variety and the primitive collection of $\Delta(1)$, for example, see [CLS, Theorem 6.4.11].*

Definition 1.3.11. *Let Δ be a fan in a lattice N, we call a subset $S \subset \Delta(1)$* movable if there exists $\{a_\rho\}_{\rho\in S} \subset \mathbb{Z}_{>0}$ such that $\sum_{\rho\in S} a_\rho\,\mathrm{u}_\rho = 0$, where $\rho = \mathbb{Z}_{\geq 0}\,\mathrm{u}_\rho$. We call a movable subset S of $\Delta(1)$ *irreducible* if it contains no proper movable subset.

Example 1.3.12. *(Example 1.3.8 continued). The set $\Delta(1)$ consists 5 rays. The set $\{\rho_1, \rho_2\}$ is not movable but sets $\{\rho_2, \rho_5\}$ and $\{\rho_1, \rho_2, \rho_4, \rho_5\}$ are movable. The* *set $\{\rho_2, \rho_5\}$ is irreducible but the set $\{\rho_1, \rho_2, \rho_4, \rho_5\}$ is not as it contains $\{\rho_2, \rho_5\}$ as a subset.*



Irreducible movable set are $S_1 = \{\rho_1, \rho_3\}$, $S_2 = \{\rho_2, \rho_5\}$ and $S_3 = \{\rho_1, \rho_2, \rho_4\}$; in fact we have $S_1 = \Sigma_{D_{\rho_2}}$, $S_2 = \Sigma_{D_{\rho_1}}$ and $S_3 = \Sigma_{D_{\rho_2}+D_{\rho_3}}$. We will see later that D_{ρ_1}, D_{ρ_2} and $D_{\rho_2} + D_{\rho_3}$ are extremal points of $\overline{\mathrm{Mov}}_1(S)$. Furthermore, we have $\overline{\mathrm{Mov}}_1(S) = \mathrm{Cone}(D_{\rho_1}, D_{\rho_2}, D_{\rho_2} + D_{\rho_3})$.

Remark 1.3.13. *We see that if C is a movable curve, then Σ_C is a movable subset of $\Delta(1)$ by Proposition 1.3.6. Conversely, if $S \subset \Delta(1)$ is a movable subset, then we can find a movable curve C such that $S = \Sigma_C$. We note that there may be two different movable curve generate same movable set S. If C_i are movable curves, then $\Sigma_{C_1+C_2} = \Sigma_{C_1} \cup \Sigma_{C_2}$.*

Proposition 1.3.14. *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety associated to a fan $\Delta \subset N$. Then there exists a one to one correspondence between irreducible movable subsets and extremal rays of $\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$.*

$$\begin{array}{ccc} \left\{\textit{Extremal rays of } \overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)\right\} & \to & \{\textit{Irreducible movable subsets of } \Delta(1)\} \\ \mathcal{R} & \to & \Sigma_{\mathcal{R}} \end{array}$$

Proof. We start with the correspondence between irreducible movable subset of $\Delta(1)$ and extremal rays $\mathcal{R}$ of $\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$. If $\mathcal{R}$ is not extremal, then there exists $C_1 \notin \mathcal{R}$ and $C_2 \in \overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ such that $C_1 + C_2 \in \mathcal{R}$. Then Σ_{C_1} is a proper subset of $\Sigma_{\mathcal{R}}$, and $\Sigma_{\mathcal{R}}$ is not irreducible. If Σ_C is not irreducible, there exists a proper movable set $\Sigma_{C'}$, then $\beta(C) - \epsilon\beta(C')$ lies in the first quadrant of $\mathbb{R}^{\Delta(1)}$ for some rational number $0 < \epsilon \ll 1$, thus $C - \epsilon C' \in \overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$. Hence the ray $\mathcal{R}$ is not extremal. $\square$

Corollary 1.3.15. *There is a one-to-one correspondence between proper movable subsets of $\Delta(1)$ and faces of $\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$.*

Proof. Let $S \subset \Delta(1)$ be a proper movable set. The definition of movable subset guarantees there is a relation $\sum_{\rho\in S} a_\rho \mathrm{u}_\rho = 0$ and $a_\rho \in \mathbb{Z}_{>0}$. Lemma 1.3.5 implies that there is a curve C with $\Sigma_C = S$. We consider the minimal face F of $\partial\,\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ such that $C \in F$, such F exists since $S = \Sigma_C \subsetneq \Delta(1)$. Converse is clear. $\square$

Remark 1.3.16. *During the preparation of this paper, the author learned that Douglas Monôres has proved a similar result about the description of the extremal rays* *of* $\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ *in terms of the elements in lattice N cf. [Mon13, Theorem 5.3.3]. The argument we present here is based on Theorem 1.3.2 and Proposition 1.3.6. Douglas Monôres started with the Theorem of Araujo instead, cf Theorem 3.2.4.*

Remark 1.3.17 (Forms of irreducible movable sets, [Mon13, Theorem 5.3.3])**.** *With the help of Theorem 3.2.4. Monôres showed that irreducible movable sets are of the form* $S = \{u_{\rho_0}, u_{\rho_1}, \ldots, u_{\rho_k}\}$ *and* $\dim \mathrm{Cone}(u_{\rho_0}, \ldots, u_{\rho_k}) = k$*, where* $\mathbb{R}_{\geq 0} u_{\rho_i} = \rho_i \in \Delta(1)$*.*

As a corollary, we have an estimate of numbers of the faces of the cone of movable curves.

Corollary 1.3.18 (Numbers of faces)**.** *We can calculate the number of faces of* $\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ *directly via the structure of* $\Delta(1)$*.* □

1.3.2 Harder-Narasimhan filtration

Given a movable curve class, we can associate the notion of stability and use the stability to filtrate a torsion free sheaf, and such stability condition provide a unique filtration of a torsion free sheaf.

Definition 1.3.19 (Slope and stability, cf. [CP11, Section 1])**.** *Let C be a movable curve class of a normal, $\mathbb{Q}$-factorial, projective variety X and $\mathcal{F}$ a torsion-free, coherent sheaf of positive rank on X. We define the* slope *of $\mathcal{F}$ with respect to C to be*

$$\mu_C(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot C}{\mathrm{rank}(\mathcal{F})}.$$

We call $\mathcal{F}$ semi-stable (resp. stable) with respect to C *or* C-semi-stable (resp. stable) *if for any nonzero proper subsheaf $\mathcal{G}$ of $\mathcal{F}$ with $\mathrm{rank}(\mathcal{G}) < \mathrm{rank}(\mathcal{F})$ holds, we have $\mu_C(\mathcal{G}) \leq (<) \mu_C(\mathcal{F})$. If there exists a nonzero subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mu_C(\mathcal{G}) \geq \mu_C(\mathcal{F})$, we call $\mathcal{G}$ a* destabilizing subsheaf *of $\mathcal{F}$. If there is a nonzero subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mu_C(\mathcal{G}) > \mu_C(\mathcal{F})$, we call $\mathcal{F}$* unstable.

Theorem 1.3.20 (Harder-Narasimhan filtration, cf. [CP11, Prop. 1.3])**.** *Let $\mathcal{F}$ be a torsion-free coherent sheaf on a normal, $\mathbb{Q}$-factorial, projective variety X and C a nonzero movable curve class on X. There exists a unique filtration of $\mathcal{F}$, the* Harder-Narasimhan filtration with respect to C*, depending on the chosen movable curve class*

$$\mathrm{HNF}_C(\mathcal{F}) := 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F},$$

with the following properties:

(i) The sheaves $\mathcal{F}_i$ are saturated in $\mathcal{F}$, i.e., $\mathcal{F}/\mathcal{F}_i$ is torsion free.

(ii) The quotients $\mathcal{G}_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ are torsion-free and semi-stable.

(iii) The slopes of the quotients satisfy

$$\mu_{C,\max}(\mathcal{F}) = \mu_C(\mathcal{G}_1) > \ldots > \mu_C(\mathcal{G}_k) = \mu_{C,\min}(\mathcal{F}).$$

□

Remark 1.3.21. *If $C = H^{n-1}$ is a complete intersection curve, where H is an ample divisor on X. Then we also use the notation HNF_H. When $\dim X = 1$, then we also use the notation* HNF*, since the notation of the Harder-Narasimhan filtration is canonical.*

Remark 1.3.22 (Maximal destabilizing subsheaf)**.** *It is clear that $\mathrm{HNF}_C(\mathcal{F})$ only depends on the ray generated by the numerical class of C in $N_1(X)_{\mathbb{R}}$. For a given sheaf $\mathcal{F}$, the slopes of all subsheaves of $\mathcal{F}$ are bounded from above and there exists a subsheaf $\mathcal{G} \subset \mathcal{F}$ such that $\mu_C^{\max}(\mathcal{F}) := \mu_C(\mathcal{G}) \geq \mu_C(\mathcal{F}')$ for all subsheaves $\mathcal{F}' \subset \mathcal{F}$. There exists a unique maximal element with respect to inclusion among all subsheaves with slope $\mu_C^{\max}(\mathcal{F})$, which coincides with the first nonzero term of $\mathrm{HNF}_C(\mathcal{F})$. It is called the* maximal destabilizing subsheaf *of $\mathcal{F}$. We refer to [GKP12, Appendix A] for the proof of boundedness and the existence of the maximal destabilizing subsheaf. It is clear that $\mathcal{F}$ is unstable if and only if the maximal destabilizing subsheaf of $\mathcal{F}$ is not equal to $\mathcal{F}$.*

Remark 1.3.23. *In fact, the sheaf $\mathcal{F}_i$ is the preimage of the maximal destabilizing subsheaf of $\mathcal{F}/\mathcal{F}_{i-1}$ in $\mathcal{F}$.*

Definition 1.3.24 (G-sheaf, cf. [Kol, Definition 1.2.1])**.** *Let G be an algebraic group acting on a normal variety X, and F a sheaf on X. We call $\mathcal{F}$ a G-sheaf if there is an isomorphism $\sigma : m_X^* \cong p_2^* F$, where $m_X : G \times X \to X$ is the action and $p_2 : G \times X \to X$ is the second projection, such that $\sigma(g) : g^* F \cong F$ for all $g \in G$ and σ satisfies the usual compatible condition.*

Definition 1.3.25 (G-invariant subsheaf)**.** *Let G be an algebraic group acting on a normal variety X, and $\mathcal{F}$ a G-sheaf with isomorphism σ. A G-invariant subsheaf $\mathcal{G}$ of $\mathcal{F}$ is a subsheaf of $\mathcal{F}$ such that the isomorphism σ induces an isomorphism of $\mathcal{G}$.*

Remark 1.3.26 (Invariability of the maximal destabilizing subsheaf)**.** *We assume that there is a connected algebraic group G acting on X and let $\mathcal{F}$ be a torsion-free G-sheaf. Let C be an arbitrary movable curve class. Then the uniqueness of maximal destabilizing subsheaf $\mathcal{G}$ of $\mathcal{F}$ implies that $\mathcal{G}$ of $\mathcal{F}$ is a G-invariant. We do not assume the movable curve class C to be G-invariant, since G acts trivially on $N_1(X)$, therefore G acts trivially on $N_1(X)_{\mathbb{R}}$, which follows, because that G is connected, acts continuously on $N_1(X)$ and $N_1(X)$ is discrete. Therefore, the Harder-Narasimhan filtration of $\mathcal{F}$ is G-invariant. In particular, the tangent sheaf of a toric variety is clearly a T_N-sheaf, where T_N is its big torus, and terms of the Harder-Narasimhan filtration with respect to a movable curve class of the tangent sheaf are T_N-invariant.*

Theorem 1.3.27 (Metha-Ramanathan theorem, cf.[Fle84, Theorem 1.2])**.** *Let X be a normal, projective variety over the complex numbers, H an ample divisor on X, and E a torsion free coherent sheaf on X. Then there exists a $m_0 > 0$ such that for all $m > m_0$ there exists a curve of the form $C = H_1 \cap \ldots \cap H_{\dim X - 1}$, H_i are general member of $|mH|$, we have $\mathrm{HNF}_H(E)|_C = \mathrm{HNF}(E|_C)$.*

1.3.3 Foliation and leaves

Definition 1.3.28 (Integrable subsheaves, [Pe00, Definition 1.2])**.** *Let X be a complex manifold, $E \subset \mathcal{T}_X$ a subbundle. E is called* integrable *if E is closed under the Lie bracket, i.e., $[E, E] \subset E$. More generally, let X be a complex variety, $E \subset \mathcal{T}_X$ a coherent subsheaf(of positive rank). Let X_{reg} be the set of smooth points of X and* $\mathrm{Sing}(\mathcal{T}_X/E)$ *the set of points where $\mathcal{T}_X/E$ is not locally free. Let*

$$X^\circ := X \setminus \mathrm{Sing}(\mathcal{T}_X/E).$$

Then E is integrable if $E|_{X^\circ} \subset \mathcal{T}_{X^\circ}$ is integrable.

Definition 1.3.29 (Foliation, [Neu10, Definition 1.1.2])**.** *Let X be a complex, normal, projective variety. A* foliation *$\mathcal{F}$ is a non-zero, coherent, saturated, integrable subsheaf of the tangent sheaf.*

Definition 1.3.30 (Leaf, [KSC, Definition 1.9])**.** *Let p be a general point on a complex, normal, projective variety X and $\mathcal{F} \subset \mathcal{T}_X$ a foliation. Let $X^\circ := X \setminus \mathrm{Sing}(\mathcal{T}_X/E)$, i.e., $\mathcal{F}|_{X^\circ} \subset \mathcal{T}_{X^\circ}$ is a subbundle. The* leaf *of $\mathcal{F}$ through p is the union $\bigcup_{p \in M} M$ of all connected manifold containing p which is contained in X°. A leaf is called* algebraic *if it is open in its Zariski closure.*

Theorem 1.3.31 (Algebraicity of leaves, cf. [KSCT, Theorem 1])**.** *Let X be a complex, normal, projective variety, $C \subset X$ a complete curve which is entirely contained in the smooth locus $\mathrm{X_{reg}}$ and $\mathcal{F} \subset \mathcal{T}_X$ a (possibly singular) foliation which is regular along C. Assume that the restriction $\mathcal{F}|_C$ is an ample vector bundle on C. If $x \in C$ is any point, the leaf through x is algebraic. If $x \in C$ is general, the closure of the leaf is rationally connected.* □

Proposition 1.3.32 (HNF and foliation)**.** *Let X be a normal, $\mathbb{Q}$-factorial, projective variety and C a movable curve class. Let*

$$0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k \subset \mathcal{T}_X$$

be the Harder-Narasimhan filtration of $\mathcal{T}_X$ with respect to C. If $\mu_C(\mathcal{F}_l/\mathcal{F}_{l-1}) > 0$, then $\mathcal{F}_l$ is a foliation.

Proof. If C is a general fibre in a general fibre of morphism $X \to Y$ between normal, projective varieties. Then the proof can be found in [KSCT], as a combination of Proposition 30, [KSCT] and an argument used in 6.2 in the same paper. This is enough for our purpose. The proof of general form go along a different line, a proof can be found in [Neu10, Theorem 4.3.3] and [Neu10, Remark 4.3.4]. For the reader's convenience, we reproduce the proof here with small modification. First of all $\{\mathcal{F}_i\}$s are all coherent, non-zero, saturated subsheaves of $\mathcal{T}_X$, hence it suffices to show $\mathcal{F}_l$ is integrable. Second we observe that if $\mu_C(\mathcal{F}_l/\mathcal{F}_{l-1}) > 0$, then $\mu_C(\mathcal{F}_k/\mathcal{F}_{k-1}) > 0$ for all $1 \leq k \leq l$. Here we prove the integrability of $\mathcal{F}_l$ by ascending induction. First step, we shall show that $\mathcal{F}_1$ is integrable. Let $X_1^\circ = X_{reg} \backslash \operatorname{Sing}(\mathcal{T}_X/\mathcal{F}_1)$, we shall show that $\mathcal{F}_1|_{X_1^\circ}$ is closed under Lie bracket. We observe that the normality of X and torsion-freeness of $\mathcal{T}_X/\mathcal{F}_1$ implies $\operatorname{Codim}(X\backslash X_1^\circ) \geq 2$. Now we can associate Lie bracket operator $[,]$ on $\mathcal{F}_1|_{X_1^\circ}$ to a morphism

$$\psi_1 : \bigwedge^2 \mathcal{F}_1|_{X_1^\circ} \to \mathcal{T}_X/\mathcal{F}_1|_{X_1^\circ},$$

the integrability of $\mathcal{F}_1|_{X_1^\circ}$ is equivalent to $\psi = 0$. Hence it suffices to show

$$\operatorname{Hom}(\bigwedge^2 \mathcal{F}_1|_{X_1^\circ}, \mathcal{T}_X/\mathcal{F}_1|_{X_1^\circ}) \subset \operatorname{Hom}(\bigotimes^2 \mathcal{F}_1|_{X_1^\circ}, \mathcal{T}_X/\mathcal{F}_1|_{X_1^\circ}) = 0.$$

We have an isomorphism between sheaves

$$\operatorname{Hom}(\bigotimes^2 \mathcal{F}_1|_{X_1^\circ}, \mathcal{T}_X/\mathcal{F}_1|_{X_1^\circ}) = \operatorname{Hom}((\bigotimes^2 \mathcal{F}_1)^{**}, (\mathcal{T}_X/\mathcal{F}_1)^{**}),$$

where

$$\mathcal{G}^{**} = \mathcal{H}om(\mathcal{H}om(\mathcal{G}, \mathcal{O}_X), \mathcal{O}_X)$$

is the double dual, by the fact that $\operatorname{Codim}(X\backslash X_1^\circ) \geq 2$ and [Har80, Proposition 1.6]. Hence we reduce to show

$$\operatorname{Hom}((\bigotimes^2 \mathcal{F}_1)^{**}, (\mathcal{T}_X/\mathcal{F}_1)^{**}) = 0.$$

Now we are ready to use the properties of $\operatorname{HNF}_C(\mathcal{T}_X)$ to establish the nullity of this group. The sheaf $\mathcal{F}_1$ is C-semistable which follows from property (ii) of 1.3.20, then we get that $(\bigotimes^2 \mathcal{F}_1)^{**}$ is also C-semistable by [GKP15, Theorem 4.2] with $\mu_C((\bigotimes^2 \mathcal{F}_1)^{**}) = 2\mu_C(\mathcal{F}_1)$. Then we have the following inequalities

$$\begin{aligned}\mu_{C,\min}((\bigotimes^2 \mathcal{F}_1)^{**}) = \mu_C((\bigotimes^2 \mathcal{F}_1)^{**}) = 2\mu_C(\mathcal{F}_1) > \\ \mu_C(\mathcal{F}_1) > \mu_{C,\max}(\mathcal{T}_X/\mathcal{F}_1) = \mu_{C,\max}((\mathcal{T}_X/\mathcal{F}_1)^{**}).\end{aligned}$$

Then $\operatorname{Hom}((\bigotimes^2 \mathcal{F}_1)^{**}, (\mathcal{T}_X/\mathcal{F}_1)^{**}) = 0$ immediately follows from inequality

$$\mu_{C,\min}((\bigotimes^2 \mathcal{F}_1)^{**}) > \mu_{C,\max}((\mathcal{T}_X/\mathcal{F}_1)^{**})$$

and [HL97, Lemma 1.3.3], and we complete proof of integrability of $\mathcal{F}_1$. For $l \geq 2$, we may assume $\mathcal{F}_1, \ldots, \mathcal{F}_{l-1}$ are integrable by induction. We set $X_l^\circ = \mathrm{X}_{\mathrm{reg}} \setminus \operatorname{Sing}(\mathcal{T}_X/\mathcal{F}_l)$ and aiming to show that the induced morphism

$$\psi_l : \bigwedge^2 \mathcal{F}_l|_{X_l^\circ} \to \mathcal{T}_X/\mathcal{F}_l|_{X_l^\circ}$$

is zero. But this morphism factor through

$$\psi_1 : \bigwedge^2 \mathcal{F}_l/\mathcal{F}_{l-1}|_{X_l^\circ} \to (\mathcal{T}_X/\mathcal{F}_{l-1})/(\mathcal{F}_l/\mathcal{F}_{l-1})|_{X_l^\circ}$$

by the fact that $\mathcal{F}_{l-1}$ is integrable. Then we reduce to the case of $l = 1$. Hence we complete the proof. □

1.3.33. *Let X be a complex, normal, projective variety and assume the conditions of Proposition 1.3.32 are fulfilled. We obtain a foliation $\mathcal{F}_l$ whose leaves are algebraic and rationally connected by Theorem 1.3.31. By setting*

$$\begin{array}{rccl} q_{\mathcal{F}_l}: & X & \dashrightarrow & \subset \mathrm{Chow}(X) \\ & x & \to & \mathcal{F}_l\text{-}\textit{leaf through } x, \end{array}$$

we obtain a rational map such that the closure of the general fibre is rationally connected, see [KSCT, Section 7].

Proposition 1.3.34 (cf. [KSCT, Proposition 29])**.** *Let C be a smooth complex projective curve and E a vector bundle on C with Harder-Narasimhan filtration with respect to $\mathcal{O}_C(1)$*

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E$$

and $\mu_i := \mu(E_i/E_{i-1})$. Suppose that $\mu_1 > 0$ and let $k := \max\{i \mid \mu_i > 0\}$. Then E_i is ample for all $1 \leq i \leq k$. □

1.4 Rational contraction and Toric varieties

1.4.1 Rational contractions

Definition 1.4.1 (Rational contraction, cf. [HK00, Definition 1.1])**.** *Let $f : X \dashrightarrow Y$ be a dominant rational map between normal projective varieties. We say that f is* contracting*, or a* rational contraction*, if there exists a resolution of f*

$$\begin{tikzcd} X' \arrow[d, "\mu"] \arrow[dr, "f'"] & \\ X \arrow[r, dashed, "f"] & Y, \end{tikzcd}$$

where X' is smooth and projective, μ is birational, and for every μ-exceptional effective divisor E on X' we have

$$f'_*\mathcal{O}_{X'}(E) = \mathcal{O}_Y.$$

If f is birational, we say that f is birational contraction. Furthermore, if both f and f^{-1} are rational contractions, then we call f a small map. If f is regular, then we call f a regular contraction. If $\dim Y < \dim X$, then we say it is a contraction of fibre type.

Remark 1.4.2. *The definition of rational contraction is independent of the choice of resolution. Furthermore, if we take $E = 0$ in the definition, we get $f'_*\mathcal{O}_{X'} = \mathcal{O}_Y$, hence the map f' has connected fibres. If f is birational, then birational contraction is equivalent to that f^{-1} does not contract any divisor.*

Remark 1.4.3. *The natural maps in the minimal model program are all rational contractions.*

Example 1.4.4. *Here we provide a non-example for rational contraction. Let $f : X \to \mathbb{P}^n$ be the blowing up a point, then $f^{-1} : \mathbb{P}^n \dashrightarrow X$ is not a rational contraction.*

1.4.2 Mori dream spaces

We showed that rational contraction between toric varieties are compositions of small maps and a contraction. The main result of this subsection is Corollary 1.4.15. Corollary 1.4.15 can be showed directly by toric method, however, we believe that the theory of Mori dream spaces make this statement more clear.

Definition 1.4.5 (Small $\mathbb{Q}$-factorial modification, cf. [HK00, Definition 1.8])**.** *Let X be a projective variety. A* small $\mathbb{Q}$-factorial modification*(SQM) of X is a small map $g : X \dashrightarrow X'$, with X' normal, projective and $\mathbb{Q}$-factorial.*

Remark 1.4.6. *Let Δ, Δ' be two fans in N and X_Δ, and $X_{\Delta'}$ the toric varieties associated with the corresponding fans. If $\phi : \mathrm{X}_\Delta \to X_{\Delta'}$ is a birational contraction, then we have $\Delta'(1) \subset \Delta(1)$; if $\phi : \mathrm{X}_\Delta \dashrightarrow X_{\Delta'}$ is a small map, then we have $\Delta(1) = \Delta'(1)$, i.e., they share same one cones.*

Lemma 1.4.7. *Let X be the $\mathbb{Q}$-factorial, projective, toric variety associated with fan Δ. Then every SQM X' of X is also toric.*

Proof. The proof can be found in [Pay06, Section 4]. □

Definition 1.4.8 (Cones of divisors)**.** *Let X be a $\mathbb{Q}$-factorial, normal, projective variety. The* effective cone $\mathrm{Eff}(X)$ *is the convex cone generated by classes of effective divisors. The* nef cone $\mathrm{Nef}(X)$ *is the cone of classes in $N^1(X)$ generated by nef divisors. The* movable cone $\mathrm{Mov}^1(X)$ *is the convex cone in $N^1(X)$ generated by the classes of movable divisors, where a Cartier divisor D is called* movable *if the base locus of $|mD|$ has codimension ≥ 2 for $m \gg 0$ and sufficiently divisible.*

Definition 1.4.9 (Mori dream space, [HK00, Definition 1.10])**.** *Let X be a normal, $\mathbb{Q}$-factorial, projective variety. We call X a* Mori dream space *if the following properties hold:*

(1) $\mathrm{Pic}(X)$ *is finitely generated (equivalently, $h^1(\mathcal{O}_X) = 0$).*

(2) $\mathrm{Nef}(X)$ *is generated by the classes of finitely many semiample divisors; and*

(3) there is a finite collection of SQMs $g_i : X \dashrightarrow X_i$ such that each X_i satisfies (2) and $\mathrm{Mov}^1(X)$ *is the union of the* $f_i^*(\mathrm{Nef}(X_i)$

Remark 1.4.10. *If X is a MDS, then the finite collection $\{X_i\}_i$ in property (3) is the set of all SQMs of X.*

Proposition 1.4.11 (cf. [HK00])**.** *A $\mathbb{Q}$-factorial, projective, toric variety X is a Mori dream space.* □

Remark 1.4.12 (Closedness of cones)**.** *If X is a MDS, then all the* $\mathrm{Eff}(X)$, $\mathrm{Nef}(X)$, *and* $\mathrm{Mov}^1(X)$ *are closed, rational polyhedral cones in* $N^1(X)_{\mathbb{R}}$.

Proposition 1.4.13 (cf [HK00, Proposition 1.11])**.** *Let X be a MDS and $f : X \dashrightarrow Y$ a rational contraction. Then there exists a SQM X' of X such that f factors as:*

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[dr, dashed, "f"'] & X' \arrow[d, "h"] \\ & Y, \end{tikzcd}$$

where h is a regular contraction. □

Remark 1.4.14. *The statement was implicit stated in the last paragraph of the proof of [HK00, Proposition 1.11].*

Corollary 1.4.15. *Let* X_Δ *be a $\mathbb{Q}$-factorial, projective, toric variety and* $f : \mathrm{X}_\Delta \dashrightarrow Y$ *a rational contraction. Then there exists a $\mathbb{Q}$-factorial, projective, toric variety X' and a small map* $g : \mathrm{X}_\Delta \to X'$ *such that f factors as:*

$$\begin{tikzcd} \mathrm{X}_\Delta \arrow[r, dashed, "g"] \arrow[dr, dashed, "f"'] & X' \arrow[d, "h"] \\ & Y, \end{tikzcd}$$

where h is a regular contraction. □

Remark 1.4.16. *The toric version of Proposition 1.4.15 can be showed by the fan structure of toric variety itself, c.f.[CLS, Theorem 15.1.10].*

Theorem 1.4.17 (D-minimal model, [HK00, Proposition 1.11],)**.** *Let X be a Mori dream space, then we may run minimal model program for any divisor on X.* □

Remark 1.4.18. *The proof of toric version of Theorem 1.4.17 can be found in [CLS, Section 15.5], [FS04, Section 4].*

1.5 Geometry of the cone of movable curves

We define the adjoint map of the pull-forward map of divisors on the space of curves. Furthermore, we identify rational contraction and faces of the cone of movable curves.

1.5.1 Numerical pullback of curves

1.5.1. *Let X be a $\mathbb{Q}$-factorial, projective, normal variety, $N^1(X)$ the formal group of irreducible divisors on X modulo numerical equivalence, and $N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R}$. Let $\varphi : X \dashrightarrow Z$ be a birational contraction between normal, $\mathbb{Q}$-factorial, projective varieties. Taking the pullback of Cartier divisors on Z defines an injective linear map $\varphi^* : N^1(Z)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$. Taking the pushforward of Weil divisors on X defines a surjective linear map $\varphi_* : N^1(X)_{\mathbb{R}} \to N^1(Z)_{\mathbb{R}}$.*

Definition 1.5.2 (Numerical pullback, cf. [A, Definition 4.1])**.** *Let $\varphi : X \dashrightarrow Z$ be a birational contraction between $\mathbb{Q}$-factorial, projective varieties. Then the numerical pullback $\varphi^*_{num} : N_1(Z)_{\mathbb{R}} \to N_1(X)_{\mathbb{R}}$ is the dual linear map of $\varphi_* : N^1(X)_{\mathbb{R}} \to N^1(Z)_{\mathbb{R}}$. It is the unique injective linear map with the following properties*

(1) If $z \in N^1(Z)_{\mathbb{R}}$ and $l \in N_1(Z)_{\mathbb{R}}$, then $\varphi^*(z).\phi^*_{num}(l) = z.l$.

(2) If $\beta \in \ker \varphi_*$ and $m \in \operatorname{Im} \phi^*_{num}$, then $\beta.m = 0$.

Proposition 1.5.3. *Let $\varphi : X \dashrightarrow Y$ be as in Definition 1.5.2. We have*

(1) If $\gamma \in N_1(X)_{\mathbb{R}}$ and $D \in N^1(Y)_{\mathbb{R}}$, then $\varphi_*\gamma.[D] = \gamma.\varphi^*[D]$.

(2) If $\gamma \in N_1(Y)_{\mathbb{R}}$ and $D \in N^1(X)_{\mathbb{R}}$, then $\varphi^*_{num}\gamma.[D] = \gamma.\varphi_*[D]$.

□

Remark 1.5.4. *If we have birational contractions $\varphi_i : X_i \dashrightarrow X_{i+1}$ between $\mathbb{Q}$-factorial projective varieties, then the numerical pullbacks are functorial in the sense that we have $(\varphi_{i-1})^*_{num} \circ (\varphi_i)^*_{num} = (\varphi_i \circ \varphi_{i-1})^*_{num}$.*

1.5.2 The geometry of the cone of movable curves

By using numerical pull back of curves, Araujo identify part of extremal rays of the cone of movable curves and $(K_X + D)$-Minimal Model Program with scaling. In fact, she showed the following theorem.

Theorem 1.5.5 ([A, Theorem 1.1])**.** *Let (X, D) be a $\mathbb{Q}$-factorial, klt pair[1]. Let $\Sigma \subset N_1(X)_{\mathbb{R}}$ be the set of classes of curves on X that are numerical pullbacks of curves lying on general fibres of Mori Fibre spaces obtaining from X by a $(K_X + D)$-Minimal Model Program with scaling. Then*

$$\overline{\mathrm{NE}}_1(X)_{(K_X+D)\geq 0} + \overline{\mathrm{Mov}}_1(X) = \overline{\mathrm{NE}}_1(X)_{(K_X+D)\geq 0} + \overline{\sum_{[C]\in\Sigma} \mathbb{R}_{\geq 0}[C]} \tag{1.2}$$

□

[1] A klt pair (X, D) consists of a normal variety X and an effective Weil divisor D on X. For the precise definition, we refer to [KM98, Section 2.3]

In the case of toric varieties (more generally, Mori dream spaces), a stronger result is also known. We can identify faces of the cone of movable curves with rational contractions. Specifically, we have following theorem.

Theorem 1.5.6 (cf. Theorem 3.4.2)**.** *Let* X_Δ *be a* $\mathbb{Q}$*-factorial, projective, toric variety associated to fan* $\Delta \subset N$*. Then for every proper face of* $\overline{\mathrm{Mov}_1}(\mathrm{X}_\Delta)$*, we can associate a rational contraction* $\phi_F : \mathrm{X}_\Delta \dashrightarrow X_{\Delta_F}$*, such that* F *is the numerical pull back of curves classese of a face* F' *of* $\overline{\mathrm{Mov}}_1(X_{\Delta_F}) \cap \overline{\mathrm{NE}}_1(X_{\Delta_F})$*. Furthermore, we have* $\dim F = \dim F'$*.*

We postpone the proof to Section 3.4.

2 Algorithm of calculation of Harder-Narasimhan filtration

In this chatper, we give an algorithm to calculate Harder-Narasimhan filtration with respect to a given movable curve class. This chapter is divided into three sections. We present preparatory lemmas of the proof of Theorem 0.0.11 in Section 2.1, then we prove Theorem 0.0.11 in Section 2.2. As an application, we give a simple way to determine the Harder-Narasimhan filtration of $\mathbb{Q}$-factorial, projective, toric varieties with Picard number one in the last section.

2.1 Preparatory lemmas for Theorem 0.0.11

Setting 2.1.1. *Throughout this section, we let* X_Δ *be a* $\mathbb{Q}$*-factorial, toric variety without torus factors, with fan* Δ *in the lattice* $N \cong \mathbb{N}^n$*. The affine variety* U_ρ *will be* $\operatorname{Spec} \mathbb{C}[\rho^\vee \cap M]$ *for* $\rho \in \Delta(1)$*, which is an open, smooth, affine, toric subvariety of* X_Δ*. For each* $\rho \in \Delta(1)$*, we let* u_ρ *be the element in* N *with* $\mathbb{Z}_{\geq 0}\,\mathrm{u}_\rho = \rho$*.*

2.1.1 Invariant vector fields

The following proposition establishes a relation between torus invariant vector fields of $\mathcal{T}_{X_\Delta}$ and sub vector spaces of $N_\mathbb{R}$.

Proposition 2.1.2 ([Oda, Proposition 3.1], [CLS, Theorem 8.16])**.** *Let* X_Δ *be a* $\mathbb{Q}$*-factorial, toric variety without torus factors. We can associate to each* $n \in N$ *a torus invariant derivation on* $\mathbb{C}[M]$ *by*

$$\delta_n : \mathbb{C}[M] \to \mathbb{C}[M],\ \chi^m \to \langle m, n\rangle \cdot \chi^m.$$

Furthermore, the morphism $\delta : N \to \operatorname{Der}(\mathbb{C}[M])$*,* $n \to \delta_n$ *gives an isomorphism between*

$$N \otimes \mathcal{O}_{\mathrm{X}_\Delta} \cong \mathcal{T}_{\mathrm{X}_\Delta}(-\log D),$$

where $D = \sum_{\rho \in \Delta(1)} D_\rho$*.* □

Remark 2.1.3. *We extend the definition of* δ_n *to elements in* $N_\mathbb{R}$ *by simply taking the real extension.*

2.1.2 Invariant saturated subsheaves

In this subsection, we establish an equivalence between the category of saturated torus invariant subsheaves of $\mathcal{T}_{X_\Delta}$ and sub vector spaces of $N_\mathbb{R}$. The inclusion maps of sub vector spaces give morphisms between saturated invariant subsheaves.

Definition 2.1.4 (Torus invariant saturated subsheaf $\mathcal{F}_V$)**.** *We follow the notation in Setting 2.1.1. For a given sub vector space $V \subset N_\mathbb{R}$, we can associate a torus invariant saturated subsheaf $\mathcal{F}_V$ of $\mathcal{T}_{X_\Delta}$ as follows. Recall the isomorphism $N \otimes \mathcal{O}_{\mathrm{X}_\Delta} \cong \mathcal{T}_{X_\Delta}(-\log D)$, which is induced by the morphism δ in Proposition 2.1.2. Via this isomorphism, a vector subspace $V \subset N_\mathbb{R}$ uniquely determines a torus invariant subsheaf $V \otimes \mathcal{O}_{\mathrm{X}_\Delta} \subset \mathcal{T}_{X_\Delta}(-\log D) \subset \mathcal{T}_{X_\Delta}$. We let $\mathcal{F}_V$ be the saturation of $V \otimes \mathcal{O}_{\mathrm{X}_\Delta}$ in $\mathcal{T}_{X_\Delta}$. The saturatedness of $\mathcal{F}_V$ is automatically satisfied, the invariance of $\mathcal{F}_V$ follows, because both $V \otimes \mathcal{O}_{\mathrm{X}_\Delta}$ and $\mathcal{T}_{X_\Delta}$ are torus invariant and the uniqueness of saturation.*

Lemma 2.1.5. *We follow the notation in Setting 2.1.1. Then every torus invariant saturated subsheaves of $\mathcal{T}_{X_\Delta}$ on X_Δ are of the form $\mathcal{F}_V$ for some sub vector space $V \subset N_\mathbb{R}$.*

Proof. Given a torus invariant saturated subsheaf $\mathcal{F}$ of $\mathcal{T}_{X_\Delta}$, the restriction of $\mathcal{F}$ on big torus T_N is a torus invariant saturated subsheaf of $\mathcal{T}_{T_N}$, therefore we have $\mathcal{F}|_{T_N} \cong V_\mathcal{F} \otimes \mathcal{O}_{T_N}$ for some sub vector space $V_\mathcal{F}$ of $N_\mathbb{R}$ via the isomorphism δ given in Proposition 2.1.2. We find that $\mathcal{F}_{V_\mathcal{F}}$ and $\mathcal{F}$ coincide on T_N, thus $\mathcal{F} \cong \mathcal{F}_{V_\mathcal{F}}$. □

Theorem 2.1.6. *There is an equivalence between the category of saturated invariant subsheaves of $\mathcal{T}_{X_\Delta}$ with inclusion morphism and the category of sub vector spaces of $N_\mathbb{R}$ with inclusion morphism between sub vector spaces of $N_\mathbb{R}$.*

Proof. If we have $\mathcal{F}_{V_1} \subset \mathcal{F}_{V_2} \subset \mathcal{T}_{X_\Delta}$, then we have $\mathcal{F}_{V_1}|_{T_N} \subset \mathcal{F}_{V_2}|_{T_N}$, therefore $V_1 \subset V_2$.

Conversely, we have two sub vector spaces V and W with $V \subset W \subset N_\mathbb{R}$. Then the isomorphism δ in Proposition 2.1.2 gives you

$$V \otimes \mathcal{O}_{\mathrm{X}_\Delta} \subset W \otimes \mathcal{O}_{\mathrm{X}_\Delta} \subset \mathcal{T}_{X_\Delta}.$$

The inclusion $V \otimes \mathcal{O}_{\mathrm{X}_\Delta} \subset W \otimes \mathcal{O}_{\mathrm{X}_\Delta} \subset \mathcal{T}_{X_\Delta}$ implies $\mathcal{T}_{X_\Delta}/(W \otimes \mathcal{O}_{\mathrm{X}_\Delta}) \subset \mathcal{T}_{X_\Delta}/(V \otimes \mathcal{O}_{\mathrm{X}_\Delta})$. Hence we have

$$\frac{\mathcal{T}_{X_\Delta}/(W \otimes \mathcal{O}_{\mathrm{X}_\Delta})}{T_{n-1}(\mathcal{T}_{X_\Delta}/(W \otimes \mathcal{O}_{\mathrm{X}_\Delta})} \subset \frac{\mathcal{T}_{X_\Delta}/(V \otimes \mathcal{O}_{\mathrm{X}_\Delta})}{T_{n-1}(\mathcal{T}_{X_\Delta}/(V \otimes \mathcal{O}_{\mathrm{X}_\Delta}))},$$

where $T_{n-1}(F)$ is the maximal subsheaf of F of dimension $\leq n-1$. Since the saturation $\mathcal{F}_V$ of $V \otimes \mathcal{O}_{\mathrm{X}_\Delta}$ in $\mathcal{T}_{X_\Delta}$ is just the kernel of

$$\mathcal{T}_{X_\Delta} \to \frac{\mathcal{T}_{X_\Delta}/(V \otimes \mathcal{O}_{\mathrm{X}_\Delta})}{T_{n-1}(\mathcal{T}_{X_\Delta}/(V \otimes \mathcal{O}_{\mathrm{X}_\Delta})},$$

we have $\mathcal{F}_V \subset \mathcal{F}_W$. □

Corollary 2.1.7. *Let* X_Δ *be a* $\mathbb{Q}$*-factorial, projective, toric variety with no torus factors. Then a* Harder-Narasimhan filtration *of* $\mathcal{T}_{X_\Delta}$ *corresponds to an increasing filtration of sub vector spaces of* $N_\mathbb{R}$*.* □

2.1.3 Chern classes

In this subsection, we calculate the first Chern class of a given invariant saturated subsheaf of $\mathcal{T}_{X_\Delta}$. In fact, for any given $V \subset N_\mathbb{R}$ we are able to write down the local generators of $\mathcal{F}_V$ in U_ρ for each $\rho \in \Delta(1)$. The main result of this subsection is the following Theorem which immediately follows from Lemma 2.1.12.

Theorem 2.1.8. *Let* X_Δ *be a* $\mathbb{Q}$*-factorial, toric variety without torus factors and* V *a sub vector space of* $N_\mathbb{R}$*. Then the first Chern class of the invariant saturated subsheaf* $\mathcal{F}_V \subset \mathcal{T}_{X_\Delta}$ *is* $[\sum_{\rho \subset V} D_\rho]$*.*

Proof. We observe that $\operatorname{Codim} \mathrm{X}_\Delta \setminus \bigcup_{\rho\in\Delta(1)} \mathrm{U}_\rho \geq 2$, hence it suffices to calculate the first Chern class of $\mathcal{F}_V|_{\bigcup_{\rho\in\Delta(1)} \mathrm{U}_\rho}$ on $\bigcup_{\rho\in\Delta(1)} \mathrm{U}_\rho$. Then Theorem immediately follows from Lemma 2.1.12, $\mathcal{O}_{\mathrm{X}_\Delta}(D_\rho)|_{\mathrm{U}_\rho} = \frac{1}{\chi^{m_\rho}} \mathcal{O}_{\mathrm{U}_\rho}$. □

The rest of this subsection is devoted to the proof of Lemma 2.1.12. Here we start with a remark of $V \otimes \mathcal{O}_{\mathrm{X}_\Delta}$.

Remark 2.1.9. *Let* $V \subset N_\mathbb{R}$ *a sub vector space and* $\{v_1, \ldots, v_k\}$ *its* $\mathbb{R}$*-basis. Then* $V \otimes \mathcal{O}_{\mathrm{X}_\Delta}$ *is a free sheaf with generator* δ_{v_i}*.*

Somehow, sheaf $V \otimes \mathcal{O}_{\mathrm{U}_\rho}$ may not saturated in $\mathcal{T}_{X_\Delta}|_{\mathrm{U}_\rho}$. We would like to know what is the difference. The following lemma gives an expression of $\mathcal{T}_{X_\Delta}|_{\mathrm{U}_\rho}$ and an explicit express of derivations δ_n on U_ρ.

Lemma 2.1.10. *For* $\rho = \mathbb{Z}_{\geq 0} u_\rho \in \Delta(1)$*, we have*

$$\rho^\vee \cap M \cong \mathbb{Z}_{\geq 0} m_\rho \oplus \mathbb{Z}_{\geq 0} m_2 \oplus \mathbb{Z}_{\geq 0} m_2^{-1} \oplus \cdots \oplus \mathbb{Z}_{\geq 0} m_n \oplus \mathbb{Z}_{\geq 0} m_n^{-1},$$

where $m_2, \ldots, m_n$ *are generators of* $\rho^\perp \cap M$ *as a lattice and* $\langle m_\rho, u_\rho \rangle = 1$*.*

Then

$$\mathrm{U}_\rho = \operatorname{Spec} A_\rho, \textit{where } A_\rho = \mathbb{C}[m_\rho, m_2, m_2^{-1}, \ldots, m_n, m_n^{-1}],$$

$\mathcal{T}_{X_\Delta}|U_\rho$ *is generated by* $\frac{\partial}{\partial \chi^{m_\rho}}$*,* $\chi^{m_2} \frac{\partial}{\partial \chi^{m_2}}, \ldots, \chi^{m_n} \frac{\partial}{\partial \chi^{m_n}}$ *and we have*

$$\delta_{u_\rho}|_{\mathrm{U}_\rho} = \chi^{m_\rho} \frac{\partial}{\partial \chi^{m_\rho}}.$$

Furthermore, we have for all $v \in N_\mathbb{R}$

$$\delta_v|_{\mathrm{U}_\rho} = \langle m_\rho, v \rangle \cdot \chi^{m_\rho} \frac{\partial}{\partial \chi^{m_\rho}} + \sum_{i=2}^{n} \langle m_i, v \rangle \cdot \chi^{m_i} \frac{\partial}{\partial \chi^{m_i}}.$$

□

Remark 2.1.11. *Lemma 2.1.10 shows that the derivation* $\delta_{u_\rho}|_{U_\rho} = \chi^{m_\rho}\frac{\partial}{\partial\chi^{m_\rho}}$ *vanishing along divisor* $(\chi^{m_\rho} = 0)$ *and suggesting a candidate of generators of* $\mathcal{F}_V|_{U_\rho}$.

Lemma 2.1.12 (Local generator)**.** *We follow notation in Lemma 2.1.10. For a given vector sub space* $V \subset N_{\mathbb{R}}$ *of dimension* k *and* $\rho \in \Delta(1)$ *we assign a vector space basis of* V *as follow: If* $\rho \subset V$, *we choose* $\{v_2, \dots, v_k\}$ *such that* $\{u_\rho, v_2, \dots, v_k\}$ *is a basis of vector space* V. *If* $\rho \not\subset V$, *we take one basis* $\{v_1, \dots, v_k\}$ *of vector space* V. *Then*

$$\mathcal{F}_V|_{U_\rho} = \begin{cases} \left\langle \frac{1}{\chi^{m_\rho}}\delta_{u_\rho}, \delta_{v_2}, \dots, \delta_{v_k} \right\rangle_{\mathcal{O}_{U_\rho}} & \textit{if } \rho \subset V \\ \left\langle \delta_{v_1}, \dots, \delta_{v_k} \right\rangle_{\mathcal{O}_{U_\rho}} & \textit{if } \rho \not\subset V. \end{cases}$$

as a $\mathcal{O}_{U_\rho}$*-module.*

Proof. In the course of the proof, we follow the calculation of Remark 2.1.10. We set an auxiliary sheaf $\mathcal{G}_{V,U_\rho}$ by following

$$\mathcal{G}_{V,U_\rho} := \begin{cases} \left\langle \frac{1}{\chi^{m_\rho}}\delta_{u_\rho}, \delta_{v_2}, \dots, \delta_{v_k} \right\rangle_{\mathcal{O}_{U_\rho}} & \text{if } \rho \subset V \\ \left\langle \delta_{v_1}, \dots, \delta_{v_k} \right\rangle_{\mathcal{O}_{U_\rho}} & \text{if } \rho \not\subset V. \end{cases}$$

It suffice to show that $\mathcal{G}_{V,U_\rho} \cong \mathcal{F}_V|_{U_\rho}$. We check equality $\mathcal{G}_{V,U_\rho}|_{T_N} \cong V \otimes \mathcal{O}_{T_N}$ and torsionfreeness of $\mathcal{T}_{X_\Delta}|_{U_\rho}/\mathcal{G}_{V,U_\rho}$. The isomorphism $\mathcal{G}_{V,U_\rho}|_{T_N} \cong V \otimes \mathcal{O}_{T_N}$ is clear. It remains to show that $\mathcal{G}_{V,U_\rho}$ is saturated, i.e., that $\mathcal{T}_{X_\Delta}|_{U_\rho}/\mathcal{G}_{V,U_\rho}$ is torsion free.

Case 1 ($\rho \not\subset V$)**.** *We prove the case* $\rho \not\subset V$ *first. On*

$$U_\rho = \mathbb{C}[m_\rho, m_2, m_2^{-1}, \dots, m_n, m_n^{-1}],$$

the tangent sheaf $\mathcal{T}_{X_\Delta}|_{U_\rho}$ *is generated by*

$$\frac{\partial}{\partial\chi^{m_\rho}}, \chi^{m_2}\frac{\partial}{\partial\chi^{m_2}}, \dots, \chi^{m_n}\frac{\partial}{\partial\chi^{m_n}}$$

and we can write

$$\delta_{v_j} = \langle m_\rho, v_j\rangle \cdot \chi^{m_\rho}\frac{\partial}{\partial\chi^{m_\rho}} + \sum_{l=2}^{n}\langle m_l, v_j\rangle \cdot \chi^{m_l}\frac{\partial}{\partial\chi^{m_l}}, j = 1, \dots, k.$$

Then we consider the short exact sequence (2.1) *with respect to the chosen bases of* $\mathcal{G}_{V,U_\rho}$ *and* $\mathcal{T}_{X_\Delta}|_{U_\rho}$*:*

$$0 \to \mathcal{G}_{V,U_\rho} \xrightarrow{\phi} \mathcal{T}_{X_\Delta}|_{U_\rho} \to \mathcal{T}_{X_\Delta}|_{U_\rho}/\mathcal{G}_{U_\rho} \to 0 \tag{2.1}$$

The exact sequnce (2.1) *can be written as follows*

$$0 \to A_\rho^k \xrightarrow{\Phi} A_\rho^n \to \operatorname{Coker}\Phi \to 0 \tag{2.2}$$

where

$$\Phi = \begin{pmatrix} \langle m_\rho, v_1\rangle \cdot \chi^{m_\rho} & \langle m_\rho, v_2\rangle \cdot \chi^{m_\rho} & \dots & \langle m_\rho, v_k\rangle \cdot \chi^{m_\rho} \\ \langle m_2, v_1\rangle & \langle m_2, v_2\rangle & \dots & \langle m_2, v_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle m_n, v_1\rangle & \langle m_n, v_2\rangle & \dots & \langle m_n, v_k\rangle \end{pmatrix}_{n\times k}.$$

The condition $\rho \not\subset V$ *is equivalent to* $\{\mathrm{u}_\rho, v_1, \dots, v_k\}$ *are linearly independent, thus the matrix*

$$\Psi = \begin{pmatrix} 1 = \langle m_\rho, \mathrm{u}_\rho\rangle & \langle m_\rho, v_1\rangle & \langle m_\rho, v_2\rangle & \dots & \langle m_\rho, v_k\rangle \\ 0 = \langle m_2, \mathrm{u}_\rho\rangle & \langle m_2, v_1\rangle & \langle m_2, v_2\rangle & \dots & \langle m_2, v_k\rangle \\ \vdots & \vdots & \ddots & \vdots & \\ 0 = \langle m_k, \mathrm{u}_\rho\rangle & \langle m_n, v_1\rangle & \langle m_n, v_2\rangle & \dots & \langle m_n, v_k\rangle \end{pmatrix}_{n\times (k+1)}$$

has the rank $k+1$*. Therefore we can find a* $(k+1)\times(k+1)$ *submatrix of* Ψ *which is invertible.*

Hence, we can run column operations and permutations, i.e., a change of the basis, to convert Ψ *into a matrix* Ψ' *of the following form*

$$\Psi' = \begin{pmatrix} 1 & b_1 & b_2 & \dots & b_k \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & & * \\ 0 & * & * & \dots & * \end{pmatrix}_{n\times (k+1)}.$$

Thus with respect to this new basis $\{v_i'\}_{i=1}^k$ *of* V *the short exact sequence* (2.2) *becomes*

$$0 \to A_\rho^k \xrightarrow{\Phi'} A_\rho^n \to \operatorname{Coker}\Phi' \to 0$$

with

$$\Phi' = \begin{pmatrix} b_1 \cdot \chi^\rho & b_2 \cdot \chi^\rho & \dots & b_k \cdot \chi^\rho \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix}_{n\times k}.$$

Now we can directly check that $\operatorname{Coker}\Phi'$ *is torsion-free. If there exists an* $\boldsymbol{f} = f_1 \cdot \frac{\partial}{\partial\chi^{m_\rho}} + \sum_{i=2}^{n} f_i \cdot \chi^{m_i} \cdot \frac{\partial}{\partial\chi^{m_i}} \in \mathcal{T}_{X_\Delta}|_{\mathrm{U}_\rho}$ *and a* $g \in A_\rho$ *such that* $g \cdot \boldsymbol{f} \in \mathcal{G}_{V,\mathrm{U}_\rho}$*, then there exist* $\{g_i\} \in A_\rho$ *and we can write*

$$g \cdot \boldsymbol{f} = \sum_{j=1}^{k} g_i \cdot \delta_{v'_j} = (\sum_{i=1}^{k} g_i b_i) \cdot \chi^{m_\rho} \frac{\partial}{\partial\chi^{m_\rho}} + g_1 \cdot \chi^{m_2} \frac{\partial}{\partial\chi^{m_2}} + \dots + g_k \cdot \chi^{m_{k+1}} \frac{\partial}{\partial\chi^{m_{k+1}}} + \sum_{j=k+2}^{n} c_{ij} \cdot \chi^{m_j} \frac{\partial}{\partial\chi^{m_j}}. \quad (2.3)$$

This shows $f_1 = (\sum_{i=1}^{k} b_i f_{i+1}) \cdot \chi^{m_\rho}$*. Thus we have* $\boldsymbol{f} = \sum_{i=1}^{k} f_{i+1}\delta_{v'_i} \in \mathcal{G}_{V,\mathrm{U}_\rho}$*. Hence* $\mathcal{G}_{V,\mathrm{U}_\rho}$ *is saturated in* $\mathcal{T}_{X_\Delta}|_{\mathrm{U}_\rho}$ *and it follows that* $\mathcal{F}_V|_{\mathrm{U}_\rho} \cong \mathcal{G}_{V,\mathrm{U}_\rho}$*.*

Case 2 ($\rho \subset V$)**.** *For the case* $\rho \subset V$*, we observe* $\frac{\partial}{\partial\chi^{m_\rho}} \in \mathcal{G}_{V,\mathrm{U}_\rho}$*, since* $\delta_{u_\rho}|_{\mathrm{U}_\rho} = \chi^{m_\rho}\frac{1}{\partial\chi^{m_\rho}}$*. Hence, after choosing a new basis* $\{v''_i\}_{i=1}^{k}$ *of* V*, the short exact sequence (2.1) becomes*

$$0 \to A_\rho^k \xrightarrow{\Upsilon} A_\rho^n \to \operatorname{Coker}\Upsilon \to 0 \quad (2.4)$$

with a matrix

$$\Upsilon = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \langle m_2, v''_2\rangle & \dots & \langle m_2, v''_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \langle m_n, v''_2\rangle & \dots & \langle m_n, v''_k\rangle \end{pmatrix}_{n\times k},$$

with coefficient in $\mathbb{C}$*. Thus* (2.4) *is induced by the exact sequence*

$$0 \to \mathbb{C}^k \xrightarrow{\Upsilon} \mathbb{C}^n \to \mathbb{C}^{n-k} \to 0.$$

Since A_ρ *is flat over* $\mathbb{C}$*, the* $\operatorname{Coker}\Upsilon$ *in* (2.4) *is locally free. Thus* $\mathcal{G}_{V,\mathrm{U}_\rho}$ *is saturated in* $\mathcal{T}_{X_\Delta}|_{\mathrm{U}_\rho}$*.*

□

2.1.4 Special form

To determine the stability of $\mathcal{T}_{X_\Delta}$ on a $\mathbb{Q}$-factorial, projective, toric variety X_Δ with respect to a movable curve class C it suffices to compare $\mu_C(\mathcal{F}_V)$ and $\mu_C(\mathcal{T}_{X_\Delta})$ for all $V \subset N_\mathbb{R}$, which consist of infinitely many objects. In this subsection we shall show that the stability of $\mathcal{T}_{X_\Delta}$ with respect to C can be determined by comparing finitely many $\mu_C(\mathcal{F}_V)$'s and $\mu_C(\mathcal{T}_{X_\Delta})$.

Definition 2.1.13. *Let* S *be a subset of* $\Delta(1)$ *and* V_S *to be the minimal vector space containing all elements in* S*. We call* V_S *the vector space associated to* $S \subset \Delta(1)$*.*

Remark 2.1.14. *A sub vector space $V \subset N_{\mathbb{R}}$ is V_S for some $S \subset \Delta(1)$ if and only if there exists a vector space basis of V which consists of primary generators of one-ray's in the fan Δ. For different subsets S and S' of $\Delta(1)$, we may have $V_S = V_{S'}$. E.g. if the lattice N is $\mathbb{Z}$, and the fan consists of $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, and 0, then both of $V_{\{R_{\geq 0}\}}$ and $V_{\{R_{\leq 0}\}}$ are $\mathbb{R} \cong N_{\mathbb{R}}$.*

There are infinity many sub vector spaces $V \subset N_{\mathbb{R}}$, hence infinity many invariant saturated subsheaves $\mathcal{F}_V \subset \mathcal{T}_{X_\Delta}$. Then following Lemma assert that the maximal destabilizing subsheaf must be of the form $\mathcal{F}_{V_S}$ for some $S \subset \Delta(1)$.

Lemma 2.1.15. *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety and C a non-zero movable curve class of X_Δ. Then the maximal destabilizing subsheaf of $\mathcal{T}_{X_\Delta}$ is of the form $\mathcal{F}_{V_{S'}}$ for some $S' \subset \Delta(1)$.*

Proof. We shall show that the slope of a C-destabilizing subsheaf $\mathcal{F}_V$ does not achieve the maximal value if the vector space V is not of form V_S for some $S \subset \Delta(1)$. First we observe that since $\mathcal{F}_V$ is a destabilizing subsheaf of $\mathcal{T}_{X_\Delta}$ and C is not numerically trivial, then $\mu_C(\mathcal{F}_V) > 0$. If V contains no V_S except for $V_\varnothing$, then $\rho \not\subset V$ for all $\rho \in \Delta(1)$. Therefore $\mu_C(\mathcal{F}_V) = 0$, which contradicts to the assumption that $\mathcal{F}_V$ is a C-destabilizing subsheaf of $\mathcal{T}_{X_\Delta}$. Now we take the maximal subset $S' \subset \Delta(1)$ with $V_{S'} \subsetneq V$. Then we have $\{\rho \in \Delta(1) : \rho \subset V\} = \{\rho \in \Delta(1) : \rho \subset V_{S'}\}$. Therefore $\mu_C(\mathcal{F}_{V_{S'}}) > \mu_C(\mathcal{F}_V)$, since $\mu_C(\mathcal{F}_V) > 0$. □

Corollary 2.1.16. *The stability of $\mathcal{T}_{X_\Delta}$ with respect to a movable curve class C can be verified by comparing $\mu_C(\mathcal{T}_{X_\Delta})$ and slopes of finitely many subsheaves of $\mathcal{T}_{X_\Delta}$, namely $\{\mu_C(\mathcal{F}_{V_S}) : S \subset \Delta\}$. In particular, every term of the Harder-Narasimhan filtration of $\mathcal{T}_{X_\Delta}$ are of the form $\mathcal{F}_{V_S}$ for some $S \subset \Delta(1)$.* □

With the Theorem 2.1.8, we have the following corollary.

Corollary 2.1.17. *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety and C a movable curve class. Recall that for each movable curve class α, we can associate a movable subset $\Sigma_\alpha := \{\rho \in \Delta(1) : \alpha.D_\rho > 0\}$. Then $\mathcal{F}_{V_{\Sigma_C}}$ is a term of the Harder-Narasimhan filtration with respect to C. In fact, we have*

$$\mathrm{HNF}_C(\mathcal{T}_{X_\Delta}) = 0 \subset \mathcal{F}_1 \ldots \subset \mathcal{F}_{V_{\Delta_C}} \subset \mathcal{T}_{X_\Delta}.$$

Proof. It immediately follows from $\deg_C(\mathcal{T}_{X_\Delta}/\mathcal{F}_{V_{\Delta_C}}) = 0$, which can be derived by Theorem 2.1.8. □ □

Definition 2.1.18 (Destabilizing chamber, [Neu10, Definition 3.3.1])**.** *Let X be a $\mathbb{Q}$-factorial, projective, normal variety and C a movable curve class. We define an equivalence relationship on $\overline{\mathrm{Mov}_1}(X)$, say $C \sim C'$ if and only that $\mathrm{HNF}_C(\mathcal{T}_X) = \mathrm{HNF}_{C'}(\mathcal{T}_X)$. We call the set of a fixed equivalence class a* destabilizing chamber*.*

Proposition 2.1.19 (Polyhedral property of destabilizing chambers)**.** *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety. Then there are finitely many destabilizing chambers, and destabilizing chambers are convex cones whose closures are polyhedral cones.*

Proof. Finiteness follows from Corollary 2.1.17. The polyhedral property are consequence that each inequality $\mu_C(\mathcal{F}_{V_S}) \leq \mu_C(\mathcal{F}_{V_{S'}})$, $S, S' \subset \Delta(1)$ defines a half plane on $\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ and each destabilizing chambers are defined by these finitely many inequalities $\mu_C(\mathcal{F}_{V_S}) \leq \mu_C(\mathcal{F}_{V_{S'}})$. □

Remark 2.1.20. *Neumann showed that if $\overline{\mathrm{Mov}}_1(X)$ is a closed polyhedral cone, then the number of chambers is finite, and locally polyhedral property holds in the interior of $\overline{\mathrm{Mov}}_1(X)$, cf. [Neu10, Theorem 3.3.4, Proposition 3.3.5]. For the toric variety, $\overline{\mathrm{Mov}}_1$ is closed polyhedral, hence both properties have been proved in the paper [Neu10]. The reason we represent Corollary 2.1.19 is that both statements are much clear in the toric content.*

2.2 Proof of Theorem 0.0.11

Proof. By Remark 1.3.23, to determine the Harder-Narasimhan filtration of a given torsion-free sheaf $\mathcal{F}$ with respect to a movable class C, it is equivalent to find an increasing filtration of sheaves $\mathcal{F}_i$'s such that each $\mathcal{F}_i$ is the preimage of the maximal destabilizing subsheaf of $\mathcal{F}/\mathcal{F}_{i-1}$ in $\mathcal{F}$. Hence the algorithm to find the maximal destabilizing subsheaf is the algorithm to calculate the Harder-Narasimhan filtration. The algorithm is the following:

Step 1: We calculate the slope $\mu_C(\mathcal{T}_{X_\Delta})$ of the tangent sheaf with respect to C via the *Euler sequence* in Theorem 1.2.18.

Step 2: By comparing $\mu_C(\mathcal{F}_{V_S})$ for all $S \subset \Delta(1)$ and $\mu_C(\mathcal{T}_{X_\Delta})$, we are able to determine the stability of $\mathcal{T}_{X_\Delta}$.

Step 3: If $\mathcal{T}_{X_\Delta}$ is unstable, then we have a non-trivial Harder-Narasimhan filtration. The first term of the Harder-Narasimhan filtration, which is the maximal destabilizing subsheaf of $\mathcal{T}_{X_\Delta}$, was determined in Step 2.

Step 4: Assume that we have already determined the first l-terms $0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_l = \mathcal{F}_{V_l}$ of $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta})$. The next term of $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta})$ is the preimage of the maximal destabilizing subsheaf of $\mathcal{T}_{X_\Delta}/\mathcal{F}_l$ in $\mathcal{T}_{X_\Delta}$. To determine the stability of $\mathcal{T}_{X_\Delta}$ we need to compare the slopes of all the invariant saturated subsheaves $\overline{\mathcal{G}} \subset \mathcal{T}_{X_\Delta}/\mathcal{F}_l$ and $\mathcal{T}_{X_\Delta}/\mathcal{F}_l$. But the preimage $\mathcal{G}$ of $\overline{\mathcal{G}}$ in $\mathcal{T}_{X_\Delta}$ is also invariant and saturated, thus $\mathcal{G} = \mathcal{F}_W$ for some $W \subset N_\mathbb{R}$. Due to the short exact sequence

$$0 \to \mathcal{F}_l = \mathcal{F}_{V_l} \to \mathcal{G} \to \overline{\mathcal{G}} \to 0,$$

we see that $\mu(\overline{\mathcal{G}}) = \frac{1}{\dim W/V_l} \sum_{\rho \subset W,\ \rho \not\subset V_k} D_\rho.C$. Therefore the stability of $\mathcal{T}_{X_\Delta}/\mathcal{F}_l$ can be tested by comparing $\mu_C(\mathcal{T}_{X_\Delta}/\mathcal{F}_l)$ and

$$\max_{V_l \subset V_S, S \subset \Delta(1)} \frac{1}{\dim V_S/V_l} \sum_{\rho \in S, \rho \not\subset V_l} D_\rho.C.$$

Since $\{S \subset \Delta(1) : V_l \subset V_S\}$ is a finite set, we can also determine the maximal destabilizing subsheaf of $\mathcal{T}_{X_\Delta}/\mathcal{F}_l$.

There are only finitely many calculations in each step. Therefore we can use computer to calculate the $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta})$ for any given $\mathbb{Q}$-factorial, projective, toric variety X_Δ and a movable curve class C. □

2.3 The Harder-Narasimhan filtration of $\mathbb{Q}$-factorial, toric variety with Picard number one

Adopting the algorithm to weighted projective spaces, we have a relatively simple form in this case. Let $\mathbb{P}(a_0, \ldots, a_n)$ be a weighted projective space with the toric structure of 1.2.23 for some $a_0 \geq \ldots \geq a_n$.

Definition 2.3.1. *For any set of decreasing integers $b_0 \geq \ldots \geq b_n$ there are integers $\{k_j\}$ with the following property:*

$$b_0 = \ldots = b_{k_1} > b_{k_1+1} = \ldots = b_{k_2} > \ldots = b_{k_l} > \ldots = b_n, \text{ and } b_{-1} := 0$$

And there exists a number e such that

$$\sum_{i=k_e+1}^{n} b_i/(n-k_e) \geq b_{k_e+1} \text{ and } \sum_{i=k_{e-1}+1}^{n} b_i/(n-k_{e-1}) < b_{k_e} = b_{k_{e-1}+1} \tag{2.5}$$

We call such set $\{e, \{k_i\}\}$ a mark of decreasing integers $\{b_i\}_{i=1}^n$.

Proposition 2.3.2 (Simple formulation)**.** *Consider any given weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ with $a_0 \geq \ldots \geq a_n$. Let $\{e, \{k_i\}\}$ be the mark of decreasing integers $\{a_i\}_{i=0}^n$, then the Harder-Narasimhan filtration of $\mathcal{T}_{X_\Delta}$ is*

$$0 \subset \mathcal{F}_1 \subset \ldots \mathcal{F}_e \subset \mathcal{T}_{X_\Delta}, \tag{2.6}$$

where $\mathcal{F}_i = \mathcal{F}_{V_{S_i}}$ with $S_i = \{\rho_0, \ldots, \rho_{k_i}\}$.

Proof. We adopt the general algorithm and show that its output is exactly the filtration (2.6) of 2.3.2. Since the *Picard number* of $\mathbb{P}(a_0, \ldots, a_n)$ is one, the Harder-Narasimhan filtration is independent of the choice of the curve class. By Proposition 1.2.21we have the ratios $D_{\rho_i}.C/D_{\rho_j}.C = a_i/a_j$, we may pick a curve class C such that $D_{\rho_i}.C = a_i$ for all i and we simply denote the slope function as μ.

Step 1: The *Euler sequence* of Theorem 1.2.18 and $D_{\rho_i}.C = a_i$ give us $\mu(\mathcal{T}_{X_\Delta}) = \sum_{i=0}^n a_i$.

Step 2: For $S = \{\mathbb{Z}_{\geq 0}u_{i_1}, \ldots, \mathbb{Z}_{\geq 0}u_{i_k}\}$ we have $\mu(\mathcal{F}_{V_S}) = \sum_{j=1}^k a_{i_j}/k$ for $k < n$, and $\mathcal{F}_{V_S} = \mathcal{T}_{X_\Delta}$ for $k \geq n$. Hence we have the following result

$$\max_{\mathcal{G} \subsetneq \mathcal{T}_{X_\Delta}, \text{invariant saturated}} \mu(\mathcal{G}) = a_0.$$

Therefore, $\mathcal{T}_{X_\Delta}$ is semistable if $a_0 \leq \sum a_i/n$ and unstable if $a_0 > \sum a_i/n$. We note that the inequality $e \geq 1$ in (2.5) is equivalent to the unstability of $\mathcal{T}_{X_\Delta}$.

Step 3: If $\mathcal{T}_{X_\Delta}$ is unstable, then it is clear that $\mathcal{F}_{V_{S_1}}$ is the maximal destabilizing subsheaf of $\mathcal{T}_{X_\Delta}$ with slope $\mu(\mathcal{F}_{V_{S_1}}) = a_0$ such that

$$S_1 := \{\mathbb{Z}_{\geq 0}u_1, \mathbb{Z}_{\geq 0}u_2, \ldots, \mathbb{Z}_{\geq 0}u_{k_1}\}.$$

Step 4: Assume that the first l-terms $0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}_{V_k}$ of $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta})$ have been determined and $\mathcal{F}_i = \mathcal{F}_{V_{S_i}}$ with $S_i = \{\mathbb{Z}_{\geq 0}u_1, \ldots, \mathbb{Z}_{\geq 0}u_{k_l}\}$. Then

$$\max_{V_l \subset V_S, S \subset \Delta(1)} \frac{1}{\dim V_S/V_l} \sum_{\rho \in S, \rho \not\subset V_l} D_\rho.C = a_{k_l+1} = a_{k_{l+1}}.$$

If $\sum_{k_l+1}^{n} a_i/(n-k_l) \geq a_{k_l+1}$, then we have $l = e$ and $\mathrm{HNF}(\mathcal{T}_{X_\Delta}) : 0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}_{V_k} \subset \mathcal{T}_{X_\Delta}$. If $\sum_{k_l+1}^{n} a_i/(n-k_l) < a_{k_l+1}$, then $\mathcal{F}_{S_{l+1}}$ is the preimage of maximal destabilizing subsheaf of $\mathcal{T}_{X_\Delta}/\mathcal{F}_l$ in $\mathcal{T}_{X_\Delta}$. This is the next term of $\mathrm{HNF}(\mathcal{T}_{X_\Delta})$.

Therefore, the filtration (2.6) equals the output of the general algorithm. $\square$

Corollary 2.3.3. *We also have a similarly simple algorithm for the fake weighted projective space* $\mathbb{P}(u_0, \ldots, u_n)$. $\square$

3 Reconstruction of rational contractions

In this chapter, we prove our main result: the relative tangent sheaf $\mathcal{T}_f$ of a rational contraction f between projective, toric varieties is a term of the Harder-Narasimhan filtrtion with respect to a suitable movable curve class. Furthermore, the choice of such movable curve class is natural in the sense that we can choose the general complete intersection curve in a general fibre of such rational map. We also show the positivity of the relative tangent sheaf along the complete intersection curve in a general fibre of rational map. We use the positivity to construct a rational map

$$\begin{array}{cccc} q_f: & X & \dashrightarrow & \mathrm{Chow}(X) \\ & x & \rightarrow & \mathcal{T}_f\text{-leaf through x'} \end{array}$$

then q and f coincide on an open set of variety X. Thus we can reconstruct any rational contraction birationally via the geometry of the cone of movable curves and the Harder-Narasimhan filtration of the tangent sheaf.

The organization of this chapter is as follows. In the first three sections, we prove that the relative tangent sheaf of Mori fibre space(Rational maps in Minimal Model program/rational contractions) is a term of the Harder-Narasimhan filtration of the tangent sheaf with respect to general complete intersection curve in a general fibre of such rational map. Then we showed that the restriction of the relative tangent sheaf on general complete intersection curve in a general fibre is an ample bundle. Then we concludes that the rational contractions we considered coincides with the rational map associated to foliations. During the course of proof, we also derived a Mehta-Ramanathan type theorem for the movable curve in the case of Mori fibre space. In the fourth section, we give a structure theory of the cone of movable curves. We discuss when a invariant subsheaf of the tangent sheaf is the relative tangent sheaf of some rational contraction in the last section.

3.1 Reconstruction of Mori fibre space

In this section we prove that the relative tangent sheaf of the Mori fibre space appears as a term of Harder-Narasimhan filtration of tangent sheaf of the total space with respect to a contracted curve, and the contraction map can be recovered from the foliation associated to such Harder-Narasimhan filtration. We recall the following necessary results of the toric minimal model program from Reid ([Reid83]).

Setting 3.1.1. *Let $X = \mathrm{X}_\Delta$ be a $\mathbb{Q}$-factorial, projective, toric variety. We pick $\omega = \langle e_1, \ldots, e_{n-1}\rangle \in \Delta(n-1)$ and let $\rho_i \in \Delta(1)$ be the 1-ray generated by e_i. Since*

Δ is simplicial, there are exactly two rays $\rho_n = \mathbb{Z}_{\geq 0} e_n$, $\rho_{n+1} = \mathbb{Z}_{\geq 0} e_{n+1} \in \Delta(1)$ such that

$$\delta_{n+1} = \langle e_1, \ldots, e_{n-1}, e_n \rangle \text{ and } \delta_n = \langle e_1, \ldots, e_{n-1}, e_{n+1} \rangle$$

are in Δ. After arranging the indices and rescaling the numbers, we have the following relations

$$\sum_{i=1}^{n+1} a_i e_i = 0, a_i \in \mathbb{Q}, \text{ and } a_{n+1} = 1$$

with two positive integers β_ω and α_ω such that

$$\begin{array}{lll} a_i < 0 & \text{for} & 1 \leq i \leq \alpha_\omega \\ a_i = 0 & \text{for} & \alpha_\omega + 1 \leq i \leq \beta_\omega \\ a_i > 0 & \text{for} & \beta_\omega + 1 \leq i \leq n+1 \end{array} \tag{3.1}$$

Lemma 3.1.2 ([Reid83, 2.11])**.** *Let X be a $\mathbb{Q}$-factorial, projective, toric variety, and suppose that $R = \mathbb{R}_{\geq 0} V(\omega)$ is an extremal ray of $\overline{\mathrm{NE}}_1(\mathrm{X}_\Delta)$. If $\omega' \in \Delta(n-1)$ with $V(\omega') \in R$, then $\alpha_\omega = \alpha_{\omega'}$ and $\beta_\omega = \beta_{\omega'}$. In fact, we have equalities of the sets of vectors $\{e_1, \ldots, e_{\alpha_\omega}\} = \{e_1, \ldots, e_{\alpha_{\omega'}}\}$ and $\{e_{\beta_\omega+1}, \ldots, e_{n+1}\} = \{e_{\beta_{\omega'}+1}, \ldots, e_{n+1}\}$.* □

Notation 3.1.3. *Under the assumptions and notation of Setting 3.1.1 we set*

$$\delta(\omega) = \langle e_1, \ldots, e_n, e_{n+1} \rangle.$$

and

$$U(\omega) = \langle e_1, \ldots, e_\alpha, e_{\beta+1}, \ldots, e_{n+1} \rangle$$

By Lemma 6.2 $U(\omega)$ only depends on the extremal ray, so we denote it as $\mu(R)$. We also observe that $U(R)$ is a vector space, if $\alpha = 0$.

Theorem 3.1.4 ([Reid83, Section 2])**.** *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety and suppose that R is an extremal ray of X_Δ whose contraction is of Mori fibre space type. Let us remove an $\omega \in \Delta(n-1)$ with $V(\omega) \in R$ and replace $\delta_n \cup \delta_{n+1}$ by $\delta(\omega)$. Then we get a complete fan Δ^*_R and we have:*

(1) $\alpha = 0$,

(2) $\Delta_R = \Delta^*_R / U(R)$ *is a complete simplicial fan,*

(3) The Mori fiber space is $\phi_R : \mathrm{X}_\Delta \to X_{\Delta_R}$, where the morphism is induced by the projection $N_\mathbb{R} \to N_\mathbb{R}/U(R)$. Therefore it is denoted by $\overline{\phi_R}$.

□

Any curve C with $C = V(\omega) \in R$ is movable. We want to calculate $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta})$. Recall that in Definition 2.1.4 we associate each sub vector space $V \subset N_\mathbb{R}$ with an invariant saturated subsheaf $\mathcal{F}_V$ of $\mathcal{T}_{X_\Delta}$.

Lemma 3.1.5. *Here we follow the notation of the Theorem 3.1.4. Let* X_Δ *be a* $\mathbb{Q}$*-factorial, projective, toric variety with the Mori fibre space structure* $\phi_R : \mathrm{X}_\Delta \to X_{\Delta_R}$ *for an extremal ray* R*. Let* $C = V(\omega) \in R$ *be a contracted curve. Then* $\mathcal{F}_{U(R)}$ *is a term of* $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta})$*. In fact, we have*

$$\mathrm{HNF}_C(\mathcal{T}_{X_\Delta}) : 0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{U(R)} \subset \mathcal{T}_{X_\Delta}.$$

Proof. First of all, in our case the cone $U(R)$ is a sub vector space, therefore the notation $\mathcal{F}_{U(R)}$ is valid. We observe that by (3.1) and Proposition 1.2.21 we have $D_\rho.C > 0$ if and only if that $\rho \subset U(R)$. Therefore $U(R)$ is the minimal sub vector space containing all $\rho \in \Delta(1)$ with $D_\rho.C > 0$ and $\mu_C(\mathcal{T}_{X_\Delta}/\mathcal{F}_{U(R)}) = 0$. Thus both statement follow from Corollary 2.1.17. □

Proposition 3.1.6. *We follow the notation of Theorem 3.1.4. Let* $\phi_R : \mathrm{X}_\Delta \to X_{\Delta_R}$ *be a* $\mathbb{Q}$*-factorial, projective, toric, Mori fibre space with extremal ray* R*. Then* $\mathcal{F}_{U(R)} = \mathcal{T}_{\mathrm{X}_\Delta/X_{\Delta_R}}$*.*

Proof. We observe that both $\mathcal{T}_{\mathrm{X}_\Delta/X_{\Delta_R}}$ and $\mathcal{F}_{U(R)}$ equal the saturation of $U(R) \otimes \mathcal{O}_{\mathrm{X}_\Delta} \subset \mathcal{T}_{X_\Delta}$. □

Corollary 3.1.7. *We follow the notation of Theorem 3.1.4. The sheaf* $\mathcal{F}_{U(R)}$ *is a foliation.*

Remark 3.1.8. *If* $0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}_{U(R)} \subset \mathcal{T}_{X_\Delta}$ *is the Harder-Narasimhan filtration of* $\mathcal{T}_{X_\Delta}$ *with respect to a movable curve* C*, then by Proposition 1.3.32 every* $\mathcal{F}_i$ *is a foliation .*

Lemma 3.1.9 (Ampleness of $\mathcal{F}_{U(R)}|_C$)**.** *Here we follow the assumptions and notation of Theorem 3.1.4. Let* C *be a general complete intersection curve in a general fibre* F *of the morphism* $\phi_R : \mathrm{X}_\Delta \to X_{\Delta_R}$ *and* $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta}) = 0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{U(R)} \subset \mathcal{T}_{X_\Delta}$*. Then the restriction* $\mathcal{F}_{U(R)}|_C$ *is an ample vector bundle on* C*.*

Proof. Recalling that the general fibre F of ϕ_R is $\mathbb{Q}$-factorial, projective, normal with Picard number one (see Proposition 1.2.20). Sheaf $\mathcal{T}_{X_\Delta}|_F$ is torsionfree which follows from the restriction of torsion free sheaves on a general fibre remains torsion free, cf. [HL97, Corollary 1.1.3]. Now, we can pick a sufficiently general complete intersection curve $C \subset F$ such that C satisfies the following conditions.

(1) $C \subset F_{reg}$, therefore $C \subset X_{reg}$.

(2) the sheaf $\mathcal{F}_{U(R)}|_C$ is a subbundle of $\mathcal{T}_{X_\Delta}|_C$.

(3) The curve satisfies the condition of Mehta-Ramanathan theorem for $\mathcal{T}_{X_\Delta}|_F$, i.e., we have $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta}|_F)|_C = \mathrm{HNF}(\mathcal{T}_{X_\Delta}|_C)$.

Since the Picard number of F is one, there is only one Harder-Narasimhan filtration of $\mathcal{T}_{X_\Delta}|_F$ on F. Theorem 3.1.11 and condition (3) of the curve imply that $\mathrm{HNF}_C(\mathcal{T}_{X_\Delta})|_C = \mathrm{HNF}(\mathcal{T}_{X_\Delta}|_C)$. Therefore Proposition 1.3.34 ensures that $\mathcal{F}_{U(R)}|_C$ is an ample vector bundle on C. □

Corollary 3.1.10 (Reconstruction)**.** *Here we follow the assumptions and notation of Theorem 3.1.4. Then the morphism $\phi_R : \mathrm{X}_\Delta \to X_{\Delta_R}$ can be realized as a rational map*

$$\begin{array}{cccc} q_R: & \mathrm{X}_\Delta & \dashrightarrow & \mathrm{Chow}(\mathrm{X}_\Delta) \\ & x & \to & \mathcal{T}_{\mathrm{X}_\Delta/X_{\Delta_R}}\textit{-leaf through } x \end{array}$$

Proof. We see that $\mathcal{F}_{U(R)}|_C$ is an ample bundle on C. Therefore Theorem 1.3.31 implies that the general leaves of $\mathcal{F}_{U(R)}$ are algebraic, hence we can construct the map

$$\begin{array}{cccc} q_R: & \mathrm{X}_\Delta & \dashrightarrow & \mathrm{Chow}(\mathrm{X}_\Delta) \\ & x & \to & \mathcal{T}_{\mathrm{X}_\Delta/X_{\Delta_R}}\text{-leaf through x} \end{array}$$

Let $Q_R = \mathrm{Im}(q_R)$, then we have rational map $q_R : \mathrm{X}_\Delta \dashrightarrow Q_R$, the map ϕ_R coincide with q_R on an open set of X_Δ. ☐

We may regard the following theorem as a variation of the Mehta-Ramanathan Theorem.

Theorem 3.1.11 (Mehta-Ramanathan type theorem)**.** *Here we follow the assumptions and notation of Theorem 3.1.4. Let $\phi : \mathrm{X}_\Delta \to X_{\Delta_R}$ be a Mori fibre space of $\mathbb{Q}$-factorial, projective, complex, toric varieties contracting extremal ray $R = \mathbb{R}_{\geq 0}C$ with a curve C in a general fibre F. Then we have*

$$\mathrm{HNF}_C(\mathcal{T}_{\mathrm{X}_\Delta})|_F = \mathrm{HNF}_C(\mathcal{T}_{\mathrm{X}_\Delta}|_F).$$

Proof. The Harder-Narasimhan filtration of $\mathcal{T}_{X_\Delta}$ is

$$\mathrm{HNF}_C(\mathcal{T}_{X_\Delta}) : 0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{U(R)} \subset \mathcal{T}_{X_\Delta}$$

by Remark 3.1.5. First of all, we note that the restrictions of reflexive(torsion-free) sheaves on a general fibre remains reflexive(torsion free)[cf.[HL97, Corollary 1.1.14]. Furthermore, the restriction of short exact sequences $0 \to \mathcal{F}_i \to \mathcal{T}_{X_\Delta} \to \mathcal{T}_{X_\Delta}/\mathcal{F}_i \to 0$ and $0 \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \mathcal{F}_{i+1}/\mathcal{F}_i \to 0$ to F remain exact by [HL97, Lemma 1.1.12] and the fact that if we have $0 \to F \to E$ and H such that H avoids the associated points of E/F then $0 \to F|_H \to E|_H$ is exact. Second, by Proposition 1.2.20 a general fibre F is a $\mathbb{Q}$-factorial, projective, toric variety with Picard number one, hence there is only one Harder-Narasimhan filtration of $\mathcal{T}_{X_\Delta}|_F$ on F. Hence it is suffices to show that the restriction of $0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{U(R)} \subset \mathcal{T}_{X_\Delta}$ satisfies the properties of the Harder-Narasimhan filtration in Theorem 1.3.20. For torsion-freeness, the sheaves $\mathcal{T}_{X_\Delta}/\mathcal{F}_i|_F = \mathcal{T}_{X_\Delta}|_F/\mathcal{F}_i|_F$ and $\mathcal{F}_i|_F/\mathcal{F}_{i-1}|_F = \mathcal{F}_i/\mathcal{F}_{i-1}|_F$ are torsion free by the choice of F. We note when C is general complete intersection curve on F, we may regard that C is also a curve on X_Δ, and $\deg_C(\mathcal{F}) = \deg_C(\mathcal{F}|_F)$. For the semi-stability of $\mathcal{F}_i|_F/\mathcal{F}_{i-1}|_F$, if $\mathcal{F}_i/\mathcal{F}_{i-1}|_F$ is not semi-stable, then we can find a subsheaf $\mathcal{G}_F \subset \mathcal{F}_i|_F/\mathcal{F}_{i-1}|_F$ with larger slope. We recall that $\phi^{-1}(T_{N_R}) \cong T_{N_R} \times F$, cf. Proposition 1.2.15, where T_{N_R} is the big torus of X_{Δ_R}. Then we consider G the saturation of coherent extension of $\mathrm{pr}_2^*(G_F)$ in $\mathcal{F}_i/\mathcal{F}_{i-1}$, cf [Har77, Exercise II.5.15]. We have $\mu_C(G) = \mu_C(G_F)$ by the fact that $D_\rho.C > 0$ if and only if $\rho \subset U(R)$. Then $\mu_C(G|_F) = \mu_C(G_F) > \mu_C(\mathcal{F}_i/\mathcal{F}_{i-1}) = \mu_C(\mathcal{F}_i|_F/\mathcal{F}_{i-1}|_F)$ contradicts to the semi-stability of $\mathcal{F}_i/\mathcal{F}_{i-1}$. The strictly decreasing of slopes are clear.

□

3.2 Reconstruction of Minimal Model

In this section, we show that we can recover the maps in minimal model program.

Definition 3.2.1. *Let X be a normal, projective variety. We say that X is* log Fano*, if there exists a $\mathbb{Q}$-divisor D such that $-(K_X + D)$ is an ample divisors on X and (X, D) has at worst Kawamata log terminal singularity(klt). We also call such pair (X, D) a log Fano pair.*

Remark 3.2.2. *It is well-known that a $\mathbb{Q}$-factorial, projective, toric variety X is log Fano. A proof can be found, e.g. [Mon13, Proposition 5.2.5].*

Remark 3.2.3. *A pair (X, D), where X a $\mathbb{Q}$-factorial, projective, toric variety and D an invariant $\mathbb{Q}$-divisor, has at worst klt singularity if and only if $D = \sum a_\rho D_\rho$ with $0 \leq a_\rho < 1$, cf. [CLS, Section 15.5].*

We recall the a structure theory of the cone of movable curves for the varieties with following result of Araujo.

Theorem 3.2.4 ([A, Corollary 1.2])**.** *Let X be a $\mathbb{Q}$-factorial, projective, toric variety and (X, D) a log Fano pair. Let $\Sigma \subset N_1(X)$ be the set of the classes of curves on X that are numerical pullbacks of curves lying on general fibres of Mori fibre space obtained from X by a $(K_X + D)$ -minimal model program with scaling. Then we have*

$$\overline{\mathrm{Mov}}_1(X) = \sum_{[C]\in\Sigma} \mathbb{R}_{\geq 0}[C] \tag{3.2}$$

□

Remark 3.2.5. *Araujo showed the theorem in a general situation, we only state the necessary part from it.*

Remark 3.2.6. *A $K_X + D$ negative curve may not be K_X negative, hence we are unable to remove the boundary divisor D.*

Definition 3.2.7 (Relative tangent sheaf of rational contraction map)**.** *Let $f : X \dashrightarrow Y$ be a rational contraction between normal, projective varieties. Let U be the largest open subset of X where f is well-defined, we observe that* $\mathrm{Codim}(X\backslash U) \geq 2$, *since X is normal and Y is projective, cf. [Har77, Lemma 5.1]. Then we define the relative tangent sheaf to be $i_*(\mathcal{T}_{U/Y}) \subset \mathcal{T}_X$, where $i : U \to X$ is the inclusion map, and denote it as $\mathcal{T}_{X/Y}$.*

Remark 3.2.8. *Since* $\operatorname{Codim}(X \backslash U) \geq 2$, $i_*(\mathcal{T}_{U/Y})$ *is a reflexive sheaf, in particular, it is coherent, cf. [S, Proposition 2.12].*

Theorem 3.2.9. *Let* X *be a* $\mathbb{Q}$*-factorial, projective, toric variety,* D *a* $\mathbb{Q}$*-divisor on* X *with coefficients between 0 and 1 such that* $K_X + D$ *ample, and* X' *an end result of* $(K_X + D)$*-minimal model program with scaling with Mori fibre space structure* $f : X' \to Z$ *, i.e., we have diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[dr, dashed, "\psi"'] & X' \arrow[d, "f"] \\ & Z, \end{tikzcd}$$

then there exists a movable curve class C *such that the relative tangent sheaf of* $\psi = f \circ \phi : X \dashrightarrow Z$ *is a term of* $\mathrm{HNF}_C(\mathcal{T}_X)$*. In fact, we can choose* C *to be an extremal ray of* $\overline{\mathrm{Mov}}_1(X)$*.*

Proof. For a given $(K_X + D)$-minimal model program with scaling $\phi : X \dashrightarrow X'$ and Mori fibre space $f : X' \to Z$. We observe that the Mori fibre space $f : X' \to Z$ comes from contracting a $K_X + D$-negative extremal ray R of $\overline{\mathrm{NE}}_1(X)$ which is also an extremal ray of $\overline{\mathrm{Mov}}_1(X)$, since D is non-negative on this ray, therefore R is also a K_X-negative ray. Now we pick a general complete intersection curve C' lies in a general fibre of f such that C' moves in a dominating family of curves on X' and C' avoids the indeterminacy locus of ϕ^{-1}. We let C be the birational transform of C', therefore it moves in a dominating family of curves on X, hence a movable curve. In fact, we have $C = \phi^*_{num}(C')$ and $K_X.C < 0$. Furthermore, C is an extremal point of $\overline{\mathrm{Mov}}_1(X)$ by Theorem 3.2.4. We observe that $\mathcal{T}_{X/Z}$ and $\mathcal{T}_{X'/Z}$ equal the saturation of $V \otimes \mathcal{O}_X$ and $V \otimes \mathcal{O}_{X'}$ in $\mathcal{T}_X$ and $\mathcal{T}_{X'}$ respectively for the same sub vector space V of $N_{\mathbb{R}}$. We note that $\mathrm{HNF}_C(\mathcal{T}_X)$ corresponds to an increasing filtration of sub vector spaces of $N_{\mathbb{R}}$ by Corollary 2.1.7, hence it suffices to show that V belongs to this filtration of sub vector spaces of $N_{\mathbb{R}}$. We have that $\mathcal{T}_{X'/Z}$ is a term of $\mathrm{HNF}_{C'}(\mathcal{T}_{X'})$ by Proposition 3.1.6. Therefore it is sufficient to show that $\mathrm{HNF}_C(\mathcal{T}_X)$ and $\mathrm{HNF}_{C'}(\mathcal{T}_{X'})$ assign same filtration of sub vector spaces of $N_{\mathbb{R}}$. But this is clear since that C does not intersects exceptional locus of ϕ, hence $C.E = 0$ for any exceptional divisor E. Therefore $\deg_C(\mathcal{F}'_V) = \deg_{C'}(\mathcal{F}'_V)$ for any sub vector space $V' \subset N_{\mathbb{R}}$, $\mathrm{HNF}_C(\mathcal{T}_X)$ and $\mathrm{HNF}_{C'}(\mathcal{T}_{X'})$ determine same filtration of sub vector spaces of $N_{\mathbb{R}}$. □

3.2.10. *We need to show that* $\mathcal{T}_{X/Z}|_C$ *is an ample bundle on* C *in order to prove the rational map* ϕ *and the rational map*

$$\begin{array}{rccc} q_{\mathcal{T}_{X/Z}} : & X & \dashrightarrow & \mathrm{Chow}(X) \\ & x & \to & \mathcal{T}_{X/Z}\text{-leaf through } x \end{array}$$

are birational equivalent, as we did in Corollary 3.1.10.

Remark 3.2.11. *The rational map* $X \overset{\phi}{\dashrightarrow} X'$ *is a birational contraction, hence we can find open sets* $U_X \subset X$ *and* $U_{X'} \subset X'$ *with the complement of* $U_{X'}$ *in* X' *has codimension* ≥ 2*, such that* $\phi : U_X \to U_{X'}$ *is an isomorphism. Since we can choose the complete intersection curve* C' *avoids the complement of* $U_{X'}$*, hence* $C' \subset U_{X'}$*, therefore the birational transform* $C \subset U_X$*. The restriction* $\mathcal{T}_{X'/Z}|_{C'}$ *is an ample vector bundle on* C' *by the choice of* C'*, but* $\mathcal{T}_{X'/Z}|_{U_{X'}} \cong \mathcal{T}_{X/Z}|_{U_X}$*, therefore the ampleness of* $\mathcal{T}_{X/Z}|_C$ *follows immediately.*

Corollary 3.2.12 (Reconstruction'). *Here we follow the assumptions and notation of Theorem 3.2.9. Then the rational contraction* $\psi : X \dashrightarrow Z$ *can be realized as a rational map*

$$\begin{array}{cccc} q: & X & \dashrightarrow & \mathrm{Chow}(X) \\ & x & \to & \mathcal{T}_{X/Z}\text{-leaf through } x \end{array}$$

Proof. Since we have the ampleness of $\mathcal{T}_{X/Z}|_C$ for general complete intersection curve C. Then the conclusion follows along the same line as Corollary 3.1.10. $\square$

Remark 3.2.13 (Reinterpretation). *We can restate the Proposition 3.1.6 and Theorem 3.2.9 by following: Combination of the geometry of* $\overline{\mathrm{Mov}}_1(X)$ *and* $\mathrm{HNF}_C(\mathcal{T}_X)$*, we can recover all the rational map in the minimal model program in the case of toric geometry.*

3.3 Reconstruction of contractions

In this section, we extend the study to arbitrary rational contraction of fibre type. We prove that the relative tangent sheaf of a rational contraction of fibre type is a term of the Harder-Narasimhan filtration with respect to a suitable movable curve.

Setting 3.3.1. *Through this section,* X *will be a* $\mathbb{Q}$*-factorial, projective, toric variety associated to a fan in a lattice* N*. For a subset* F *of a vector space, we denote the relative interior of* F *by* F°*.*

3.3.1 Regular contraction

In this subsection, we prove that for a given regular contraction $\phi : X \to Z$. The relative tangent sheaf is a term of $\mathrm{HNF}_C(\mathcal{T}_X)$ for a suitable movable curve class C, and the morphism ϕ can be reconstructed from the foliation of $\mathcal{T}_{X/Z}$. From the viewpoint of the cone of movable curves, Section 3.1 deal with extremal rays of $\overline{\mathrm{Mov}}_1(X)$. We prove that the similar result also hold for the higher dimensional faces of $\overline{\mathrm{Mov}}_1(X)$.

Remark 3.3.2. *A morphism between projective varieties is uniquely determined by the curves it contracted. Hence we can describe the morphism by the geometry of the cone of the curves. A regular contraction corresponds to a face* $F \subset \overline{\mathrm{Mov}}_1 \cap \overline{\mathrm{NE}}_1$*.*

Remark 3.3.3. *A contraction $f : X \to Y$ equals to $X \to \operatorname{Proj}(H^0(X, L))$, where $L = f^*H$, and H an ample divisor on Y. If X is toric, then Y is also toric and f is a morphism between toric varieties whose restriction on big torus is a homomorphism between tori by this description.*

Theorem 3.3.4. *Let X be a $\mathbb{Q}$-factorial, projective, toric variety,*

$$F \subset \partial\overline{\mathrm{Mov}}_1(X) \bigcap \partial\,\overline{\mathrm{NE}}_1(X)$$

a face of $\overline{\mathrm{NE}}_1(X)$, and $\phi_F : X \to X_F$ the regular contraction contracts all curves of F. Then there is a movable curve class C such that the relative tangent sheaf of ϕ_F is a term of $\mathrm{HNF}_C(X)$. In fact, the curve class C can be chosen to be the curve class lies in the relative interior of F.

Proof. We start with a footnote that a general complete intersection curve in a general fibre is movable. Let set

$$F^{\perp} := \{D | D \in \mathrm{Eff}(X), D.C = 0 \text{ for } C \in F^{\circ}\},$$

then by the definition of F we see that every element $D \in F^{\perp}$ is nef, hence semi-ample. Furthermore, the morphism ϕ_F is the morphism associated divisor $D \in (F^{\perp})^{\circ}$. We set $\Sigma_F := \{\rho \in \Delta(1) | D_\rho.C > 0 \text{ for } C \in F^{\circ}\}$. It is clear that if we pick $C \in F^{\circ}$, then $\mathcal{F}_{V_{\Sigma_F}}$ is a term of $\mathrm{HNF}_C(\mathcal{T}_X)$ and $\mathrm{HNF}_C(\mathcal{T}_X) = 0 \subset \mathcal{F}_1 \ldots \subset \mathcal{F}_i \subset \mathcal{F}_{V_{\Sigma_F}} \subset \mathcal{T}_X$. Hence it suffices to show that $\mathcal{T}_{\mathrm{X}_\Delta/Z} = \mathcal{F}_{V_{\Sigma_F}}$. We prove the theorem in the case that Z is $\mathbb{Q}$-factorial first, then show that the general case can be reduced to $\mathbb{Q}$-factorial case.

3.3.5 (Assuming Z is $\mathbb{Q}$-factorial)**.** *The morphism ϕ_F is the contraction map of F, hence D_ρ is $\mathbb{Q}$-linearly equivalent to a pull back of a Cartier divisor from Z if and only if $\rho \notin \Sigma_F$ via the Cone theorem cf. [KM98, Theorem 3.7]. Since we can choose a fan structure on Y such that the morphism ϕ_F arise from a fan map $\varphi : N \to N'$ by Theorem 1.2.13.*

Claim 3.3.6 implies that the kernel of $\varphi_{\mathbb{R}}$, the $\mathbb{R}$-extension of φ, equal to the minimal sub vector space containing all rays of Σ_F. Therefore we have

$$0 \to V_{\Sigma_F} \to N_{\mathbb{R}} \to N'_{\mathbb{R}} \to 0,$$

and $\mathcal{T}_{X/Z} = \mathcal{F}_{V_{\Sigma_F}}$ immediately follows from Lemma 1.2.16.

Claim 3.3.6. *The divisor D_ρ is $\mathbb{Q}$-linearly equivalent to a a pull back of a Cartier divisor on Z if and only if $\varphi(\rho) = 0$ in N'.*

Proof of Claim 3.3.6. If $D_\rho = \phi_F^*(L)$ for some Cartier divisor L on Z, then we have $\phi_{F*}(D_\rho)$ is a proper torus invariant divisor of Z, hence $\phi(\rho) \neq 0$. If $\phi(\rho) \neq 0$, then $\phi_{F*}(D_\rho)$ is a divisor. Since Z is $\mathbb{Q}$-factorial, we may assume that $\phi_{F*}(D_\rho)$ is Cartier. Then we consider $\phi_F^*\phi_{F*}(D_\rho)$ which is an effective divisor whose support containing D. From the fact that $C \cdot \phi_F^*\phi_{F*}(D_\rho) = 0$, and C is a movable curve. We get that $D_\rho \cdot C = 0$, hence D_ρ comes from Z. □

3.3.7 (Reduce to the case Z is $\mathbb{Q}$-factorial)**.** *If Z is not $\mathbb{Q}$-factorial, since Z is a toric variety, hence there exists a small $\mathbb{Q}$-factorialization $f : Z' \to Z$ of Z by Theorem 1.2.22. Then the composition of $f^{-1} \circ \phi_F : X \dashrightarrow Z'$ is a rational contraction. Hence there exists a* SQM *$X' \dashrightarrow X$ such that the composition $g : X' \to Z'$ is a regular contraction by Proposition 1.4.15. Thus we have commutative diagram*

$$\begin{tikzcd} X' \arrow[r, dashed, "f'"] \arrow[d, "g"] & X \arrow[d, "\phi_F"] \\ Z' \arrow[r, "f"] & Z, \end{tikzcd}$$

and there are open sets $U_X \subset X$, $U_{X'} \subset X'$ such that the complements are of codimension ≥ 2, $f' : U_{X'} \to U_X$ is an isomorphism. We observe that

(1) $\mathcal{T}_{X'/Z'}$ and $\mathcal{T}_{X/Z}$ are the saturation of $V \otimes \mathcal{O}_{X'} \subset \mathcal{T}_{X'}$ and $V \otimes \mathcal{O}_X \subset \mathcal{T}_X$ with same $V \subset N_{\mathbb{R}}$; and

(2) once we choose the general complete intersection curve C in side a general fibre of ϕ_F, and C' be its birational transform in X'. Then $\mathrm{HNF}_C(\mathcal{T}_X)$ *and* $\mathrm{HNF}_{C'}(\mathcal{T}_{X'})$ *determine same filtration of sub vector spaces of $N_{\mathbb{R}}$. And both C and C' lies in the cone of contracted curves respectively.*

Therefore we reduce to Z is $\mathbb{Q}$-factorial.

$\square$

Remark 3.3.8. *For a regular contraction ϕ_F, we found that*

$$\{\rho \in \Delta(1) : \rho \subset V_{\Sigma_F}\} = \Sigma_F.$$

We have observed such phenomenon in the case of Mori fibre space case, cf. Theorem 3.1.4.

3.3.9. *In the Section 3.1, we use a Mehta-Ramanathan type theorem to show the ampleness of the restriction of $\mathcal{T}_{X/Z}$ on the general complete intersection curve on a general fibre. Nevertheless, the Picard number of the general fibre G of ϕ_F is not one, hence we do not have Theorem 3.1.11 in this content. Furthermore a general fibre G of ϕ_F may not be $\mathbb{Q}$-factorial, we need to take care what is the intersection number between a Weil divisor and a movable curve in this setting. Second, we have following weak version of Theorem 3.1.11.*

Remark 3.3.10 (Harder-Narasimhan filtration on non $\mathbb{Q}$-factorial spaces)**.** *In the following theorem, we need to consider Harder-Narasimhan filtration on non-$\mathbb{Q}$-factorial spaces. The major issue lies in how to define intersection number between Weil divisors and movable curves. Nevertheless, the intersection theory between Weil divisors* *and complete intersection curves remain valid as follows: C is a complete intersection curve, we may write $C = D_1 \cap \ldots \cap D_{\dim X-1}$. Then*

$$D.C := D_1 \cdot \ldots \cdot D_{\dim X-1} \cdot D$$

the intersection of $\dim X - 1$ Cartier divisors and a codimension one irreducible subvariety, cf. [Lar04, Section 1.1.C].

Theorem 3.3.11. *We follow the notation of Theorem 3.3.4. Let G be a general fibre of ϕ_F which is projective, toric and C a general complete intersection curve on G. Then $\mathcal{T}_{X/Z}|_G$ is a term of $\mathrm{HNF}_C(\mathcal{T}_X|_G)$.*

Proof. As G is general, we have $\mathcal{T}_X|_G$ is torsion free, hence we can talk about $\mathrm{HNF}_C(\mathcal{T}_X|_G)$. As we have $\mathrm{HNF}_C(\mathcal{T}_X|_G) = \{\mathcal{G}_{G,i}\}$, we can extend this filtration by taking consecutive saturation $\mathcal{G}_i$ of coherent extensions of sheaves of $\mathrm{HNF}_C(\mathcal{T}_X|_G)$ as we did in the course of the proof of Theorem 3.1.11 and we have $\deg_C(\mathcal{G}_i) = \deg_C(\mathcal{G}_{G,i})$, $\mathcal{G}_i|_G = \mathcal{G}_{G,i}$. We check that $\{\mathcal{G}_i\} = \mathrm{HNF}_C(\mathcal{T}_X)$. Furthermore, we have equality $\mathcal{T}_{X/Z} \in \mathrm{HNF}_C(\mathcal{T}_X)$ by Corollary 2.1.17, then conclusion follows. □

Remark 3.3.12. *If C is a general complete intersection curve on G, then $C \in F^\circ$.*

Remark 3.3.13. *By the formula $\deg_C(\mathcal{F}) = \deg_C(\mathcal{F}|_G)$, we also know that*

$$\mathrm{HNF}_C(\mathcal{T}_X|_G) = 0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k \subset \mathcal{T}_{X/Z}|_G \subset \mathcal{T}_X|_G,$$

and $\mu_C(\mathcal{T}_{X/Z}|_G/\mathcal{F}_k) > 0$.

Corollary 3.3.14 (Ampleness of relative tangent sheaf, II)**.** *We follow the assumptions and notation of Theorem 3.3.4. Let C be a sufficiently general complete intersection curve of a general fibre G of ϕ_F. Then $\mathcal{T}_{X/Y}|_C$ is an ample vector bundle on C.*

Proof. It is along the same line of Proof of Lemma 3.1.9, except that Theorem 3.1.11 is replaced by Theorem 3.3.11. □

Corollary 3.3.15 (Reconstruction II)**.** *Here we follow the assumptions and notation of Theorem 3.3.4. Then the morphism $\phi_F : X \to Z$ can be realized as a rational map*

$$\begin{array}{cccc} q: & X & \dashrightarrow & \mathrm{Chow}(X) \\ & x & \to & \mathcal{T}_{X/Z}\text{-leaf through } x \end{array}$$

Proof. It is along the same ling of the proof of Corollary 3.1.10. □

3.3.2 Rational contraction

In this subsection, we prove that the relative tangent sheaf of a rational contraction $\phi : X \dashrightarrow Z$ is a term of the Harder-Narasimhan filtration of the tangent sheaf with respect to a suitable movable curve, and we can reconstruct the rational map ϕ by the foliation associated to $\mathcal{T}_{X/Z}$.

Theorem 3.3.16. *Let $\phi : X \dashrightarrow Z$ be a rational contraction, Z is normal, projective. Then the relative tangent sheaf of ϕ is a term of the Harder-Narasimhan filtration of the tangent sheaf $\mathcal{T}_X$ with respect to a suitable movable curve class.*

Proof. Since X is a $\mathbb{Q}$-factorial, projective, toric variety, and ϕ is a rational contraction. Proposition 1.4.15 ensures that there exists a SQM X' of X filling the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[dr, dashed, "\phi"'] & X' \arrow[d, "h"] \\ & Z, \end{tikzcd}$$

and h is a regular contraction. By Theorem 3.3.4, we have $\mathcal{T}_{X'/Z}$ is a term of $\mathrm{HNF}_{C'}(\mathcal{T}_{X'})$, where C' is a general complete intersection curve in a general fibre of h. The rational map g is small, hence the birational transform C of C' is also a general complete intersection curve of the fibre of ϕ, and movable. The $\mathrm{HNF}_C(\mathcal{T}_X)$ and $\mathrm{HNF}_{C'}(\mathcal{T}_{X'})$ determine same sub vector spaces of $N_\mathbb{R}$ by the fact that C is the birational transform of C' and g is small. The sheaves $\mathcal{T}_{X/Z}$ and $\mathcal{T}_{X'/Z}$ also determine same sub vector space of $N_\mathbb{R}$, therefore we derive the desired conclusion. □

Corollary 3.3.17 (Reconstruction)**.** *Here we follow the assumptions and notation of Theorem 3.3.16. Then the rational contraction ϕ can be realized as a rational map*

$$\begin{array}{cccc} q: & X & \dashrightarrow & \mathrm{Chow}(X) \\ & x & \to & \mathcal{T}_{X/Z}\text{-leaf through } x \end{array}$$

Proof. The ampleness of $\mathcal{T}_{X/Z}|_C$ follows from ampleness of $\mathcal{T}_{X'/Z}|_{C'}$, the rest is along the same line of the proof of Corollary 3.1.10. □

3.4 Geometry of the cone of movable curves

In this section, we prove a promised Araujo type theorem of the cone of movable curves. We start with a remark of cones between small $\mathbb{Q}$-factorial modification.

Remark 3.4.1. *If X is $\mathbb{Q}$-factorial, projective, normal variety, and X' a SQM of X. Then birational transform of divisors gives the isomorphism between* $\mathrm{Eff}(X)$ *and* $\mathrm{Eff}(X')$*; the numerical pull back of curves gives the isomorphism between* $\overline{\mathrm{Mov}}_1(X)$ *and* $\overline{\mathrm{Mov}}_1(X')$.

Theorem 3.4.2 (Contraction)**.** *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety associated to fan $\Delta \subset N$. Then for every proper face of $\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$, we can associate a birational contraction $\phi_F : \mathrm{X}_\Delta \dashrightarrow X_{\Delta_F}$, such that F is the numerical pull back of a face F' of $\overline{\mathrm{Mov}}_1(X_{\Delta_F}) \cap \overline{\mathrm{NE}}_1(X_{\Delta_F})$. Furthermore, we have $\dim F = \dim F'$.*

Proof. Let $\Sigma_F \subset \Delta(1)$ be the movable set associated to the face $F \subset \overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ and $\Sigma_F = \{\rho \in \Delta(1) : D_\rho.C > 0, \text{ for all } C \in F\} \subset \Delta(1)$. Since F is a face, we have that $\Sigma_F \subsetneq \Delta(1)$ is a proper set. Then the we see that $E := \sum_{\rho\in\Delta(1)\setminus\Sigma_F} D_\rho$ is the defining effective divisor of $F \subset \overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$. If E is nef, then it is semi-ample, and we can build up a regular contraction, which contracts the face F, via the linear system $|E|$. If E is not nef, then we can run the E-minimal model program,

$$\phi_F : X \dashrightarrow X_F,$$

and $E_F := (\phi_F)_*(E)$ is nef. We note that X_F is also a toric variety, and we denote its fan by Δ_F. We also observe that the flip does not change the set of one cones; the divisorial contraction contracts the invariant divisors which covered by a E-negative curve C, then $C \subset \mathrm{Supp}(E)$ and $C \subset D_\rho$ for some $\rho \in \Delta(1)\setminus\Sigma_F$. Hence we have $\Sigma_F \subset \Delta_F(1)$, and $E_F = \sum_{\rho'\in\Delta_F(1)\setminus\Sigma_F} D_{\rho'}$ on X_F. The nef divisor E_F defines a face $F' \subset \overline{\mathrm{Mov}}_1(X_F) \cap \overline{\mathrm{NE}}_1(X_F)$. We observe that $C \subset F'$ if and only if $E_F \cdot C = 0$, if and only if $\Sigma_C \subset \Sigma_F$, therefore we have $\Sigma'_F = \Sigma_F$ as set of rays in $N_\mathbb{R}$.

Claim 3.4.3. *If $\phi : X \dashrightarrow X'$ a birational contraction between $\mathbb{Q}$-factorial, projective, toric varieties. Let $C' \subset X'$ a movable curve and $C = \phi^*_{num}C'$ its numerical pull back. Then we have $\Sigma_C = \Sigma_{C'}$*

Proof of Claim. It follows from $E.C = \phi_* E.C'$. □

Hence we find that $(\phi^*_F)_{num}(F') = F$ which follows from $\Delta_F = \Delta_{F'}$ and the claim. For the dimension property, it follows from that the dimension of F be can read from Σ_F. □

Remark 3.4.4. *Since E_F is nef on X_F, and X_F is a toric variety. Therefore E_F is basepoint-free. We set $\psi_F : X_F \to Z_F$ the induced morphism by E_F, and $\pi_F = \psi_F \circ \phi_F$. Then we have following diagram.*

$$\begin{tikzcd} X \arrow[r, dashed, "\phi_F"] \arrow[dr, dashed, "\pi_F"'] & XF \arrow[d, "\psi_F"] \\ & Z_F, \end{tikzcd}$$

Remark 3.4.5. *For a fixed face F of the cone of movable curves, it may has different birational contraction ϕ_F.*

Remark 3.4.6. *The existence of birational contraction ϕ_F a face F of $\overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ for toric varieties or log Fano varieties is well-known, e.g.[BCHM, Corollary 1.3.5]. We* *reproduce the proof here for reader's convenience and use the property of Mori dream space instead, e.g. Theorem 1.4.17. Furthermore, we use that ϕ_F is a composition of E-minimal model program maps to get the information of contracted divisors in order to have a simple proof of the dimension condition.*

Remark 3.4.7. *A weak toric(or Mori dream spaces) version of Araujo theorem(cf. Theorem 1.5.5) can be recovered by Theorem 3.4.2. In fact, Theorem 3.4.2 gives a directly proof of the equality (1.2). We identify the extremal rays of $\overline{\mathrm{Mov}}_1(X)$ at very beginning, then build up the desired rational contraction, and show this extremal ray is indeed a numerical pull back of the contracting curve of Mori fibre space.*

Remark 3.4.8. *In the proof we also showed that the numerical pull back of an extremal face of $\overline{\mathrm{Mov}}_1(X_F)$ remains an extremal face for $\overline{\mathrm{Mov}}_1(X)$. During the preparation of this work, the author learned that Monôres showed similar result for Log Fano varieties(which can be easily extend to Mori dream spaces). Our methods is explicit.*

Corollary 3.4.9 (Lower bound of necessary divisorial contractions)**.** *Let $F \subset \overline{\mathrm{Mov}}_1(\mathrm{X}_\Delta)$ be a face, then we have a lower bound of necessary divisorial contractions in the rational contraction ϕ_F.*

Proof. We observe that for the face $F' \subset \overline{\mathrm{Mov}}_1(X_F)$, $\left\{\rho : \rho \subset V_{\Delta_{F'}}\right\} = \Delta_{F'}$ by Remark 3.3.8. Hence the rational contraction ϕ_F has at least eliminated $\#(\rho \in \Delta(1) : \rho \subset V_{\Delta_F},\ \rho \notin \Delta_F)$ many divisors. □

3.5 When $\mathcal{F}_V$ is a relative tangent sheaf

It is natural to ask what kind of subsheaves can be realized as a tangent sheaf for some rational contraction. We give a necessary and sufficient condition.

Definition 3.5.1. *Let $\Delta \subset N$ be a fan. We say $V \subset N_{\mathbb{R}}$ is a cone sub vector space with respect to Δ, if $V = \mathrm{Cone}(\mathrm{u}_\rho : \mathbb{Z}_{\geq 0}\,\mathrm{u}_\rho = \rho \subset V,\ \rho \in \Delta(1))$*

We have shown the following proposition.

Proposition 3.5.2. *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety associated to fan $\Delta \subset N$. If $\mathcal{F} \subset \mathcal{T}_{X_\Delta}$ is the relative tangent sheaf for a rational contraction, then $\mathcal{F} = \mathcal{F}_V$ for some cone sub vector space $V \subset N_{\mathbb{R}}$.* □

Proposition 3.5.3. *Let X_Δ be a $\mathbb{Q}$-factorial, projective, toric variety associated to fan $\Delta \subset N$, and $W \subset N_{\mathbb{R}}$ a cone sub vector space with respect to Δ. Then there exists a rational contraction ϕ such that $\mathcal{F}_V \subset \mathcal{T}_{X_\Delta}$ is the relative tangent sheaf of ϕ.*

Proof. Since W is a cone sub vector space, then there exists a movable subset $S \subset \Delta(1)$ such that $W = V_S$, we refer Definition 1.3.11 and Definition 2.1.13 for the definitions of movable subset and V_S. Then Corollary 1.3.15 implies that if $S \subsetneq \Delta(1)$, then $S = \Sigma_F$ for some face $F \subset \partial\overline{\mathrm{Mov}_1}(\mathrm{X}_\Delta)$, and Theorem 3.4.2 implies that $W = V_S$ is the relative tangent of π_F in Remark 3.4.4,

$$\begin{tikzcd} X \arrow[r, dashed, "\phi_F"] \arrow[dr, dashed, "\pi_F"'] & X_F \arrow[d, "\psi_F"] \\ & Z_F \end{tikzcd} .$$

If $S = \Delta(1)$, then we take $X \to$ point. □

Remark 3.5.4. *Neumann showed that for smooth Fano 3-folds, every term of the Harder-Narasimhan filtration of the tangent sheaf is relative tangent for some regular contraction, cf. [Neu10, Theorem 4.1]. The proposition implies that this is wrong for the singular Fano 3-folds.*

We give an example for the the existence of terms of the Harder-Narasimhan filtration which are not relative tangent sheaves.

Example 3.5.5. *Let X be the weighted projective space $\mathbb{P}(1,1,1,2)$, then the Harder-Narasimhan filtration is not trivial, but $\mathbb{P}(1,1,1,2)$ admit only one rational contraction of fibre type, the trivial one $f : \mathbb{P}(1,1,1,2) \to \{pt\}$.*

Remark 3.5.6. *In fact, $\mathbb{P}(1,1,1,2)$ has at worst terminal singularity by Reid's theorem(cf. [Reid87, Section 4.11]), hence we are unable to remove the smoothness condition in Neumann's theorem to the other singularities types in the minimal model program.*

Remark 3.5.7. *The foliation $\mathcal{O}(2) \to \mathcal{T}_X$ corresponding the ruling of projective cone over Veronese surface $v_2(\mathbb{P}^2)$, and associated rational map is not a rational contraction. Nevertheless, if we blow up the only singular point of $P(1,1,1,2)$, then we get $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(2))$, and ruling becomes the fibres of projection $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(2)) \to \mathbb{P}^2$*

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