Eta-forms and adiabatic limits for fibrewise Dirac operators with varying kernel dimension

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Contents

1 Introduction

The η-invariant was introduced by Atiyah, Patodi and Singer [APS75a] as a correction term from the boundary in their generalization of the Atiyah-Singer-Index theorem to manifolds with boundary. It is a spectral invariant of the Dirac operator on the boundary or more generally for any Dirac operator D on an odd-dimensional manifold. It is defined to be the value at $s = 0$ of the meromorphic continuation of

$$\eta(D, s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda) |\lambda|^{-s} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s-1}{2}} \operatorname{tr}\left(De^{-tD^2}\right) dt,$$

where the summation is taken over all eigenvalues λ of D. Thus formally

$$\eta(D) = \eta(D, 0) = \#\{|\lambda| > 0\} - \#\{|\lambda| < 0\}$$

measures the asymmetry of the spectrum $\operatorname{spec} D$. One can define η-invariants for a larger class of operators than Dirac operators, namely for any elliptic self-adjoint pseudo-differential operator of positive order. In the present article we will focus on Dirac operators though. These operators are by definition directly coupled to the geometry which makes them easier to study. On the other hand they still give a very large class of examples and are important for many applications.
Since then many more applications of the η-invariant besides its boundary contribution in the index theorem have been discovered. Many differentialtopological invariants, as for example the ρ-invariant [APS75b] or the Eells-Kuiper-invariant [EK62], are defined or in certain cases can be calculated with the help of η-invariants. A very nice survey about these applications can be found in [Goe12, Section 4]. Goette [Goe14] also used an adiabatic limit formula for η-invariants to compute Eells-Kuiper invariants for certain seven-dimensional manifolds which possibly allow positive sectional curvature metrics. Recently Tang and Zhang [TZ14] gave an application into another direction. They computed η-invariants to obtain a result on closed geodesics of the Eells-Kuiper quaternionic projective plane.
For an arbitrary Riemannian manifold M and Dirac operator D it is very hard to compute the η-invariant since one needs knowledge about the whole spectrum of D. As already pointed out in the introduction of [APS75a] specific examples show that it cannot be written down in terms of local curvature expressions, since it does not behave multiplicatively with respect to finite coverings. Therefore the η-invariant is indeed a global metric invariant, it has just been calculated in few examples and one would like to have formulas which simplify the calculation.
Bismut and Cheeger [BC89] proved, that for the total space M of a Riemannian fibre

bundle $X \hookrightarrow M \overset{\pi}{\twoheadrightarrow} B$ of compact spin manifolds the *adiabatic limit* of the η-invariant at least partially localizes if the fibrewise Dirac operators D_X are invertible. Dai [Dai91] generalized their result to the case of constant kernel dimension of D_X. Passing to the adiabatic limit means that we scale the product metric along the base directions $g_\varepsilon = g_X \oplus \frac{1}{\varepsilon^2}\pi^* g_B$ and consider the limiting behaviour as $\varepsilon \to 0$. Therefore the metric along the base directions is blown up whereas the metric along the fibre directions is fixed. Note that for the η-invariant we could equivalently fix the metric along the base and shrink the fibres, since it is invariant under scaling of the metric of the total space M. For convenience we state the theorem of Bismut, Cheeger and Dai for $\dim X$ odd and $\dim B$ even, the other case is up to some small technical details analogous.

1.0.1 Theorem ([BC89, Theorem 4.35],[Dai91, Theorem 0.1])**.** *If $\ker D_X \to B$ forms a smooth vector bundle and there exists an $\varepsilon_0 > 0$ such that $\dim\ker D_{M,\varepsilon}$ is constant for all $\varepsilon \in (0, \varepsilon_0)$, the adiabatic limit of the η-invariant can be computed by*

$$\lim_{\varepsilon\to 0} \eta\left(D_{M,\varepsilon}\right) = 2\int_B \hat{A}\left(TB\right)\tilde{\eta} + \eta\left(D_B\right) + \sum_{\nu=1}^{\dim\ker D_B} \operatorname{sign}\left(\lambda_\nu(\varepsilon)\right),$$

where $\lambda_\nu(\varepsilon)$ denote the finitely many eigenvalues of $D_{M,\varepsilon}$ which decay at least quadratically and D_B denotes the twisted Spin-Dirac operator on $\Sigma B \otimes \ker D_X \to B$. The $\tilde{\eta}$-form for the Bismut superconnection $\mathbb{A}_t = \sqrt{t} D_X + \nabla^{\pi_* V} - \frac{c(T)}{4\sqrt{t}}$ is defined to be

$$\tilde{\eta} = \sum_j (2\pi i)^{-j} \left[\frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2}\right) dt\right]_{[2j]} \in \Omega^{\mathrm{ev}}(B)$$

and its exterior differential makes the cohomological index exact

$$d\tilde{\eta} = \int_{M/B} \hat{A}\left(TX\right) \operatorname{ch}\left(V/\Sigma X\right).$$

The $\tilde{\eta}$-form is actually a generalization of the η-invariant of the fibres. In degree 0 it is the smooth function $\tilde{\eta}_{[0]}\colon B \to \mathbb{R}, b \mapsto \eta\left(\left(D_X\right)_b\right)$, where $\left(D_X\right)_b$ is the Dirac operator on the fibre M_b. The $\tilde{\eta}$-form also plays the role of the correction term of the boundaries in the family index theorem for manifolds with boundary, which was proven by Bismut and Cheeger [BC90].

Both proofs of the adiabatic limit theorem above rely on a good knowledge of the behaviour of the eigenvalues $\lambda(\varepsilon)$ of $D_{M,\varepsilon}$ as $\varepsilon \to 0$. The first term $2\int_B \hat{A}(TB)\tilde{\eta}$ comes from the bounded eigenvalues, the second term $\eta\left(D_B\right)$ from the eigenvalues that decay at least linearly in ε and as already explained, the last term comes from the finitely many that decay at least as ε^2. Dai proved that if $\dim\ker D_X \equiv \text{const}$ these are the only possible behaviours of eigenvalues of $D_{M,\varepsilon}$.

The behaviour of eigenvalues of certain geometric operators in the adiabatic limit is also a topic of interest in mathematical physics. For example very recently in [HLT15], Haag,

Lampart and Teufel considered so-called generalized quantum waveguides, where quantum waveguides are ε-tubular neighbourhoods of curves in $\mathbb{R}^3$. They investigated the spectral properties of the Dirichlet Laplacian and even computed explicitly the adiabatic operator to all significant orders.
However, if the fibres X are odd-dimensional, the assumption that the kernels of the fibrewise Dirac operators have constant dimension is very restrictive. There are topological obstructions to this. For odd-dimensional fibres the family of fibrewise Dirac operators defines an element in the K-group $\mathrm{K}^{-1}(B)$ and if this cohomology class is not trivial, the dimension does indeed vary. So the obvious question to ask would be „Does there exist a formula for $\lim_{\varepsilon\to 0}\eta(D_{M,\varepsilon})$ if the kernel dimension of D_X might vary?“. But before one can answer this question there are mainly two questions for which one needs to find an answer first:

- We know that $\int_0^T \mathrm{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt}e^{-\mathbb{A}_t^2}\right)dt$ cannot converge to a smooth differential form $\tilde{\eta}\in\Omega^{\mathrm{ev}}(B)$ as $T\to\infty$, if the kernel dimension of D_X varies, since already the η-invariant $\eta(D_X)=\tilde{\eta}_{[0]}$ has integer jumps at points where eigenvalues cross 0. But does the limit exist in a weaker sense? Does $\tilde{\eta}$ exist at least as a current in $(\Omega^{\mathrm{ev}}(B))^*$?

- Which eigenvalue behaviour as $\varepsilon\to 0$ can occur and which operators model these small eigenvalues? So in particular what η-invariants of which operators do we see in the large time contribution, $0<\alpha<1$

$$\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^{\alpha-2}}^{\infty}\mathrm{tr}\left(D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}\right)\frac{dt}{\sqrt{t}}$$

  to $\eta\left(D_{M,\varepsilon}\right)$ as $\varepsilon\to 0$?

These are the problems we are concerned with in this thesis and we will solve them in Chapter 3 and Chapter 4 in a special case. Considering the most general case without any further assumptions on $\ker D_X$ seems impossible to handle since the eigenvalues of D_X can behave very wildly. We will still consider a case where we have a good control on how they vanish. So we will focus on the next interesting case after constant kernel dimension in which locally one eigenvalue of multiplicity one of D_X crosses 0 transversally:

1.0.2 Assumption. We assume that we can find a covering $\{U_i\}_{1\le i\le k}$ for B such that on each U_i either $(D_X)_b$ is invertible or we have a smooth function $f_i\colon U_i\to(-K,K)$ which has 0 as a regular value, such that for all $b\in U_i$, $\mathrm{spec}\,(D_X)_b\cap(-K-\delta,K+\delta)=\{f_i(b)\}$ and $f_i(b)$ is of multiplicity 1.

One should note that our assumption excludes two rather prominent examples of Dirac operators. The first one is the signature operator. Its kernel is given by the deRham-cohomology $H^\bullet_{\mathrm{dR}}(M_b)$ of the fibres $M_b\cong X$, $b\in B$ and hence the dimension cannot vary. The second one is the Dolbeault operator. Its kernel is given by the Dolbeault cohomology $H^\bullet(M_b, \mathcal{O}(\mathcal{W}_b))$ of the fibres and the dimension can indeed vary. Yet a corollary of the proper base change theorem implies that the dimension can just increase on complex subvarieties which cannot have real codimension one as in our assumption. We will see an example satisfying the assumption in Section 3.1.

1.0.3 Assumption. Furthermore we assume that the metric g_B on B is such that if we consider a tubular neighbourhood of the hypersurface $B_0 = \cup_i f_i^{-1}(\{0\})$, $\exp\colon B_0 \times (-a, a) \to N_a$ the functions f_i describing the small eigenvalue of D_X are given by

$$f_i(\exp(x, y)) = y.$$

This assumption will not be a restriction in Chapter 3, where we just consider the fibrewise situation and where we are allowed to change a chosen metric on the base. On the other hand in Chapter 4, where we consider the Dirac operator on the total space, this is of course a restriction. However if one wants to calculate η-invariants there is a formula connecting η-invariants of different metrics. Let D_M^0, D_M^1 be Dirac operators associated to metrics g_M^0 and g_M^1. Then a corollary of the Atiyah-Patodi-Singer index theorem and [APS76, Theorem 7.4] proves

$$\begin{aligned}
&\left(\eta\left(D_M^1\right) + h\left(D_M^1\right)\right) - \left(\eta\left(D_M^0\right) + h\left(D_M^0\right)\right) \\
&\quad = \mathrm{sf}\left(\left(D_M^s\right)_{s\in[0,1]}\right) + \int_M \Big(\tilde{\hat{A}}\left(TM, \nabla^{TM,0}, \nabla^{TM,1}\right) \mathrm{ch}\left(E/\Sigma, \nabla^{E,0}\right) \\
&\qquad\qquad - \hat{A}\left(TM, \nabla^{TM,1}\right) \tilde{\mathrm{ch}}\left(E/\Sigma, \nabla^{E,0}, \nabla^{E,1}\right)\Big),
\end{aligned}$$

where h denotes the dimension of the kernel of the inserted operator, $\mathrm{sf}\left(\left(D_M^s\right)_{s\in[0,1]}\right)$ denotes the spectral flow and $\tilde{\hat{A}}$ and $\tilde{\mathrm{ch}}$ denote the transgression forms of Chern-Weil theory.

Outline of the dissertation

In Chapter 2 we will give the basic definitions and preliminaries to understand the later chapters and set notation. We will introduce families of manifolds and associated curvature tensors as well as the Dirac operators we need and the notion of adiabatic limits. We will also explain in more detail why we are interested in varying kernel dimension of D_X.

Chapter 3 is concerned with the fibrewise situation. Before considering the question of existence of the $\tilde{\eta}$-form we will investigate the behaviour of $\mathrm{tr}^{\mathrm{odd}}\left(\exp\left(-\mathbb{A}_t^2\right)\right)$ as $t \to \infty$. This is the first step to prove that $\tilde{\eta}$ exists, see also [BGV04, Section 9,10]. Furthermore we already know by [BF86, Theorem 2.10] that $\mathrm{tr}^{\mathrm{odd}}\left(\exp\left(-\mathbb{A}_t^2\right)\right)$ is a representative for the odd Chern character of the family D_X. But we do not have a concrete analytical representative just depending on the spectral properties of D_X as in the even-dimensional case. The question for such a representative was already raised in [DK10] where in contrast to the present article they investigated the influence of multiplicities on the analytical index. In Theorem 3.2.17, assuming Assumption 1.0.2, we calculate the limit

$$\lim_{t\to\infty}\sum_k (2\pi i)^{-k}\operatorname{tr}^{\mathrm{odd}}\left(\exp\left(-\mathbb{A}_t^2\right)\right)_{[2k+1]} = -\delta_{B_0}\operatorname{ch}\left(\ker D_X \to B_0, \nabla^{\ker D_X}\right)$$

as a current in $(\Omega^{\mathrm{ev}}(B))^*$, where $B_0 \subset B$ is the hypersurface where the kernels of the fibrewise Dirac operators form a line bundle $\ker D_X \to B_0$. For the proof we make use of holomorphic functional calculus and generalize the ideas of [Bis90]. So in this case we really get a representative for the analytical index depending just on the kernels of the operators. In degree one we see the spectral flow of the operators which was already discovered and pointed out in [APS76, Section 7]. From the asymptotics of $\lim_{t\to\infty}\operatorname{tr}^{\mathrm{odd}}\left(\exp\left(-\mathbb{A}_t^2\right)\right)$ we can then prove in Proposition 3.2.19 that

$$\tilde{\eta} = \frac{1}{\sqrt{\pi}}\sum_k (2\pi i)^{-k}\left[\int_0^\infty \operatorname{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt}\exp\left(-\mathbb{A}_t^2\right)\right)dt\right]_{[2k]} \in L^1\left(B, \Lambda^{\mathrm{ev}}T^*B\right)$$

exists as a differential form with integrable coefficients and its differential as a current is given by

$$d\tilde{\eta} = \int_{M/B}\hat{A}\left(TX, \nabla^X\right)\operatorname{ch}\left(V/\Sigma X\right) + \delta_{B_0}\operatorname{ch}\left(\ker D_X \to B_0, \nabla^{\ker D_X}\right),$$

which represents the Atiyah-Singer family index theorem for odd-dimensional fibres, see [APS76, Theorem 3.4].
The content of this chapter was prepublished in [Wit15].

The question remains which eigenvalues we see for large times. This is dealt with in Chapter 4. We start by motivating where the small eigenvalues of $D_{M,\varepsilon}$ are supposed to come from. The operator which gives the small eigenvalues should and will be the twisted Dirac operator D_{B_0} on $\ker D_X \otimes \Sigma B_0 \to B_0$. We will define an isometry $J_\varepsilon\colon L^2\left(B_0, \ker D_X \otimes \Sigma B_0\right) \to L^2\left(M, V \otimes \pi^*\Sigma B\right)$ and prove that $\frac{1}{\varepsilon}J_\varepsilon^{-1}q_\varepsilon D_{M,\varepsilon}q_\varepsilon J_\varepsilon$ converges to $-D_{B_0}$ as $\varepsilon \to 0$, see Theorem 4.2.11, where q_ε denotes the projection onto the image of J_ε. Furthermore we develop the technical tools to use holomorphic functional calculus for proving that there exists an $0 < \alpha < 1$ such that

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{\varepsilon^{\alpha-2}}^\infty \operatorname{tr}\left(D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}\right)\frac{dt}{\sqrt{t}} = -\eta\left(D_{B_0}\right) + \sum_{\nu=1}^{\dim\ker D_{B_0}}\operatorname{sign}\left(\lambda_\nu(\varepsilon)\right),$$

for some ε small enough, provided that there exists an $\varepsilon_0 > 0$ such that $\dim\ker D_{M,\varepsilon}$ is constant for all $\varepsilon \in (0, \varepsilon_0)$, see Theorems 4.3.11 and 4.3.12. Parts of the proof are based on ideas in [BL91] and also in [Goe14].

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2 Preliminaries: Geometry of Families

In this chapter we will fix notation and introduce the situation of families of manifolds which we use in the following chapters. The definitions and the setup is based on [BC89, Chapter 4], [Goe14] and [BGV04, Chapter 9, 10], which will also provide more details and background.

We consider an oriented Riemannian fibre bundle $X \hookrightarrow M \overset{\pi}{\twoheadrightarrow} B$ where the fibres $M_b = \pi^{-1}(\{b\}) \cong X$ are closed and $n = \dim X$ is odd. We assume the base (B, g_B) also to be closed and oriented and of even dimension $m = \dim B$, such that the total space M is odd-dimensional. We denote the *vertical tangent bundle* by $TX = \ker d\pi$, choose a *horizontal distribution* $T_H M \cong \pi^* TB$ such that $TM = TX \oplus T_H M$ and consider the metric $g = g_X \oplus \pi^* g_B$. We will denote vertical local frames by $\{e_i\}_{i=1,\dots,n}$ and horizontal ones by $\{f_\alpha\}_{\alpha=1,\dots m}$. On TM we have the Levi-Civita connection ∇^M and the Euclidean connection $\nabla^\oplus = \nabla^X \oplus \pi^* \nabla^B$ where ∇^X is the projection of ∇^M onto TX and ∇^B is the Levi-Civita connection on B. The difference tensor of the two connections

$$S(Y)Z = \nabla^M_Y Z - \nabla^\oplus_Y Z$$

can be used to compute the *torsion tensor* of $\nabla^\oplus$ (or *curvature* of M/B)

$$\begin{aligned} T(U,V) &= \nabla^\oplus_U V - \nabla^\oplus_V U - [U,V] \\ &= S(U)V - S(V)U \in TX \end{aligned} \tag{2.0.1}$$

for horizontal vectors $U, V \in T_H M$. We define the *mean curvature* $k \in \Omega^1(M)$ to be

$$k(U) = \sum_{i=1}^n g\left(S(e_i)e_i, U\right)$$

for a local orthonormal frame $\{e_i\}_{i=1,\dots,n}$ of TX.

2.1 Fibrewise situation and the Bismut superconnection

Assume that we have a vertical *Dirac bundle* $\left(V, g^V, \nabla^V, c_X\right)$, by which we mean a Hermitian vector bundle $\left(V, g^V\right) \to M$, a compatible connection ∇^V and a smooth linear map $c_X \colon TX \to \operatorname{End} V$ which satisfies the following

- $c_X(Y)$ is skew-symmetric for all $Y \in TX$,
- for all $Y \in \Gamma(M, TM)$, $Z \in \Gamma(M, TX)$ and $\sigma \in \Gamma(M, V)$

$$\nabla^V_Y \left(c_X(Z)\sigma\right) = c_X\left(\nabla^X_Y Z\right)\sigma + c_X(Z)\nabla^V_Y \sigma,$$

- c_X is a *Clifford multiplication*, that means

$$c_X(Y)c_X(Z) + c_X(Z)c_X(Y) = -2g_X(Y,Z), \text{ for all } Y, Z \in \Gamma(M, TX).$$

For a vertical Dirac bundle we can associate the fibrewise *Dirac operator*

$$D_X = \sum_{i=1}^{n} c_X(e_i) \nabla^V_{e_i} \colon \Gamma(M,V) \to \Gamma(M,V),$$

where $e_1, ..., e_n$ is a local orthonormal frame of TX. We can also write

$$D_X = c_X \circ g_X^{-1} \circ \nabla^V \big|_{TX}.$$

Since we just differentiate into vertical directions, D_X restricts to fibrewise sections $\Gamma(M_b, V_b)$ and we get a family $b \mapsto (D_X)_b$ of fibrewise Dirac operators. For a vector bundle $V \to M$ we get the associated Fréchet bundle $\pi_* V \to B$ whose infinite-dimensional fibres are the fibrewise smooth sections of V. We will make use of the natural isomorphism $\Gamma(B, \pi_* V) \cong \Gamma(M, V)$ without actually mentioning it. The induced connection

$$\nabla^{\pi_* V} = \nabla^V - \frac{1}{2} k \tag{2.1.1}$$

is Euclidean with respect to the L^2-metric $g_{\pi_* V}$ on $\pi_* V$. The *Bismut superconnection* [Bis85, Definition 3.2] is then defined by

$$\mathbb{A}_t = \sqrt{t} D_X + \nabla^{\pi_* V} - \frac{1}{4\sqrt{t}} c_X(T) \colon \Omega^\bullet(B, \pi_* V) \to \Omega^\bullet(B, \pi_* V),$$

where we assume that horizontal one-forms f^α and Clifford multiplication by vertical vectors $c_X(e_i)$ anticommute. As for usual connections we define its curvature by $\mathbb{A}_t^2 \in \Omega^\bullet(B, \operatorname{End} \pi_* V)$. It follows from the transgression formula, see for example [BC89, Eq. (4.38)], that

$$d \int_T^s \operatorname{tr}^{\mathrm{ev}} \left(\frac{d\mathbb{A}_t}{dt} \exp\left(-\mathbb{A}_t^2\right) \right) dt = \operatorname{tr}^{\mathrm{odd}} \left(\exp\left(-\mathbb{A}_T^2\right) \right) - \operatorname{tr}^{\mathrm{odd}} \left(\exp\left(-\mathbb{A}_s^2\right) \right). \tag{2.1.2}$$

If the dimension of the kernels of D_X is constant, they form a smooth vector bundle $\ker D_X \to B$. In this situation we know by [BC89] and [BGV04] that

$$\hat{\eta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{tr}^{\mathrm{ev}} \left(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2} \right) dt \in \Omega^{\mathrm{ev}}(B, \mathbb{C})$$

is a smooth differential form and the differential of its rescaled version

$$\tilde{\eta} = \sum_j (2\pi i)^{-j} \hat{\eta}_{[2j]} \in \Omega^{\mathrm{ev}}(B)$$

fulfills

$$d\tilde{\eta} = \int\limits_{M/B} \hat{A}\left(TX, \nabla^X\right) \mathrm{ch}\left(L, \nabla^L\right),$$

if $V = \Sigma X \otimes L$ where we assume that TX is spin and ΣX denotes the spinor bundle for a chosen spin structure. The formula follows from the transgression formula (2.1.2) and by [BF86, Theorem 2.10] which states that

$$\frac{1}{\sqrt{\pi}} \lim_{T\to 0} \sum_k (2\pi i)^{-k} \operatorname{tr}^{\mathrm{odd}}\left(\exp\left(-\mathbb{A}_T^2\right)\right)_{[2k+1]} = \int\limits_{M/B} \hat{A}\left(TX, \nabla^X\right) \mathrm{ch}\left(L, \nabla^L\right),$$

since for constant kernel dimension $\lim_{t\to\infty} \operatorname{tr}^{\mathrm{odd}}\left(\exp\left(-\mathbb{A}_t^2\right)\right) = 0$. But also for non-constant kernel-dimension the differential form $\int\limits_{M/B} \hat{A}\left(TX, \nabla^X\right) \mathrm{ch}\left(L, \nabla^L\right)$ is always a representative for the odd Chern character of the family $\{(D_X)_b\}_{b\in B} \in K^{-1}(B)$ by the Atiyah-Singer family index theorem. One should notice that we use Chern-Weil forms of the form $P\left(F/2\pi i\right)$ for the curvature F of a connection.

2.2 Dirac operator $D_{M,\varepsilon}$ on the total space

In the following we introduce an extra parameter $\varepsilon > 0$ by which we scale the base directions. Let

$$g_\varepsilon = g_X \oplus \frac{1}{\varepsilon^2}\pi^* g_B.$$

In the limit $\varepsilon \to 0$ the metric becomes singular since the base becomes infinitely large and the fibres are separated. This limiting process is called the *adiabatic limit*.
We will assume that we have a fibration $X \hookrightarrow M \overset{\pi}{\twoheadrightarrow} B$ of spin manifolds and let $\Sigma M \cong \Sigma X \otimes \pi^*\Sigma B$ be a spinor bundle. Let $\left(L, g^L, \nabla^L\right) \to M$ be a Hermitian vector bundle with compatible connection, then we consider the twisted Dirac operator $D_{M,\varepsilon}$ on $L \otimes \Sigma X \otimes \pi^*\Sigma B = V \otimes \pi^*\Sigma B$ associated to the metric g_ε. We know by [BC89, Eq. (4.26)] that $D_{M,\varepsilon}$ decomposes as

$$D_{M,\varepsilon} = \tilde{D}_X + \varepsilon\tilde{D}_B + \varepsilon^2\tilde{T},$$

where

$$\begin{aligned}
\tilde{D}_X &= \sum_{i=1}^n c(e_i)\nabla_{e_i}^{V\otimes\pi^*\Sigma B} = D_X \otimes \omega + \sum_{i=1}^n c(e_i) 1 \otimes \nabla_{e_i}^{\pi^*\Sigma B} \\
\tilde{D}_B &= \sum_{\alpha=1}^m c(f_\alpha)\left(\nabla_{f_\alpha}^{V\otimes\pi^*\Sigma B} - \frac{1}{2}k(f_\alpha)\right) \\
\tilde{T} &= \frac{1}{4}\sum_{\alpha,\beta=1}^m c\left(T(f_\alpha, f_\beta)\right) c(f_\alpha) c(f_\beta),
\end{aligned}$$

see [Goe14, Section 2.a] for the actual calculation. $\omega \in \operatorname{End} \Sigma B$ denotes the $\mathbb{Z}_2$-grading of ΣB (remember that $\dim B = m$ is even) and Clifford multiplication is given by

$$c(e_i) = c_X(e_i) \otimes \omega$$

for e_i vertical and

$$c(f_\alpha) = 1 \otimes c_B(f_\alpha)$$

for f_α horizontal. Note that if we identify $\pi_* (V \otimes \pi^* \Sigma B) \cong \pi_* V \otimes \Sigma B$ the operator $\tilde{D}_X$ acts as $D_X \otimes \omega$.

2.2.1 Theorem ([BC89, Theorem 4.35],[Dai91, Theorem 0.1])**.** *If* $\ker D_X \to B$ *forms a smooth vector bundle and there exists an* $\varepsilon_0 > 0$ *such that* $\dim \ker D_{M,\varepsilon}$ *is constant for all* $\varepsilon \in (0, \varepsilon_0)$*, the adiabatic limit of the* η*-invariant can be computed by*

$$\lim_{\varepsilon \to 0} \eta\left(D_{M,\varepsilon}\right) = 2 \int\limits_B \hat{A}\left(TB\right) \tilde{\eta} + \eta\left(D_B\right) + \sum_{\nu=1}^{\dim \ker D_B} \operatorname{sign}\left(\lambda_\nu(\varepsilon)\right), \tag{2.2.1}$$

where $\lambda_\nu(\varepsilon)$ *denote the finitely many eigenvalues of* $D_{M,\varepsilon}$ *which decay at least quadratically,* D_B *the twisted Dirac operator on* $\ker D_X \otimes \Sigma B \to B$ *and* ε *is small enough. The* $\tilde{\eta}$*-form for the Bismut superconnection* $\mathbb{A}_t = \sqrt{t} D_X + \nabla^{\pi_* V} - \frac{c(T)}{4\sqrt{t}}$ *is defined to be*

$$\tilde{\eta} = \sum_j (2\pi i)^{-j} \left[\frac{1}{\sqrt{\pi}} \int\limits_0^\infty \operatorname{tr}^{\mathrm{ev}} \left(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2} \right) dt \right]_{[2j]} \in \Omega^{\mathrm{ev}}(B)$$

and its exterior differential makes the cohomological index exact

$$d\tilde{\eta} = \int\limits_{M/B} \hat{A}\left(TX\right) \operatorname{ch}\left(L\right).$$

Both proofs rely on a decomposition of the integral

$$\eta\left(D_{M,\varepsilon}\right) = \frac{1}{\sqrt{\pi}} \int\limits_0^\infty \operatorname{tr}\left(D_{M,\varepsilon} e^{-t D_{M,\varepsilon}^2} \right) \frac{dt}{\sqrt{t}}$$

into three parts, for a certain choice of $0 < \alpha, \beta < 1$ the integral is split into

$$\int\limits_0^{\varepsilon^{\beta-1}} + \int\limits_{\varepsilon^{\beta-1}}^{\varepsilon^{\alpha-2}} + \int\limits_{\varepsilon^{\alpha-2}}^{\infty} \tag{2.2.2}$$

Bismut and Cheeger proved that the first part converges to

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi}} \int\limits_0^{\varepsilon^{\beta-1}} \operatorname{tr}\left(D_{M,\varepsilon} e^{-t D_{M,\varepsilon}^2} \right) \frac{dt}{\sqrt{t}} = 2 \int\limits_B \hat{A}(B) \tilde{\eta},$$

see step (ii) in the proof of [BC89, Theorem 4.35]. Then they prove that if the fibrewise Dirac operators D_X are invertible, there exists a constant $C > 0$ such that all eigenvalues $\lambda(\varepsilon)$ of $D_{M,\varepsilon}$ have absolute value bounded below by C, i.e., $|\lambda(\varepsilon)| \geq C$ for all ε small enough and therefore the large time contribution is negligible (see step (i) and (iii) in the proof of [BC89, Theorem 4.35])

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi}} \int_{\varepsilon^{\beta-1}}^{\infty} \operatorname{tr}\left(D_{M,\varepsilon} e^{-tD^2_{M,\varepsilon}}\right) \frac{dt}{\sqrt{t}} = 0.$$

Dai considers the case where the dimension of the kernels of D_X is constant but not zero. In this case he proves that either

$$|\lambda(\varepsilon)| \geq C > 0$$

or

$$\lambda(\varepsilon) \sim \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + ...,$$

where $\lambda_1 \in \operatorname{spec}(D_B)$, see [Dai91, Theorem 1.5]. This proves that there are just finitely many eigenvalues that decay at least quadratically and the fact that we see the eigenvalues of $\frac{1}{\varepsilon} D_{M,\varepsilon}$ for large times $t \in (\varepsilon^{\alpha-2}, \infty)$ explains the η-invariant of D_B. Last [Dai91, Proposition 1.8] proves that the second term in (2.2.2) vanishes as $\varepsilon \to 0$.

2.3 Varying kernel dimension of the fibrewise operators D_X

We already explained in Section 2.2 the results on adiabatic limits of η-invariants in the case of constant dimension of $\ker D_X$. We want to omit that assumption since there are topological obstructions for constant kernel dimension if the fibres X are odd-dimensional. We will explain these obstructions, state our assumptions and explain heuristically what to expect as a representative for the analytical index in our case.
Let H_0 and H_1 be separable complex Hilbert spaces where furhermore H_0 is $\mathbb{Z}/2\mathbb{Z}$-graded. We denote by $\mathcal{B}(H_0)$ and $\mathcal{B}(H_1)$ bounded linear operators on H_0 and H_1. Then we define

$$\mathbb{K}^0 = \{F \in \mathcal{B}(H_0) \,|\, F \text{ self-adjoint Fredholm operator, } F \text{ odd, } F^2 - 1 \text{ compact}\},$$

and

$$\begin{aligned}\mathbb{K}^1 = &\{F \in \mathcal{B}(H_1) \,|\, F \text{ self-adjoint Fredholm operator,} \\ &\quad F \text{ has infinite dimensional positive and negative eigenspaces, } F^2 - 1 \text{ compact}\big\}.\end{aligned}$$

We consider $\mathbb{K}^*$ as topological spaces with the smallest topology such that evaluation and $F \mapsto 1 - F^2$ are continuous. Then we know by [Bun09, Section 1.1.1] or rather [BJS03] that the topological K-theory of a compact topological space B is given by homotopy classes of continuous maps from B to $\mathbb{K}^*$

$$\mathrm{K}^*(B) = [B, \mathbb{K}^*].$$

A reader not familiar with K-theory might take this as a definition. Note that we use weaker topologies than the ones of the more classical classifying spaces using Fredholm operators of Atiyah and Jänich [Ati67] and Atiyah and Singer [AS69].
A family of Dirac operators $b \mapsto (D_X)_b$ as in Section 2.1 defines an element in the topological K-theory $\mathrm{K}^*(B) = [B, \mathbb{K}^*]$ of B by taking the homotopy class of the map

$$\begin{aligned} &B \to \mathbb{K}^* \\ &b \mapsto (D_X)_b \left(1 + (D_X)_b^2\right)^{-1/2} \end{aligned}$$

where $* \equiv \dim X \mod 2$. Note that we already presume Bott periodicity at this point. The class $\frac{D_X}{\left(1+D_X^2\right)^{1/2}} \in K^*(B)$ is called *analytical index* of D_X.
We already know by [AS71, Section 2] how to modify the family D_X for even dimensional fibres to obtain a family of operators with constant kernel dimension and to define a vector bundle $\operatorname{ind} D_X$ which represents the analytical index under the identification of $\mathrm{K}^0(B)$ with the Grothendieck group of isomorphism classes of vector bundles (see also [BGV04, Section 9.5]).
In contrast to this we know by [Ebe13, Theorem 4.1] that in odd dimensions, constant kernel dimension implies that the map above is homotopic to a constant map. So if the family of fibrewise Dirac operators D_X defines a non-trivial element in $\mathrm{K}^1(B)$, one needs to consider varying kernel dimension. One can easily see such non-trivial elements if the base $B = S^1$ is a one-dimensional sphere, where the element in $\mathrm{K}^1(S^1) \cong \mathbb{Z}$ is given by the spectral flow. The example of Section 3.1 will also provide a non-trivial element in $\mathrm{K}^1(B)$.
Therefore we will consider odd-dimensional fibres and varying kernel dimension of the fibrewise Dirac operators D_X. We will not consider the most general case but focus on the next interesting case:

2.3.1 Assumption. We assume that we can find a covering $\{U_i\}_{1 \le i \le k}$ for B such that on each U_i either $(D_X)_b$ is invertible or we have a smooth function $f_i : U_i \to (-K, K)$ which has 0 as a regular value, such that $\operatorname{spec}(D_X)_b \cap (-K-\delta, K+\delta) = \{f_i(b)\}$ and $f_i(b)$ is of multiplicity 1.

We will assume this condition in Chapter 3 as well as in Chapter 4, but to remind the reader we will mention it again.

2.3.2 Remark. It follows from the assumption and the regular value theorem that we get a compact hypersurface

$$B_0 = \bigcup_{1 \le i \le k} f_i^{-1}(\{0\}) \subset B$$

where we have a line bundle $\ker D_X \to B_0$ and D_X is invertible on $B \backslash B_0$. We denote by $i\colon B_0 \to B$ the inclusion. Since we assumed B to be oriented we get an orientation on B_0 by

$$(v_2, ..., v_m) \in o_x(B_0) \Leftrightarrow (\operatorname{grad}_x f_i, v_2, ..., v_m) \in o_x(B).$$

Let $\nu B_0 \to B_0$ be the normal bundle, which is then trivial $\nu B_0 \cong B_0 \times \mathbb{R}$. Since B_0 is compact we find a constant $0 < a \leq K$ small enough, such that

$$\exp\colon B_0 \times (-a, a) \to B$$

is a diffeomorphism onto its image N_a. We will not fix a since we may take it as small as needed in the proofs.
In Chapter 3 we may assume without loss of generality that

$$f_i\,(x, y) = y.$$

To achieve that we maybe need to change the metric on B near B_0. But we know by [BGV04, Proposition 10.2] that ∇^X is independent of g_B and we can also give a g_B-independent formula for the torsion tensor T namely

$$T(U, V) = -P[U, V]$$

for $U, V \in \Gamma\,(M, T_H M)$ and $P\colon TM \to TX$ the projection. These are the only definitions in Chapter 3 where we used the metric on B. Therefore the assumption $f_i(x, y) = y$ is no restriction in Chapter 3. However, in Chapter 4 we consider not just the fibrewise but the situation on the total space, so there it will actually be a restriction. Nevertheless we already explained in the introduction, how one can receive the desired η-invariant of a metric g_B by the knowledge of the η-invariant of a different metric g'_B.

2.3.3 Remark. We can already see from our assumptions what we have to expect as analytical index of the family of fibrewise operators. Since D_X is invertible on $B \setminus N_a$ it defines an element in $\mathrm{K}^1\,(B, B \setminus N_a) = \mathrm{K}^1_c\,(N_a)$. The family of operators corresponds under desuspending twice $\mathrm{K}^1_c\,(N_a) \cong \mathrm{K}^0\,(B_0)$, by the Thom isomorphism, to the family $\overline{D}_X$ which maps $x \in B_0$ to the operator

$$\begin{pmatrix} 0 & D_{X,(x,y)} + \frac{\partial}{\partial y} \\ D_{X,(x,y)} - \frac{\partial}{\partial y} & 0 \end{pmatrix} : \Gamma\,(M_x \times (-a, a), V_x \oplus V_x) \to \Gamma\,(M_x \times (-a, a), V_x \oplus V_x)$$

where we use $B_0 \times (-a, a) \cong N_a$. We can check that the kernel dimension of this family is constant and therefore the index in $\mathrm{K}^0\,(B_0)$, seen as the Grothendieck group of isomorphism classes of vector bundles, is given by

$$\left[\ker\left(D_X + \frac{\partial}{\partial y}\right)\right] - \left[\ker\left(D_X - \frac{\partial}{\partial y}\right)\right] = 0 - \left[e^{-y^2/2} \ker D_X\right] \cong -\left[\ker D_X\right].$$

The diagram

$$
\begin{tikzcd}
\mathrm{K}^0\left(B_0\right) \arrow[r, "\mathrm{Thom}", "\sim"'] \arrow[d, "\mathrm{ch}^{\mathrm{ev}}"'] & \mathrm{K}^0_c\left(B_0 \times \mathbb{R}^2\right) \arrow[d, "\mathrm{ch}^{\mathrm{ev}}"] \\
H^{\mathrm{ev}}_{\mathrm{dR}}\left(B_0\right) \arrow[r, "\mathrm{Thom}", "\sim"'] & H^{\mathrm{ev}}_{\mathrm{dR},c}\left(B_0 \times \mathbb{R}^2\right)
\end{tikzcd}
$$

usually just commutes up to the Todd class of the vector bundle. Now we use that the correction term is $\mathrm{Td}\left(B_0 \times \mathbb{C}\right) = 1$ and the odd chern character is by construction given by

$$
\mathrm{K}^1_c\left(N_a\right) = \mathrm{K}^0_c\left(B_0 \times \mathbb{R}^2\right) \xrightarrow{\mathrm{ch}^{\mathrm{ev}}} H^{\mathrm{ev}}_{\mathrm{dR},c}\left(B_0 \times \mathbb{R}^2\right) \xrightarrow{\int_{B_0 \times \mathbb{R}^2/\mathbb{R}}} H^{\mathrm{odd}}_{\mathrm{dR},c}\left(N_a\right).
$$

This implies that

$$
\begin{aligned}
\mathrm{ch}\left(\left[\frac{D_X}{(1+D_X^2)^{1/2}}\right]\right) &= \mathrm{Thom} \circ \mathrm{ch} \circ \mathrm{Thom}^{-1}\left(\left[\frac{D_X}{\left(1+D_X^2\right)^{1/2}}\right]\right) \\
&= \mathrm{Thom} \circ \mathrm{ch}\left(-\ker D_X\right) \\
&= -\delta_{B_0}\, \mathrm{ch}\left(\ker D_X\right),
\end{aligned}
$$

where δ_{B_0} denotes the fundamental class or current of integration of B_0 and we used Poincaré duality without mentioning it.

3 A representative for the odd Chern character and existence of the $\tilde{\eta}$-form

In this chapter we will just consider the fibrewise situation. The content is up to some small changes of notation identical with that in [Wit15, Section 3,4].
The only Dirac operator occuring in this chapter will be the fibrewise operator D_X. Thus for simplicity we will just write D.
First in Section 3.1 we will consider an example where Assumption 1.0.2 from the introduction and the preliminaries is actually fulfilled. We will explicitly calculate $\hat{\eta} = \int_0^\infty \mathrm{tr}^{\mathrm{ev}} \left(\frac{d\mathbb{A}_t}{dt} \exp\left(-\mathbb{A}_t^2\right)\right) dt$ as a differential form with integrable coefficients and $\lim_{t\to\infty} \mathrm{tr}^{\mathrm{odd}} \left(\exp\left(-\mathbb{A}_t^2\right)\right)$ as a current.
In Section 3.2 we will calculate

$$\lim_{t\to\infty} \mathrm{tr}^{\mathrm{odd}} \left(\exp\left(-\mathbb{A}_t^2\right)\right) = -\delta_{B_0} \operatorname{ch}\left(\ker D_X \to B_0, \nabla^{\ker}\right)$$

in the general case, but still assuming 1.0.2. This reassures what we expected from the considerations in Remark 2.3.3 about the odd analytical index in this case. Then we will prove that $\tilde{\eta} \in L^1\left(B, \Lambda^{\mathrm{ev}} T^* B\right)$ is a differential form with integrable coefficients.

3.1 Example of a S^1-bundle

Before we start with the more general case, we will consider one special example of a family of Dirac operators. We follow the requirements in [Zha94], where we adopt the construction of the fibre bundle but change the Dirac bundle.
Let $\left(E, g^E\right) \xrightarrow{\pi} (B, g_B)$ be a real, Euclidean, oriented vector bundle of rank 2 and denote by ∇^E a Euclidean connection on it. We write $T_H E \cong \pi^* TB$ for the horizontal bundle of TE, which is specified by ∇^E. We define the metric $g_{TE} = \pi^* g_E \oplus \pi^* g_B$ on $TE = \pi^* E \oplus T_H E$. Let

$$\begin{aligned}
M &= \{v \in E \,|\, g_E(v,v) = 1\},\\
T_H M &= T_H E|_M,\\
TM &= \ker d\pi \oplus T_H M = TX \oplus T_H M,\\
g &= g_{TE}|_M = g_X \oplus \pi^* g_B.
\end{aligned}$$

$M \to B$ is an oriented, Riemannian fibre bundle with fibres $X \cong S^1$. Let $e \in \Gamma(M, TX)$ be the unique section which is positive oriented and of length

$$g_X(e,e) = 1.$$

Let $\left(V, g^V, \nabla^V\right) \to M$ be a Hermitian line bundle with compatible connection. By setting $c(e) = -i$ we make it into a vertical Dirac bundle with Dirac operator $D = -i\nabla^V_e$. The fibrewise holonomies $e^{-2\pi i \overline{f}}$ give rise to a smooth function $\overline{f}\colon B \to \mathbb{R}/\mathbb{Z}$.

3.1.1 Assumption. $\overline{f}\colon B \to \mathbb{R}/\mathbb{Z}$ crosses $[0]$ transversally.

We denote the codimension 1 submanifold $\overline{f}^{-1}([0]) \subset B$ by B_0.

3.1.2 Remark. If the holonomies give rise to a non-constant $\overline{f}\colon B \to \mathbb{R}/\mathbb{Z}$ we can always modify the connection ∇^V to fulfill Assumption 3.1.1. Sard's Theorem makes sure that there exists an element $[x] \in \operatorname{im} \overline{f}$ which is a regular value. The connection

$$\tilde{\nabla}^V = \nabla^V - ixe^*$$

then leads to

$$\tilde{f} = \overline{f} - [x]\colon B \to \mathbb{R}/\mathbb{Z}$$

which crosses zero transversally.

3.1.3 Lemma. *We have a vector bundle* $\ker D \to B_0$ *of rank* 1 *over the hypersurface* B_0 *and* D_b *is invertible for* $b \in B \backslash B_0$.

Proof. We choose open neighbourhoods $U_j \subseteq B$ where $\pi^{-1}(U_j) \cong U_j \times S^1$ such that we can find local trivializations $\sigma_0\colon U_j \times S^1 \to V|_{U_j \times S^1}$ of V on $U_j \times S^1$ coming from a local eigensection of D

$$D\sigma_0 = f_j \sigma_0,$$

where $f_j\colon U_j \to \mathbb{R}$ is smooth. For coordinates φ of S^1 such that $\frac{\partial}{\partial \varphi} = e$ we can see that $e^{ik\varphi}\sigma_0$ is an eigensection of D corresponding to the eigenvalue $k + f_j$. Therefore the spectrum of D_b for $b \in U_j$ is $(k + f_j(b))_{k \in \mathbb{Z}}$ where each eigenvalue is of multiplicity one. Since we can easily check by the same argument as above that

$$f_j \mod \mathbb{Z} = \overline{f},$$

the statement follows by [BGV04, Corollary 9.11]. $\square$

In the following we will for simplicity just write f for f_j. We orient B_0 such that

$$(v_2, \ldots, v_m) \in o_x(B_0) \Leftrightarrow (\operatorname{grad} f_x, v_2, \ldots, v_m) \in o_x(B).$$

Since $D = -i\nabla^V_e$, the connection ∇^V locally looks like

$$\nabla^V = d + ife^* + \gamma,$$

for some $\gamma \in \Gamma\left(U, T^*_H M|_U \otimes_{\mathbb{R}} \mathbb{C}\right)$. We will assume that $\gamma = \pi^*\beta$ for an element $\beta \in \Gamma\left(\pi(U), T^*B|_{\pi(U)} \otimes_{\mathbb{R}} \mathbb{C}\right)$.

3.1.4 Lemma ([Zha94, Lemma1.3]). *Let T be the torsion of $\nabla^{\oplus}$ as in (2.0.1). Then*

$$g(T(U,V),e) = de^*(U,V)$$

and hence T defines a two-form which we will also denote by $T \in \Omega^2(B)$.

3.1.5 Lemma ([Zha94, Lemma 1.6]). *The mean curvature k of the fibres vanishes and therefore (2.1.1) leads to*

$$\nabla_X^{\pi_* V}\sigma = \nabla_{X^H}^V \sigma.$$

3.1.6 Remark. To facilitate the computations for the next theorem we calculate the following summands of the curvature $\mathbb{A}_t^2$ of the Bismut superconnection. We write $[.,.]$ for the supercommutator with respect to the grading of $\Omega^{\bullet}(B)$ and keep in mind that dy_α and $c(e_i)$ anticommute.

$$\begin{aligned}[c(T), \nabla^{\pi_* V}] &= 0\\ [D, c(T)] &= 2Dc(T)\\ c(T)^2 &= -T^2.\end{aligned}$$

In our chosen trivialization

$$\begin{aligned}[D, \nabla^{\pi_* V}] &= df\\ \left(\nabla^{\pi_* V}\right)^2 &= d\beta + ifT - T\nabla_e^V.\end{aligned}$$

3.1.7 Theorem. *Set*

$$\alpha(T) := \frac{1}{\sqrt{\pi}} \int_0^T \mathrm{tr}^{\mathrm{ev}} \left(\frac{d\mathbb{A}_t}{dt} \exp\left(-\mathbb{A}_t^2\right) \right) dt \in \Omega^{2\bullet}(B, \mathbb{C}).$$

For each $b \in B$ the differential form $\alpha(T)_b$ converges as $T \to \infty$ to

$$\hat{\eta}_b = \lim_{T\to\infty} \alpha(T)_b \in \Lambda^{2\bullet} T_b^* B$$

and we get that

$$\begin{aligned}\tilde{\eta}_b &= \sum_j \frac{1}{(2\pi i)^j} \hat{\eta}_{[2j]}\\ &= \exp\left(-\frac{d\beta + ifT}{2\pi i}\right) \begin{cases} \sum\limits_{k=1}^{\infty} \frac{B_k(\overline{f})}{k!} \left(\frac{T}{2\pi}\right)^{k-1}, & \text{if } b \in B \backslash B_0\\ \sum\limits_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(\frac{T}{2\pi}\right)^{2k-1}, & \text{if } b \in B_0 \end{cases}\\ &= \exp\left(-\frac{d\beta + ifT}{2\pi i}\right) \left(-\frac{T}{2\pi}\right)^{-1} \begin{cases} \left(\frac{T/2\pi}{\exp(T/2\pi)-1} \exp\left(\frac{\overline{f}T}{2\pi}\right) - 1\right), & \text{if } b \in B\backslash B_0\\ \left(\frac{T/2\pi}{\exp(T/2\pi)-1} - 1 + \frac{T}{4\pi}\right), & \text{if } b \in B_0 \end{cases}\end{aligned}$$

where $f\colon U \to \mathbb{R}$ describes a local eigenvalue of D, $\overline{f} = f \mod \mathbb{Z}$, β is the corresponding horizontal connection form of the Dirac bundle in this trivialization and B_{2k} are the Bernoulli numbers and $B_k(\overline{f})$ the Bernoulli polynomials.

3.1.8 Remark. An easy computation shows that our formula for $\hat{\eta}$ corresponds to the one given in [Sav14, (5.23)] for $r = f$. The difference lies in the fact that in our case f is a function depending on the parameter $b \in B$ such that we get a differential form which has jumps, whereas in [Sav14] $r \in \mathbb{R}$ is seen as a fixed real number and $\hat{\eta}$ is seen as a smooth differential form for each $r \in \mathbb{R}$.

3.1.9 Remark. We prove that the right hand side of the formula in Theorem 3.1.7 is independent of the chosen trivialization. Therefore we take another local eigensection σ_1 with

$$D\sigma_1 = f_1\sigma_1.$$

Since the eigenvalues of D differ by integers, there exists a $k \in \mathbb{Z}$ such that $f_1 = f + k$ and $\sigma_1 = e^{ik\varphi}\sigma_0$. The local horizontal connection 1-form β_1 in this trivialization is then defined by

$$\beta_1 = \frac{g^V\left(\nabla^V\sigma_1, \sigma_1\right)}{g^V\left(\sigma_1, \sigma_1\right)}$$

and we can conclude that

$$\begin{aligned}\beta_1 &= d\left(e^{-ik\varphi}\right)e^{ik\varphi} + \beta \\ &= -ike^* + \beta.\end{aligned}$$

It follows by Lemma 3.1.4 that

$$d\beta_1 = -ikT + d\beta$$

and therefore

$$\exp\left(-\frac{d\beta + ifT}{2\pi i}\right) = \exp\left(-\frac{d\beta_1 + if_1T}{2\pi i}\right).$$

Proof of Theorem 3.1.7:

$$\begin{aligned}\hat{\eta} &= \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt}\exp\left(-\mathbb{A}_t^2\right)\right)dt \\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{tr}^{\mathrm{ev}}\left(\left(D - \frac{iT}{4t}\right)\right. \\ &\qquad \left.\cdot \exp\left(-tD^2 - \sqrt{t}df - d\beta - ifT + T\nabla_e^V + \frac{Dc(T)}{2} + \frac{T^2}{16t}\right)\right)\frac{dt}{2\sqrt{t}}.\end{aligned}$$

We see that df is the only odd differential form and because of $df \wedge df = 0$, it does not contribute to $\mathrm{tr}^{\mathrm{ev}}$. Since the eigenspaces of D are preserved by all occuring operators, we can write the trace as

$$\begin{aligned}\hat{\eta} &= \frac{1}{\sqrt{\pi}} \exp\left(-d\beta - ifT\right) \int_0^\infty \sum_{k\in\mathbb{Z}} \left(\left(k + f - \frac{iT}{4t} \right) \right. \\ &\qquad\qquad \left. \exp\left(-t(k+f)^2 + \frac{(k+f)iT}{2} + \frac{T^2}{16t} \right) \right) \frac{dt}{2\sqrt{t}} \\ &= \frac{1}{\sqrt{\pi}} \exp\left(-d\beta - ifT\right) \int_0^\infty \sum_{k\in\mathbb{Z}} \left(\left(k + f - \frac{iT}{4t} \right) \right. \\ &\qquad\qquad \left. \exp\left(\left(i\sqrt{t}(k+f) + \frac{T}{4\sqrt{t}} \right)^2 \right) \right) \frac{dt}{2\sqrt{t}}.\end{aligned}$$

This is why we have to calculate

$$\sum_{k\in\mathbb{Z}} \left(k + f - \frac{iT}{4t} \right) \exp\left(\left(i\sqrt{t}(k+f) + \frac{T}{4\sqrt{t}} \right)^2 \right) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} g(k+f).$$

We denote by $\hat{g}$ the Fourier transform of g and use the generalized Poisson summation formula to see that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} g(k+f) &= \sum_{k\in\mathbb{Z}} \hat{g}(k) \cdot \exp\left(2\pi i k f\right) \\ &= -\sum_{k\in\mathbb{Z}} ik \left(\frac{\pi}{t}\right)^{3/2} \exp\left(-\frac{\pi^2 k^2}{t} + 2\pi i k f + \frac{\pi k T}{2t} \right).\end{aligned}$$

We insert that into the formula of $\hat{\eta}$ and get

$$
\begin{aligned}
\hat{\eta} &= \pi \exp\left(-d\beta - ifT\right) \int_0^\infty \sum_{k\in\mathbb{Z}} \frac{k}{i} \frac{1}{t^{3/2}} \exp\left(-\frac{\pi^2 k^2}{t}\right) \exp\left(2\pi i k f + \frac{\pi k i T}{2it}\right) \frac{dt}{2\sqrt{t}} \\
&= -\pi \exp\left(-d\beta - ifT\right) \sum_{k=1}^\infty \int_0^\infty k \exp\left(-\frac{\pi^2 k^2}{t}\right) \sin\left(-2\pi k f + \frac{\pi k i T}{2t}\right) \frac{dt}{t^2} \\
&= -\pi \exp\left(-d\beta - ifT\right) \sum_{k=1}^\infty k \int_0^\infty \exp\left(-\pi^2 k^2 x\right) \sin\left(-2\pi k f + \frac{\pi k i T}{2} x\right) dx \\
&= -\pi \exp\left(-d\beta - ifT\right) \sum_{k=1}^\infty \left(\frac{4k}{4\pi^2 k^2 - T^2} \sin\left(-2\pi k f\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + i \frac{2T}{4\pi^3 k^2 - \pi T^2} \cos\left(-2\pi k f\right)\right) \\
&= \exp\left(-d\beta - ifT\right) \left(\sum_{k=1}^\infty \sum_{n=0}^{\dim B} \frac{T^{2n}}{2^{2n}\pi^{2n+1}k^{2n+1}} \sin(2\pi k f)\right. \\
&\qquad\qquad\qquad \left. - i \sum_{k=1}^\infty \sum_{n=0}^{\dim B} \frac{T^{2n+1}}{2^{2n+1}\pi^{2n+2}k^{2n+2}} \cos(2\pi k f)\right).
\end{aligned}
$$

We define

$$
g_n(x) = \begin{cases} \sum\limits_{k=1}^\infty \left(2^n \pi^{n+1} k^{n+1}\right)^{-1} \sin\left(2\pi k x\right), & \text{for } n \text{ even} \\ -i \sum\limits_{k=1}^\infty \left(2^n \pi^{n+1} k^{n+1}\right)^{-1} \cos\left(2\pi k x\right), & \text{for } n \text{ odd} \end{cases}
$$

such that

$$
\hat{\eta} = \exp\left(-d\beta - ifT\right) \sum_n g_n(f) T^n.
$$

We see that the functions g_n just depend on $\overline{f} = f - \lfloor f \rfloor \in [0,1)$.
First of all we look at the case $f(b) \in \mathbb{Z}$ and see immediately that $g_n = 0$ for $n \in 2\mathbb{N}$. If $n = 2k+1 \in 2\mathbb{N}+1$ we compute

$$
g_n(f) = -\frac{i}{2^n \pi^{n+1}} \zeta(n+1) = -\frac{i}{2^{2k+1}\pi^{2k+2}} \zeta(2k+2)
$$

and therefore

$$
\begin{aligned}
\hat{\eta}|_{B_0} &= -\exp\left(-d\beta - ifT\right) \sum_k \frac{i}{2^{2k+1}\pi^{2k+2}} \zeta(2k+2) T^{2k+1} \\
&= -\exp\left(-d\beta - ifT\right) \sum_k \frac{i^{2k+1}}{(2k+2)!} B_{2k+2} T^{2k+1},
\end{aligned}
$$

where B_i are the Bernoulli numbers, so $B_i = \frac{d^i h(x)}{dx^i}\Big|_{x=0}$ where $h(x) = \frac{x}{e^x - 1}$. We have $B_{2k+1} = 0$ if $k \geq 1$ and get

$$\begin{aligned}
\hat{\eta}|_{B_0} &= -\exp\left(-d\beta - ifT\right)(iT)^{-1} \sum_k \frac{d^{2k+2}h(x)}{dx^{2k+2}}\bigg|_{x=0} \frac{1}{(2k+2)!}(iT)^{2k+2} \\
&= \exp\left(-d\beta - ifT\right)(-iT)^{-1}\left(\frac{iT}{e^{iT}-1} - 1 + \frac{iT}{2}\right).
\end{aligned}$$

For points where $f \notin \mathbb{Z}$ up to a constant the functions $g_n \colon (0,1) \to \mathbb{R}$ are the Fourier series of the Bernoulli polynomials

$$g_n(x) = \frac{(-1)^{n+1}}{i^n (n+1)!} B_{n+1}(x) = -\frac{i^n}{(n+1)!} B_{n+1}(x).$$

For Bernoulli polynomials we know that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

where the B_k are again the Bernoulli numbers. So we get

$$\begin{aligned}
\hat{\eta}|_{B \backslash B_0} &= -\exp\left(-d\beta - ifT\right) \sum_{n=0}^{\infty} \frac{1}{(n+1)!} B_{n+1}(\overline{f})(iT)^n \\
&= -\exp\left(-d\beta - ifT\right)(iT)^{-1} \sum_{n=0}^{\infty}\sum_{k=0}^{n+1} \frac{1}{k!}\frac{d^k h(x)}{dx^k}\bigg|_{x=0} (iT)^k \frac{1}{(n+1-k)!}(i\overline{f}T)^{n+1-k} \\
&= -\exp\left(-d\beta - ifT\right)(iT)^{-1}\left(\left(\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n h(x)}{dx^n}\bigg|_{x=0}(iT)^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}(i\overline{f}T)^n\right) - 1\right) \\
&= \exp\left(-d\beta - ifT\right)(-iT)^{-1}\left(\frac{iT}{e^{iT}-1}\exp\left(i\overline{f}T\right) - 1\right).
\end{aligned}$$

It follows that

$$\hat{\eta} = \exp\left(-d\beta - ifT\right)(-iT)^{-1}\begin{cases}\left(\frac{-iT}{\exp(-iT)-1} - 1 - \frac{iT}{2}\right), & \text{for } b \in B_0 \\ \left(\frac{iT}{\exp(iT)-1}\exp\left(i\overline{f}T\right) - 1\right), & \text{for } b \in B\backslash B_0\end{cases}$$

and

$$\begin{aligned}
\tilde{\eta} &= \sum_k \frac{1}{(2\pi i)^k}\hat{\eta}_{[2k]} \\
&= \exp\left(-\frac{d\beta + ifT}{2\pi i}\right)\left(-\frac{T}{2\pi}\right)^{-1}\begin{cases}\left(\frac{-T/2\pi}{\exp(-T/2\pi)-1} - 1 - \frac{T}{4\pi}\right), & b \in B_0 \\ \left(\frac{T/2\pi}{\exp(T/2\pi)-1}\exp\left(\frac{\overline{f}T}{2\pi}\right) - 1\right), & b \in B\backslash B_0.\end{cases}
\end{aligned}$$

□

3.1.10 Theorem. *We define $d\tilde{\eta}\colon \Omega^{\mathrm{odd}}(B) \to \mathbb{R}$ as a current by*

$$\int_B (d\tilde{\eta}) \wedge \omega := -\int_B \tilde{\eta} \wedge d\omega.$$

The following formula for the differential holds

$$d\tilde{\eta} = \int_{M/B} \mathrm{ch}\left(V, \nabla^V\right) + \delta_{B_0}\, \mathrm{ch}\left(\ker D \to B_0, \nabla^{\ker}\right),$$

where $\nabla^{\ker} = P_0 \nabla^{\pi_* V} P_0$ and P_0 is the orthogonal projection $P_0\colon\ \pi_* V|_{B_0} \to \ker D$.

Proof. We have two different possibilities to calculate the differential of $\tilde{\eta}$. On the one hand we have the transgression formula (2.1.2)

$$d \int_s^T \mathrm{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2}\right) = \mathrm{tr}^{\mathrm{odd}}\left(e^{-\mathbb{A}_s^2}\right) - \mathrm{tr}^{\mathrm{odd}}\left(e^{-\mathbb{A}_T^2}\right)$$

By [BF86, Theorem 2.10] we know the limit for $s \to 0$ is

$$\lim_{s\to 0} \frac{1}{\sqrt{\pi}} \mathrm{tr}^{\mathrm{odd}}\left(e^{-\mathbb{A}_t^2}\right) = (2\pi i)^{-1} \int_{M/B} \det\left(\frac{R^{M/B}/2}{\sinh\left(R^{M/B}/2\right)}\right)^{1/2} \mathrm{tr}\left(\exp\left(-\left(\nabla^V\right)^2\right)\right)$$

and since $\hat{A}\left(TX, \nabla^X\right) = \hat{A}\left(TS^1\right) = 1$ we get the first summand. For the second we need to proof that

$$\lim_{T\to\infty} \mathrm{tr}^{\mathrm{odd}}\left(e^{-\mathbb{A}_T^2}\right) = -\sqrt{\pi}\delta_{B_0}\, \mathrm{tr}\left(\exp\left(-\left(\nabla^{\ker}\right)^2\right)\right).$$

For that we know that for all eigenvalues $k+f$, $k \neq 0$ and all $\mathcal{C}^\ell$-norms

$$\left\|\exp\left(-t(k+f)^2 - \sqrt{t}df - d\beta - ifT + i\frac{(k+f)T}{2} + \frac{T^2}{16t}\right)\right\|_{\mathcal{C}^\ell} \leq Ce^{-ct}.$$

For $k = 0$ we see that we cannot take the limit as a differential form, we have to integrate over the normal direction of a tubular neighbourhood $N_a \cong B_0 \times (-a, a)$ of B_0 where $f(x,y) = y$. Let $\omega \in \Omega^\bullet(B)$ where $\overline{\mathrm{supp}\,\omega} \subset N_a$

$$\begin{aligned}
&\int_{-a}^{a} \exp\left(-ty^2 - \sqrt{t}dy - d\beta - iyT + \frac{iyT}{2} + \frac{T^2}{16t}\right)\omega \\
&= \int_{-a\sqrt{t}}^{a\sqrt{t}} \exp\left(-y^2 - dy - f_t^* d\beta - \frac{iyf_t^* T}{2\sqrt{t}} + \frac{f_t^* T^2}{16t}\right) f_t^*\omega
\end{aligned}$$

where $f_t\colon (-a\sqrt{t}, a\sqrt{t}) \to (-a,a), x \mapsto \frac{x}{\sqrt{t}}$. Now we can see that we have a Gaussian bell curve and therefore

$$\begin{aligned}
&\lim_{t\to\infty} \int_{-a}^{a} \exp\left(-ty^2 - \sqrt{t}dy - d\beta - \frac{iyT}{2} + \frac{T^2}{16t}\right)\omega \\
&= -\sqrt{\pi} i^* \exp\left(-d\beta\right) i^*\omega,
\end{aligned}$$

where $i\colon B_0 \to B$ denotes the inclusion.
On the other hand we can directly calculate the formula for $d\tilde{\eta}$ by the formula for $\tilde{\eta}$ of Theorem 3.1.7 and

$$\begin{aligned}
\int_B \left(d\tilde{\eta}\right)\omega &= -\int_B \tilde{\eta} d\omega \\
&= -\lim_{a\to 0} \int_{B\setminus N_a} \tilde{\eta} d\omega \\
&= \lim_{a\to 0} \int_{B\setminus N_a} \left(d\tilde{\eta}\right)\omega - \lim_{a\to 0} \int_{B\setminus N_a} d\left(\tilde{\eta}\omega\right) \\
&= \lim_{a\to 0} \int_{B\setminus N_a} \left(d\tilde{\eta}\right)\omega - \lim_{a\to 0} \int_{B_0 - a} i^*\left(\tilde{\eta}\omega\right) + \lim_{a\to 0} \int_{B_0 + a} i^*\left(\tilde{\eta}\omega\right),
\end{aligned}$$

which will lead to the same formula as the reader may easily check.

□

3.2 Transversal zero-crossing of a single eigenvalue

We will now turn to a more general setting. Let $M \to B$ be a Riemannian fibre bundle and $V \to M$ a vertical Dirac bundle as in Section 2.1. The transgression formula in [BC89, Theorem 4.95] holds for invertible vertical Dirac operators, it was generalized by [BGV04, Theorem 10.32] for vertical Dirac operators with constant kernel dimension (see also [Dai91, Theorem 0.1] for odd-dimensional fibres). We want to take the next step and give a generalization for a transversal zero-crossing of one eigenvalue of multiplicity one. For the proof we adopt many ideas of the proof of [Bis90, Theorem 3.2]. However, we have to be very careful which norms we use, since our operators are endomorphisms of an infinite rank vector bundle. We also use different contours as in [Bis90] which comes from the fact that we want to use holomorphic funtional calculus of the form

$$\exp\left(-\mathbb{A}_t^2\right) = \frac{1}{2\pi i} \int_\Gamma \frac{e^{-z}}{z - \mathbb{A}_t^2} dz$$

rather than

$$\exp\left(-\mathbb{A}_t^2\right) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \frac{e^{-z^2}}{z - \mathbb{A}_t} dz.$$

3.2.1 Assumption. We assume that we can find a covering $\{U_i\}_{1\leq i\leq k}$ for B such that on each U_i either D_b is invertible or we have a smooth function $f_i : U_i \to (-K, K)$ which has 0 as a regular value, such that $\operatorname{spec} D_b \cap (-K-\delta, K+\delta) = \{f_i(b)\}$ and $f_i(b)$ is of multiplicity 1.

3.2.2 Remark. We want to remind the reader of Remark 2.3.2 in the premliniaries. By the above assumption we get a codimension 1 submanifold

$$B_0 = \bigcup_{1\leq i\leq k} f_i^{-1}(\{0\}) \subset B$$

where we have a complex line bundle $\ker D \to B_0$ of rank 1 and D_b is invertible for all $b \in B \setminus B_0$. We denote by $i\colon B_0 \to B$ the inclusion. As in section 3.1 we get an orientation on B_0 by

$$(v_2, ..., v_m) \in o_x(B_0) \Leftrightarrow (\operatorname{grad} f_x, v_2, ..., v_m) \in o_x(B).$$

Let $\nu B_0 \to B_0$ be the normal bundle, which is trivial $\nu B_0 \cong B_0 \times \mathbb{R}$. Then we find a constant $0 < a \leq K$ small enough such that

$$\exp\colon B_0 \times (-a, a) \to B$$

is a diffeomorphism onto its image N_a. We will not fix a since we may take it as small as needed in the proofs. Without loss of generality we may assume that in this identification

$$f(x, y) = y.$$

3.2.3 Proposition and Definition. *Let P_b, $b \in N_a$ be the orthogonal projections onto the spectral subspace $(-a-\delta, a+\delta)$ of D_b. Then*

$$L = \operatorname{im} P \to N_a$$

is a smooth line bundle on the tubular neighbourhood N_a of B_0. We denote the projection onto the orthogonal complement W by $Q = 1 - P$ and the projection of the connection $\nabla^{\pi_* V}$ onto the subbundles L and W by

$$\nabla^{L\oplus W} = P\nabla^{\pi_* V}P \oplus Q\nabla^{\pi_* V}Q.$$

The projections of D are denoted by $D^- = DP = yP$ and $D^+ = DQ$.

Proof. The first part follows from [BGV04, Proposition 9.10], since $\pm a \pm \delta$ is not an eigenvalue of D_b for $b \in N_a$. □

3.2.4 Lemma. *If we are working locally around B_0 we consider the isometry*

$$\left(\exp^* \pi_* V|_{N_a}, \exp^* g_{\pi_* V}|_{N_a}\right) \underset{G}{\xrightarrow{\sim}} \left(\pi_* V|_{B_0} \times (-a, a), g_{\pi_* V}|_{B_0} \times (-a, a)\right),$$

where G is given by parallel transport along normal geodesics with respect to the connection $\nabla^{\pi_* V}$. (Note that it is in general not possible to trivialize with respect to the connection $\nabla^L \oplus \nabla^W$!)

Proof. Let us fix $b \in B_0$. The lifts of the geodesic $\exp_b\colon (-a,a) \to N_a$ gives a family of geodesics $\widetilde{\exp}_b\colon M_b \times (-a,a) \to \pi^{-1}(N_a)$, see [Kli82, Corollary 1.11.11]. By taking a small enough we may assume that $\widetilde{\exp}_b(\cdot,t)\colon M_b \to M_{\exp_b(t)}$ is an isomorphism for all $t \in (-a,a)$. Therefore if $\sigma \in (\pi_* V)_b = \Gamma\left(M_b, V|_{M_b}\right)$ we can use parallel transport for each $\sigma_x \in V_{b,x}$ with respect to the connection $\nabla^V - \frac{1}{2}k$ to get a vector in $V_{\widetilde{\exp}_b(x,t)}$. This depends smoothly on $x \in M_b$ so we get a smooth section in $(\pi_* V)_{\exp_b(t)}$. □

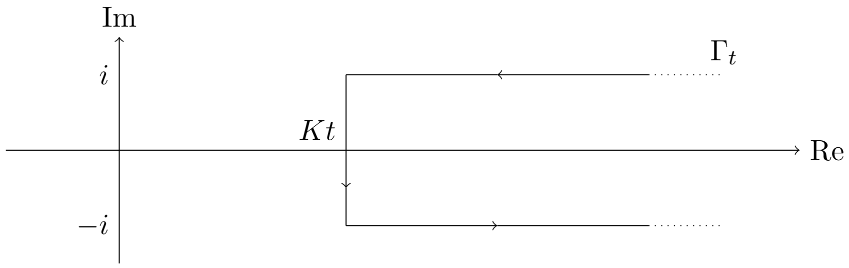
3.2.5 Definition. We denote by

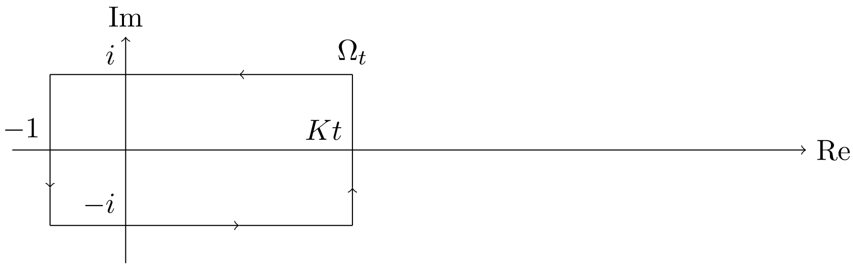
$$E_t := \mathbb{A}_t^2 - tD^2 = \sqrt{t}[\nabla^{\pi_* V}, D] + \left(\nabla^{\pi_* V}\right)^2 - \frac{[D, c(T)]}{4} - \frac{[\nabla^{\pi_* V}, c(T)]}{4\sqrt{t}} + \frac{c(T)^2}{16t}.$$

Let $\lambda_0(x,y) = f(x,y)$ be the small eigenvalue of $D_{(x,y)}$ which crosses zero and denote the other eigenvalues by λ_k for $k \in \mathbb{Z}\backslash\{0\}$. Then by our assumption

$$\exists \tilde{K} > 0 : \sup_{(x,y)\in N} \lambda_0^2(x,y) + \tilde{K} = a^2 + \tilde{K} \leq \inf_{(x,y)\in N} \lambda_k^2(x,y) \quad \forall k \neq 0.$$

Let $K := a^2 + \frac{\tilde{K}}{2}$ and define the contours $\Gamma_t, \Omega_t \subset \mathbb{C}$ by:





Since

$$\left(z - \mathbb{A}_t^2\right)^{-1} = \sum_{n=0}^{\dim B} \left(z - tD^2\right)^{-1} \left(E_t \left(z - tD^2\right)^{-1}\right)^n$$

the spectrum of $\mathbb{A}_t^2$ equals the spectrum of the generalized Laplace operator tD^2. By holomorphic functional calculus [GGK90, Chapter XV, Proposition 1.1] we know that on N_a

$$\begin{aligned}\exp\left(-\mathbb{A}_t^2\right) &= \frac{1}{2\pi i}\int\limits_{\Omega_t\cup\Gamma_t} \exp\left(-z\right)\left(z-\mathbb{A}_t^2\right)^{-1} dz \\ &= \frac{1}{2\pi i}\int\limits_{\Omega_t} \exp\left(-z\right)\left(z-\mathbb{A}_t^2\right)^{-1} dz + \frac{1}{2\pi i}\int\limits_{\Gamma_t} \exp\left(-z\right)\left(z-\mathbb{A}_t^2\right)^{-1} dz \\ &= \mathbb{P}_t\left(\exp\left(-\mathbb{A}_t^2\right)\right) + \left(1-\mathbb{P}_t\right)\left(\exp\left(-\mathbb{A}_t^2\right)\right).\end{aligned}$$

Note that the projection

$$\mathbb{P}_t = \frac{1}{2\pi i}\int\limits_{\Gamma_t} \left(z-\mathbb{A}_t^2\right)^{-1} dz \colon \Lambda^\bullet T^*B\otimes\pi_*V \to \Lambda^\bullet T^*B\otimes\pi_*V$$

coincides in degree 0 with the spectral projection $P\colon \pi_*V \to L\subset \pi_*V$.

3.2.6 Definition. We take the pullback of the bundle $\ker D\to B_0$ via $g : B_0\times\mathbb{R}\to B_0$ with the connection $g^*\nabla^{\ker}$ which, by abuse of notation, will also be denoted by $\nabla^{\ker}$. We denote the second coordinate of $B_0\times\mathbb{R}$ by y and consider the superconnection

$$y+\nabla^{\ker} : \Omega^\bullet\left(B_0\times\mathbb{R}, g^*\ker D\right)\to\Omega^\bullet\left(B_0\times\mathbb{R}, g^*\ker D\right),$$

where we assume that multiplication by y and 1-forms anticommute. Note that this differs slightly from the superconnection B introduced in [Bis90, III.a]. One should also point out at this point that the bundles $L\to N_a\cong B_0\times(-a,a)$ and $g^*\ker D = \ker D\times(-a,a)\to B_0\times(-a,a)$ correspond at B_0 but in general not on $N_a\backslash B_0$. This is because the spectral subspace $L\subset \pi_*V|_{N_a}$ is in general not constant in the trivialization of Lemma 3.2.4 where we used parallel transport with respect to the connection ∇^{π_*V}. If $|y|\le a\sqrt{t}$ we can proceed as in the previous definition and write

$$\exp\left(-\left(y+\nabla^{\ker}\right)^2\right) = \frac{1}{2\pi i}\int\limits_{\Omega_t}\exp(-z)\left(z-\left(y+\nabla^{\ker}\right)^2\right)^{-1} dz.$$

Notation. We will need different kinds of norms in the following statements and proofs which we will introduce here. See also [RS75, Appendix of IX.4, Example 2].
We denote by $W^k = W^{(k,2)}\left(M_b, V_b\right)$ the kth Sobolev space of sections with Sobolev norm $|\cdot|_k$, $W^0 = L^2\left(M_b,V_b\right)$. For a linear operator $A : W^k\to W^{k'}$ we define the operator norm

$$\|A\|_{k,k'} = \sup_{|x|_k=1} |A(x)|_{k'}.$$

We say a bounded linear operator $A\in\mathcal{B}\left(W^0\right)$ is *trace-class* if

$$\|A\|_1 = \operatorname{tr}|A| < \infty.$$

For $1 \leq p < \infty$ the *p-Schatten norm* is defined by

$$\|A\|_p = \left(\operatorname{tr}\left(|A|^p\right)\right)^{1/p}.$$

For a smooth differential form $\omega \in \Omega^\bullet(B)$ we denote by $\|\omega\|_{\mathcal{C}^\ell(B)}$ the $\mathcal{C}^\ell$-norm. For $\omega \in \Omega^\bullet(B_0 \times (-a,a))$ we see $\|\omega\|_{\mathcal{C}^\ell(B_0)}$ as a function on $(-a,a)$.

3.2.7 Remark. The trivialization of Lemma 3.2.4 provides us with an isometry

$$L^2\left(M_x, V_x\right) \cong L^2\left(M_{(x,y)}, V_{(x,y)}\right)$$

for all $(x,y) \in B_0 \times (-a,a)$. If we work with Sobolev-sections for $k > 0$ we still get an isomorphism but not an isometry. However we know that the topology of the Banach spaces is the same and therefore the Sobolev norms are equivalent. In particular since B_0 is compact and if a is small enough we find constants $C, c > 0$ such that for all $(x,y) \in B_0 \times (-a,a)$ and all sections $\sigma \in W^{k,2}\left(M_x, V_x\right) \cong W^{k,2}\left(M_{(x,y)}, V_{(x,y)}\right)$ the following estimate holds

$$C\left|\sigma\right|_{k,(x,y)} \leq \left|\sigma\right|_{k,x} \leq c\left|\sigma\right|_{k,(x,y)}.$$

So in the following estimates we will make no difference for which $y \in (-a,a)$ we use the Sobolev norms because by changing the constants the estimates hold for all points y and we get the same speed of convergence.

3.2.8 Lemma. *Let $z \in \Gamma_t$ or $z \in \Omega_t$, $p \geq \dim M_b + 1$ and t big enough, then we have the following estimates*

$$\left\|\left(z - tD_b^2\right)^{-1}\right\|_{0,0} \leq C_1, \tag{3.2.1}$$

$$\left\|\left(z - tD_b^2\right)^{-1}\right\|_{0,2} \leq C_2\left(1 + \frac{|z|}{t}\right), \tag{3.2.2}$$

$$\left\|\left(z - tD_b^2\right)^{-1}\right\|_{p} \leq C_3\left(1 + \frac{|z|}{t}\right), \tag{3.2.3}$$

for every $b \in N_a$.

Proof. (3.2.1) follows from the choice of the contours Γ_t and Ω_t.
(3.2.2) and (3.2.3) follow as in [BG00, Proposition 7.2] by writing

$$\left(z - tD^2\right)^{-1} = t^{-1}\left(i - D^2\right)^{-1} - \left(i - D^2\right)^{-1}\left(\frac{z}{t} - i\right)\left(z - tD^2\right)^{-1}.$$

We then use the well-known facts that there exist constants such that

$$\left\|\left(i - D^2\right)^{-1}\right\|_p \leq C$$

for $p \geq \dim M_b + 1$, this follows for example by [Roe98, Remark 5.32, Propostision 8.9], and

$$\left\|\left(i - D^2\right)^{-1}\right\|_{0,2} \leq C$$

see [BG00, Equation (7.7)]. Together with estimate (3.2.1) these prove the claimed inequalities (3.2.3) and (3.2.2).

□

3.2.9 Proposition. *On the tubular neighbourhood $N_a \cong B_0 \times (-a, a)$ of B_0 in B the following estimate holds true*

$$\left\| \mathrm{tr}^{\mathrm{odd}} \left((1 - \mathbb{P}_t) \left(\exp \left(-\mathbb{A}_t^2 \right) \right) \right) \right\|_{\mathcal{C}^0} \leq c f(t) \exp \left(-Ct \right),$$

where $f(t) \in \mathbb{R}[t, t^{-1}]$ is polynomial in t and t^{-1}.

Proof. By the definition of E_t and since B is compact we know that

$$\|E_t\|_{2,0} \leq C\sqrt{t}.$$

Combining this with the estimates (3.2.2) and (3.2.3) we get

$$\begin{aligned}
\left\| \left(z - \mathbb{A}_t^2 \right)^{-p} \right\|_1 &\leq \left\| \left(z - \mathbb{A}_t^2 \right)^{-1} \right\|_p^p \\
&\leq \left(\sum_{n=0}^{\dim B} \left\| \left(z - tD^2 \right)^{-1} \right\|_{0,2}^n \|E_t\|_{2,0}^n \left\| \left(z - tD^2 \right)^{-1} \right\|_p \right)^p \\
&\leq \left(\sum_{n=0}^{m} C \left(1 + \frac{|z|}{t} \right)^{n+1} t^{n/2} \right)^p \\
&\leq C \left(1 + \frac{|z|}{t} \right)^{(m+1)p} t^{mp/2},
\end{aligned}$$

where $m = \dim B$ and constants C varying from line to line. It follows that

$$\begin{aligned}
&\left\| \mathrm{tr}^{\mathrm{odd}} \left((1 - \mathbb{P}_t) \left(\exp \left(-\mathbb{A}_t^2 \right) \right) \right) \right\|_{\mathcal{C}^0} \\
&\leq \left\| (1 - \mathbb{P}_t) \left(\exp \left(-\mathbb{A}_t^2 \right) \right) \right\|_1 \\
&= \left\| \frac{1}{2\pi i} \int_{\Gamma_t} \frac{\exp(-z)}{z - \mathbb{A}_t^2} dz \right\|_1 \\
&= \frac{1}{2\pi p!} \left\| \int_{\Gamma_t} \frac{\exp(-z)}{\left(z - \mathbb{A}_t^2 \right)^p} dz \right\|_1 \\
&\leq \frac{1}{2\pi p!} \int_{\Gamma_t} |\exp(-z)| \, C \left(1 + \frac{|z|}{t} \right)^{(m+1)p} t^{mp/2} dz \\
&\leq C f(t) \exp(-Kt),
\end{aligned}$$

where $f \in \mathbb{R}[t, t^{-1}]$.

□

3.2.10 Definition. We define the functions g, f_t and i to be

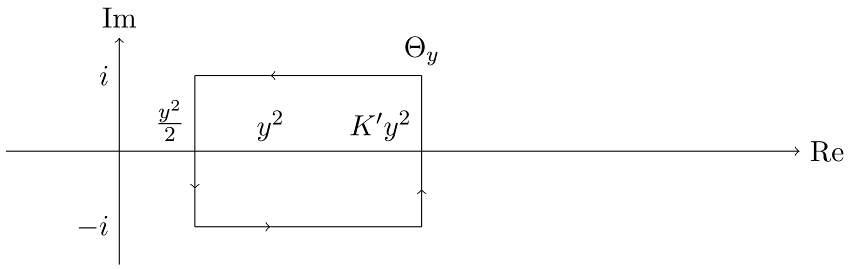
$$g\colon B_0 \times (-a\sqrt{t}, a\sqrt{t}) \to B_0, (x,y) \mapsto x,$$

$$f_t\colon B_0 \times (-a\sqrt{t}, a\sqrt{t}) \to B_0 \times (-a,a), (x,y) \mapsto \left(x, \frac{y}{\sqrt{t}}\right)$$

and

$$i\colon B_0 \to B_0 \times (-a,a), x \mapsto (x,0).$$

Note that we use g for any projection onto the first coordinate $B_0 \times I \to B_0$, independent of the interval I.
For $y \in (-a\sqrt{t}, a\sqrt{t})$ and $|y| \geq 1$ the contour $\Theta_y \subset \mathbb{C}$ is defined to be



where K' is taken small enough, such that Ω_y only contains the small eigenvalue of $D^2_{(x,y)}$ for all $x \in B_0$. Then we can write the spectral projection $\mathbb{P}_t$ as

$$\mathbb{P}_t\left(\exp\left(-f_t^*\mathbb{A}_t^2\right)_{(x,y)}\right) = \frac{1}{2\pi i}\int\limits_{\Theta_y} \exp(-z)\left(z - f_t^*\mathbb{A}_t^2\right)^{-1} dz$$

and also

$$\exp\left(-\left(y + \nabla^{\ker}\right)^2\right) = \frac{1}{2\pi i}\int\limits_{\Theta_y} \exp(-z)\left(z - \left(y + \nabla^{\ker}\right)^2\right)^{-1} dz.$$

3.2.11 Remark. It is clear by the definition of the contour Θ_y that the estimates in Lemma 3.2.8 also hold for $z \in \Theta_{y\sqrt{t}}$.

3.2.12 Lemma. *If ω is a differential form on B with support in $B_0 \times (-a,a)$ and α a multiindex of length ℓ then*

$$\left|D^\alpha\left((i \circ g)^*\omega - f_t^*\omega\right)_{(x,y)}\right| \leq \frac{C}{\sqrt{t}} \|\omega\|_{\mathcal{C}^{\ell+1}(B)}\left(1 + |y|\right).$$

Proof. This follows by a straight-forward calculation, see also [Bis90, Eq. (3.107)] for the statement. $\square$

3.2.13 Lemma (cf. [Bis90, Proposition 3.4])**.** *Let $(x,y) \in B_0 \times (-a,a)$ and $z \in \Omega_t$ or $z \in \Theta_{y\sqrt{t}}$, a small enough and t big enough. By abuse of notation we write $\left(D^+_{(x,y)}\right)^{-1}$ instead of $\left(D^+_{(x,y)}\right)^{-1} Q_{(x,y)}$. Then the following inequalities hold*

$$\left\| \left(z - t \left(D^+_{(x,y)}\right)^2 \right)^{-1} \right\|_{0,2} \leq \frac{C}{t} \left(1 + |z|\right)$$

$$\left\| \left(z - t \left(D^+_{(x,y)}\right)^2 \right)^{-1} + t^{-1} \left(D^+_{(x,0)}\right)^{-2} \right\|_{0,2} \leq \frac{C}{t} \left(|y| + t^{-1} |z| + t^{-1} |z|^2 \right).$$

Proof. The proof follows the ideas of the proof of [Bis90, Proposition 3.4]. Our constants $C > 0$ may vary from line to line but they are all indepenent of t, y and z and since B_0 is compact also of x.
For the first estimate we write

$$\left(z - t \left(D^+_{(x,y)}\right)^2 \right)^{-1} = -t^{-1} \left(1 - \frac{z}{t} \left(D^+_{(x,y)}\right)^{-2} \right)^{-1} \left(D^+_{(x,y)}\right)^{-2}. \tag{3.2.4}$$

As in [Bis90, Eq. (3.37)] we know that for $|\operatorname{Im} z| = 1$

$$\begin{aligned} \left\| \left(1 - \frac{z}{t} \left(D^+_{(x,y)}\right)^{-2} \right)^{-1} \right\|_{0,0} &\leq \sup_{x\in\mathbb{R}} |1 - xz|^{-1} \\ &= \frac{1}{\inf_{x\in\mathbb{R}} |1 - xz|} \\ &= |z|. \end{aligned}$$

If $|\operatorname{Im} z| < 1$ we know that either $\operatorname{Re} z = Kt$, $\operatorname{Re} z = -1$ or $\operatorname{Re} z = Cty^2$. We find a constant $C > 0$ such that for t big enough in each of these three cases

$$\left\| \frac{\operatorname{Re} z}{t} \left(D^+_{(x,y)}\right)^{-2} \right\|_{0,0} \leq Ca^2,$$

in particular for a small enough

$$\left\| \frac{\operatorname{Re} z}{t} \left(D^+_{(x,y)}\right)^{-2} \right\|_{0,0} \leq \frac{1}{2}$$

and therefore

$$\left\| \left(1 - \frac{z}{t} \left(D^+_{(x,y)}\right)^{-2} \right)^{-1} \right\|_{0,0} \leq 2.$$

So for all z in the contours Ω_t and $\Theta_{y\sqrt{t}}$ the inequality

$$\left\| \left(1 - \frac{z}{t} \left(D^+_{(x,y)}\right)^{-2} \right)^{-1} \right\|_{0,0} \leq C \left(1 + |z|\right)$$

holds. Also for a small enough we find a constant C sucht that for all $(x,y) \in B_0 \times (-a,a)$

$$\left\| \left(D^+_{(x,y)} \right)^{-2} \right\|_{0,2} \leq C.$$

Inserting this into equation (3.2.4) leads to

$$\left\| \left(z - t \left(D^+_{(x,y)} \right)^2 \right)^{-1} \right\|_{0,2} \leq \frac{C}{t} \left(1 + |z| \right)$$

which completes the first part of the lemma.
For the second inequality of the lemma we write

$$\begin{aligned} &\left\| \left(z - t \left(D^+_{(x,y)} \right)^2 \right)^{-1} + t^{-1} \left(D^+_{(x,0)} \right)^{-2} \right\|_{0,2} \\ &\leq \left\| \left(z - t \left(D^+_{(x,y)} \right)^2 \right)^{-1} z t^{-1} \left(D^+_{(x,y)} \right)^{-2} \right\|_{0,2} + \left\| t^{-1} \left(D^+_{(x,0)} \right)^{-2} - t^{-1} \left(D^+_{(x,y)} \right)^{-2} \right\|_{0,2}. \end{aligned}$$

By Taylor approximating, see [Růž04, Satz 2.8], we know that

$$t^{-1} \left\| \left(D^+_{(x,0)} \right)^{-2} - \left(D^+_{(x,y)} \right)^{-2} \right\|_{0,2} \leq \frac{C}{t} |y|$$

and by using the first part we have

$$\left\| \left(z - t \left(D^+_{(x,y)} \right)^2 \right)^{-1} \frac{z}{t} \left(D^+_{(x,y)} \right)^{-2} \right\|_{0,2} \leq \frac{C}{t^2} \left(|z| + |z|^2 \right).$$

Combing these leads to

$$\left\| \left(z - t \left(D^+_{(x,y)} \right)^2 \right)^{-1} - t^{-1} \left(D^+_{(x,0)} \right)^{-2} \right\|_{0,2} \leq \frac{C}{t} \left(|y| + t^{-1} |z| + t^{-1} |z|^2 \right)$$

which completes the second part of the lemma. □

3.2.14 Proposition ([Bis90, Proposition 3.5])**.** *For $x \in B_0$ and $X \in T_x B$*

$$\nabla^{\pi_* V}_X - \nabla^{L \oplus W}_X = \begin{pmatrix} 0 & P \nabla^{\pi_* V}_X (D) Q \left(D^+ \right)^{-1} \\ - \left(D^+ \right)^{-1} Q \nabla^{\pi_* V}_X (D) P & 0 \end{pmatrix}$$

with respect to the decomposition $\pi_* V|_{N_a} = L \oplus W$. Therefore on B_0

$$\left(\nabla^{\ker} \right)^2_x = P \left(\nabla^{\pi_* V} \right)^2 P - P \nabla^{\pi_* V}(D) \left(D^+ \right)^{-2} \nabla^{\pi_* V}(D) P.$$

3.2.15 Proposition. *We define for $(x,y) \in B_0 \times \left(-a\sqrt{t}, a\sqrt{t}\right)$, $z \in \Omega_t$ or Θ_y the operator α by*

$$\left(Pf_t^* E_t P + Pf_t^* E_t Q \left(z - t f_t^* D^2\right)^{-1} Q f_t^* E_t P\right)\Big|_{(x,y)} = g^*\left(dy + \left(\nabla^{\ker}\right)^2\right)\Big|_{(x,y)} + \alpha\left(x,y,z,t\right)$$

where we identify $L_{(x,y/\sqrt{t})}$ and $\ker D_{(x,0)}$ by parallel transport along the geodesic $s \mapsto (x, sy/\sqrt{t})$ with respect to ∇^L. Then there exists a constant $C > 0$ such that for t big enough

$$\left\|\alpha\left(x,y,z,t\right)\right\|_{2,0} \leq C t^{-1/2}\left(1 + |y| + |z| + |z|^2\right).$$

Proof. First we use Proposition 3.2.14 to see that

$$\begin{aligned}
&\left\|Pf_t^* E_t P + Pf_t^* E_t Q \left(z - t f_t^* D^2\right)^{-1} Q f_t^* E_t P - g^*\left(dy + \left(\nabla^{\ker}\right)^2\right)\right\|_{2,0} \\
&= \Big\|Pf_t^* E_t P + Pf_t^* E_t Q \left(z - t f_t^* D^2\right)^{-1} Q f_t^* E_t P \\
&\quad - g^*\left(dy + P\left(\nabla^{\pi_* V}\right)^2 P - P\nabla^{\pi_* V}(D)\left(D^+\right)^{-2}\nabla^{\pi_* V}(D)P\right)\Big\|_{2,0} \\
&\leq \left\|Pf_t^* E_t P - g^*\left(dy + P\left(\nabla^{\pi_* V}\right)^2 P\right)\right\|_{2,0} \\
&\quad + \left\|Pf_t^* E_t Q \left(z - t f_t^* D^2\right)^{-1} Q f_t^* E_t P + g^*\left(P\nabla^{\pi_* V}(D)\left(D^+\right)^{-2}\nabla^{\pi_*}(D)P\right)\right\|_{2,0}.
\end{aligned}$$

By definition, our trivialization, Lemma 3.2.12 and again Taylor approximation [Růž04, Satz 2.8]

$$\left\|Pf_t^* E_t P - g^*\left(dy + P\left(\nabla^{\pi_* V}\right)^2 P\right)\right\|_{2,0} \leq \frac{C}{\sqrt{t}}\left(1 + |y|\right).$$

For the second summand we have

$$\begin{aligned}
&\left\|Pf_t^* E_t Q \left(z - tD^2\left(x, \frac{y}{\sqrt{t}}\right)\right)^{-1} Q f_t^* E_t P + g^*\left(P\nabla^{\pi_* V}(D)\left(D^+\right)^{-2}\nabla^{\pi_* V}(D)P\right)\right\|_{2,0} \\
&\leq \left\|Pf_t^* E_t Q \left(z - tD^2\left(x, \frac{y}{\sqrt{t}}\right)\right)^{-1} Q\left(f_t^* E_t - \sqrt{t}\nabla^{\pi_* V}(D)\right)P\right\|_{2,0} \\
&\quad + \left\|Pf_t^* E_t Q \left(\left(z - tD^2\left(x, \frac{y}{\sqrt{t}}\right)\right)^{-1} + t^{-1}\left(D^+\right)^{-2}(x,0)\right) Q\sqrt{t}\nabla^{\pi_* V}(D)P\right\|_{2,0} \\
&\quad + \left\|P\left(-f_t^* E_t + \sqrt{t}\nabla^{\pi_* V}(D)\right)t^{-1}\left(D^+\right)^{-2}(x,0)\sqrt{t}\nabla^{\pi_* V}(D)P\right\|_{2,0} \\
&\leq C_1 t^{-1/2}\left(1 + |z| + |z|^2\right) + C_2 t^{-1/2}\left(|y| + |z| + |z|^2\right) + C_3 t^{-1/2}
\end{aligned}$$

where we used Lemma 3.2.13 and the definition of E_t. □

3.2.16 Proposition. *Let $(x,y) \in B_0 \times \left(-a\sqrt{t}, a\sqrt{t}\right)$, $z \in \Omega_t$ or $z \in \Omega_y$ and t big enough. We define*

$$\left(z - f_t^* \mathbb{A}_t^2\right)^{-1} - \left(z - \left(y + \nabla^{\ker}\right)^2\right)^{-1} =: \gamma\left(x,y,z,t\right).$$

Then there exist constants $C_1, C_2, C_3, C_4 > 0$ and polynomials p_1, p_2, p_3, p_4, p_5 such that

$$\begin{aligned}
\|P\gamma P\|_{0,0} &\leq C_1 t^{-1/2}\left(1 + p_1\left(|y|\right) + p_2\left(|z|\right)\right)\\
\|P\gamma Q\|_{0,0} &\leq C_2 t^{-1/2}\left(1 + p_3\left(|z|\right)\right)\\
\|Q\gamma P\|_{0,0} &\leq C_3 t^{-1/2}\left(1 + p_4\left(|z|\right)\right)\\
\|Q\gamma Q\|_{0,0} &\leq C_4 t^{-1}\left(1 + p_5\left(|z|\right)\right).
\end{aligned}$$

Proof. Throughout the proof we will denote by p some polynomial in $|z|$ or $|y|$ which may vary from line to line but is independent of x, t and y or z respectively. The constants $C > 0$ may also vary but again are indepenent of x, y, z and t. For simplicity but by abuse of notation we define just for this proof $A := \left(z - t f_t^* D^2\right)^{-1}$, $B := f_t^* E_t$, $X := \left(z - y^2\right)^{-1}$ and $Y := dy + \left(\nabla^{\ker}\right)^2$. Then we know that

$$\left(z - f_t^* \mathbb{A}_t^2\right)^{-1} - \left(z - \left(y + \nabla^{\ker}\right)^2\right)^{-1} = \sum_{n\geq 0} A(BA)^n - X(YX)^n,$$

where the sum is finite.
Let us first look at

$$\begin{aligned}
&P\left(\sum_{n\geq 0} A(BA)^n - X(YX)^n\right)P\\
&= \sum_{n\geq 0} XP(BA)^n P - X(YX)^n\\
&= \sum_{n\geq 0} XP((PBP + PBQ + QBP + QBQ)A)^n P - X(YX)^n
\end{aligned}$$

Since $PQ = QP = 0$ the only combination in which QBQ can occur is of the following form

$$PBQA(QBQA)^k QBP.$$

But since we know by Lemma 3.2.13 that

$$\|QAQ\|_{0,2} = \left\|\left(z - t\left(D^+\right)^2\left(x, \frac{y}{\sqrt{t}}\right)\right)^{-1}\right\|_{0,2} \leq \frac{C}{t}\left(1 + |z| + |z|^2\right)$$

and by the definition of E_t that

$$\|B\|_{2,0} = \|f_t^* E_t\|_{2,0} \leq C\sqrt{t},$$

it follows that
$$\left\|PBQA(QBQA)^kQBP\right\|_{2,0} \leq Ct^{-k/2}\left(1+p(|z|)\right).$$
By the same argument as above, PBQ and QBP can only occur as
$$PBQAQBP.$$
Combining these together with inequality (3.2.2) of Lemma 3.2.8 yields to
$$\begin{aligned}
&\left\|P\left(\left(z-f_t^*\mathbb{A}_t^2\right)^{-1}-\left(z-\left(y+\nabla^{\ker}\right)\right)^{-1}\right)P\right\|_{0,0}\\
&\leq \left\|\sum_{n\geq 0} XP((PBP+PBQ+QBP)A)^nP - X(YX)^n\right\|_{0,0} + Ct^{-1/2}\left(1+p(|z|)\right)\\
&\leq \sum_{n\geq 0}\|X((PBP+PBQAQBP)X)^n - X(YX)^n\|_{0,0} + Ct^{-1/2}\left(1+p(|z|)\right)\\
&\leq Ct^{-1/2}\left(1+p_1\left(|y|\right)+p_2\left(|z|\right)\right),
\end{aligned}$$
where we used Proposition 3.2.15 and inequality (3.2.2) in the last step.
For the other estimates we don't need $X(YX)^n$, since $PX(YX)^nP = X(YX)^n$. We know that
$$A = \begin{pmatrix} \left(z-y^2\right)^{-1} & 0 \\ 0 & \left(z-tf_t^*\left(D^+\right)^2\right)^{-1}\end{pmatrix}.$$
As before we know by Lemma 3.2.8 that
$$\|A\|_{0,2} \leq C\left(1+\frac{|z|}{t}\right)$$
and by Lemma 3.2.13
$$\left\|\left(z-tf_t^*\left(D^+\right)^2\right)^{-1}\right\|_{0,2} \leq Ct^{-1}\left(1+|z|\right).$$
In general $\|B\|_{2,0} = \|f_t^*E_t\|_{2,0} \leq Ct^{1/2}$ but for PBP we even get
$$\|PBP\|_{2,0} \leq C,$$
since the only summand involving t with a positive exponent is
$$\sqrt{t}f_t^*P\nabla^{\pi_*V}(D)P = \sqrt{t}f_t^*dy = dy.$$
Now one can easily check inductively that
$$\begin{aligned}
\|PA(BA)^nQ\|_{0,0} &\leq Ct^{-1/2}\left(1+p\left(|z|\right)\right)\\
\|QA(BA)^nP\|_{0,0} &\leq Ct^{-1/2}\left(1+p\left(|z|\right)\right)\\
\|QA(BA)^nQ\|_{0,0} &\leq Ct^{-1}\left(1+p\left(|z|\right)\right)
\end{aligned}$$
which proves the other three estimates in the statement. □

3.2.17 Theorem. *There exist constants $C, c > 0$ depending on ℓ, such that for t big enough we get the following estimates. On $B \backslash N_{a/2}$*

$$\left\| \operatorname{tr}\left(\exp\left(-\mathbb{A}_t^2\right)\right)\big|_{B\backslash N_{a/2}} \right\|_{\mathcal{C}^\ell\left(B\backslash N_{a/2}\right)} \leq Ce^{-ct}.$$

for all $\mathcal{C}^\ell$-norms on $\Omega^\bullet(B\backslash N_{a/2})$. Furthermore for all $\omega \in \Omega^\bullet(B)$ with $\operatorname{supp}\omega \in N_a$

$$\left\| \left(\int_{-a}^{a} \operatorname{tr}\left(\exp\left(-\mathbb{A}_t^2\right)\right) \right) \omega + \sqrt{\pi} \operatorname{tr}\left(\exp\left(-\left(\nabla^{\ker}\right)^2\right)\right) i^*\omega \right\|_{\mathcal{C}^0(B_0)} \leq Ct^{-1/2} \|\omega\|_{\mathcal{C}^1(B)}.$$

If we combine the estimates we have

$$\left| \int_B \operatorname{tr}^{\mathrm{odd}}\left(\exp\left(-\mathbb{A}_t^2\right)\right)\omega + \sqrt{\pi} \int_{B_0} \operatorname{tr}\left(\exp\left(-\left(\nabla^{\ker}\right)^2\right)\right) i^*\omega \right| \leq \frac{C}{\sqrt{t}} \|\omega\|_{\mathcal{C}^1(B)}.$$

Proof. In the following we have constants $C > 0$ which may vary from line to line and depend on ℓ but not on t, y, z and x.
Since D_b is invertible for all $b \in B\backslash N_{a/2}$, we know that

$$\left\| \operatorname{tr}\left(\exp\left(-\mathbb{A}_t^2\right)\right)\big|_{B\backslash N_{a/2}} \right\|_{\mathcal{C}^\ell\left(B\backslash N_{a/2}\right)} \leq Ce^{-ct}$$

on $B\backslash N_{a/2}$ for all $\mathcal{C}^\ell$-norms.
On N_a we know by Proposition 3.2.9 that

$$\left\| \operatorname{tr}\left((1-\mathbb{P}_t)\left(\exp\left(-\mathbb{A}_t^2\right)\right)\right)\right\|_{\mathcal{C}^0(N)} \leq Cf(t)\exp\left(-Kt\right)$$

where $f(t) \in \mathbb{R}[t, t^{-1}]$ is a polynomial in t and t^{-1}. It remains to prove that

$$\left(\int_{-a}^{a} \operatorname{tr}\left(\mathbb{P}_t\left(\exp\left(-\mathbb{A}_t^2\right)\right)\right) \right) \omega + \sqrt{\pi} \operatorname{tr}\left(\exp\left(-\left(\nabla^{\ker}\right)^2\right)\right) i^*\omega \in \Omega^\bullet(B_0)$$

is of $O\left(t^{-1/2}\right)$ for the $\mathcal{C}^0$-norm on $\Omega^\bullet(B_0)$.

$$
\begin{aligned}
&\left\| \left(\int_{-a}^{a} \operatorname{tr}\left(\mathbb{P}_t\left(\exp\left(-\mathbb{A}_t^2\right)\right)\right)\omega \right) + \sqrt{\pi}\operatorname{tr}\left(\exp\left(-\left(\nabla^{\ker}\right)^2\right)\right) i^*\omega \right\|_{\mathcal{C}^0(B_0)} \\
&\leq \left\| \int_{-a\sqrt{t}}^{a\sqrt{t}} \operatorname{tr}\left(\mathbb{P}_t\left(\exp\left(-f_t^*\mathbb{A}_t^2\right)\right) f_t^*\omega - \operatorname{tr}\left(\exp\left(-\left(y+\nabla^{\ker}\right)^2\right)\right)\right) g^* i^*\omega \right\|_{\mathcal{C}^0(B_0)} \\
&\quad + Ct^{-1/2}e^{-ct} \\
&\leq \int_{-a\sqrt{t}}^{a\sqrt{t}} \Big(\left\|\operatorname{tr}\left(\mathbb{P}_t\left(\exp\left(-f_t^*\mathbb{A}_t^2\right)\right)\right)\right\|_{\mathcal{C}^0(B_0\times\{y\})} \left\| f_t^*\omega - g^*i^*\omega\right\|_{\mathcal{C}^0(B_0\times\{y\})} \\
&\quad + \left\|\operatorname{tr}\left(\mathbb{P}_t\left(\exp\left(-f_t^*\mathbb{A}_t^2\right)\right) - \exp\left(-\left(y+\nabla^{\ker}\right)^2\right)\right)\right\|_{\mathcal{C}^0(B_0\times\{y\})} \left\|g^*i^*\omega\right\|_{\mathcal{C}^0(B_0\times\{y\})} \Big)\, dy \\
&\quad + Ct^{-1/2}e^{-ct}.
\end{aligned}
$$

We write the projection $\mathbb{P}_t$ via holomorphic functional calculus. We use the contour Ω_t for $|y| \leq 1$ and the contour Θ_y for $1 \leq |y| \leq a\sqrt{t}$. Since $\mathbb{P}_t$ projects our operators onto a one-dimensional subspace we make our estimates in the operator instead of the $\|.\|_1$-norm.
First case: $|y| \leq 1$.

$$
\begin{aligned}
&\left\|\operatorname{tr}\left(\mathbb{P}_t\left(\exp\left(-f_t^*\mathbb{A}_t^2\right)\right) - \exp\left(-\left(y+\nabla^{\ker}\right)^2\right)\right)\right\|_{\mathcal{C}^0(B_0\times\{y\})} \\
&\leq C\left\| \frac{1}{2\pi i}\int_{\Omega_t} e^{-z}\left(\left(z - f_t^*\mathbb{A}_t^2\right)^{-1} - \left(z - \left(y+\nabla^{\ker}\right)\right)^{-1}\right) dz \right\|_{0,0} \\
&\leq \frac{C}{2\pi}\int_{\Omega_t} \left|e^{-z}\right| \left\| \left(z - f_t^*\mathbb{A}_t^2\right)^{-1} - \left(z - \left(y+\nabla^{\ker}\right)^2\right)^{-1} \right\|_{0,0} dz \\
&\leq \frac{C}{2\pi}\int_{\Omega_t} e^{-\operatorname{Re} z} C t^{-1/2}\left(1 + p(|\operatorname{Re} z| + 1)\right) dz
\end{aligned}
$$

here we used Proposition 3.2.16, $|y| \leq 1$ and $|\operatorname{Im} z| \leq 1$. Calculating the integral leads to

$$
\left\|\operatorname{tr}\left(\mathbb{P}_t\left(\exp\left(-f_t^*\mathbb{A}_t^2\right)\right) - \exp\left(-\left(y+\nabla^{\ker}\right)^2\right)\right)\right\|_{\mathcal{C}^0(B_0\times\{y\})} \leq Ct^{-1/2}. \tag{3.2.5}
$$

Second case: $1 \leq |y| \leq a\sqrt{t}$.

$$\begin{aligned}
&\left\| \operatorname{tr}\left(\mathbb{P}_t \left(\exp\left(-f_t^* \mathbb{A}_t^2\right)\right) - \exp\left(- \left(y + \nabla^{\ker}\right)^2 \right)\right) \right\|_{\mathcal{C}^0(B_0 \times \{y\})} \\
&\leq \left\| \frac{C}{2\pi i} \int\limits_{\Theta_y} e^{-z} \left(\left(z - f_t^* \mathbb{A}_t^2 \right)^{-1} - \left(z - \left(y + \nabla^{\ker} \right)^2 \right)^{-1} \right) dz \right\|_{0,0} \\
&\leq \frac{C}{2\pi} \int\limits_{\Theta_y} e^{-\operatorname{Re} z} C t^{-1/2} \left(1 + p_1(|y|) + p_2(|\operatorname{Re} z| + 1)\right) dz \\
&\leq C t^{-1/2} e^{-y^2/2} \left(1 + p(|y|)\right).
\end{aligned}$$

If we now split the integral over $\left(-a\sqrt{t}, a\sqrt{t}\right)$ into an integral over $|y| \leq 1$ and an integral over $1 \leq |y| \leq a\sqrt{t}$ and insert the estimates respectively we obtain

$$\left\| \int\limits_{-a}^{a} \operatorname{tr}\left(\mathbb{P}_t \left(\exp\left(-\mathbb{A}_t^2\right)\right)\right) \omega + \sqrt{\pi} \operatorname{tr}\left(\exp\left(- \left(\nabla^{\ker} \right)^2 \right)\right) i^* \omega \right\|_{\mathcal{C}^0(B_0)} \leq C t^{-1/2} \|\omega\|_{\mathcal{C}^1}$$

where we used Lemma 3.2.12 which states in particular that

$$\left| \left(f_t^* \omega - g^* i^* \omega \right)_{(x,y)} \right| \leq C t^{-1/2} \|\omega\|_{\mathcal{C}^1} \left(1 + |y|\right).$$

□

3.2.18 Remark. D. Cibotaru calculated explicitly $\lim_{t\to\infty} \operatorname{ch}(A_t)$ for superconnections $A_t = \nabla + tA$ on finite rank vector bundles $E \to B$, see [Cib14, Theorem 9.4, 9.5]. Theorem 3.2.17 can be seen as a generalization to infinite dimensions. In exchange we restrict ourselves to a vector bundle of rank one $\ker D \to B_0$. In any case the currents we obtain are not surprising considering what we know from finite dimensions.
The top cohomology class of our representative $-\delta_{B_0} \operatorname{ch}\left(\ker D \to B_0, \nabla^{\ker}\right)$ of the analytical index also agrees with the formula given in [Cib11, Proposition 1.1] for $\dim B = 3$.

3.2.19 Proposition.

$$\beta := \operatorname{tr}^{\mathrm{ev}} \left(\frac{d\mathbb{A}_t}{dt} \exp\left(-\mathbb{A}_t^2\right)\right) dt \in \Omega^\bullet \left(B \times (0,\infty), \mathbb{C}\right)$$

is an integrable differential form.

Proof. We know from [BGS88, Theorem 2.11] that $\|\beta\|_{\mathcal{C}^\ell(B)} \leq C$ for small t and therefore $\operatorname{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt}\exp\left(-\mathbb{A}_t^2\right)\right) dt$ is integrable as $t \to 0$.
Since D_b is invertible for all $b \in B \setminus N_a$ we know that β is integrable on $B \setminus N_a \times (0,\infty)$ [BC89, p. 57]. So let us now consider β on $N_a \cong B_0 \times (-a,a)$ as $t \to \infty$. Set $S = (1-\delta, 1+\delta)$ and consider the fibre bundle $\widetilde{M} = M|_{N_a} \times S \to \widetilde{N_a} = N_a \times S$ as

in the proof of [BGV04, Theorem 10.32]. We denote the extra coordinate in S by s and define the vertical metric by $g_{\widetilde{M}/\widetilde{B}} = s^{-1}g_{M/B}$. The vertical Dirac bundle will be $\widetilde{V} = V \times S \to \widetilde{M}$, where we take the natural extensions of the given connections. We will write $\sim$ over all induced objects on this family. So let $\widetilde{\mathbb{A}}$ be the Bismut superconnection in this situation which we scale again by the parameter $t \in (0,\infty)$ as follows

$$\widetilde{\mathbb{A}}_t = \sqrt{t}\widetilde{D} + \widetilde{\nabla^{\pi_* V}} - \frac{1}{4\sqrt{t}}\widetilde{c(T)}.$$

We made assumption 3.2.1 for the Dirac operators D, but

$$\widetilde{D}_{(b,s)} = \sqrt{s}D_b$$

implies that it also holds for $\widetilde{D}$. The assumption on the small eigenvalue ($\tilde{\lambda}_0\left(\exp_{b,s}(y)\right) = y$) is fullfilled for $\tilde{D}$ if we choose the metric

$$\frac{1}{s}g_B \oplus 1$$

on $N_a \times S$, where g_B is the metric on B such that the small eigenvalue of D is given by $\lambda_0\left(\exp_b(y)\right) = y$.
We have a bundle $\ker \widetilde{D} \to \widetilde{B}_0 = B_0 \times S$ which is just the pullback of $\ker D \to B_0$. The submanifold $B_0 \times S$ is of course not compact, but if we allow δ to become smaller, we get the same uniform estimates as in Theorem 3.2.17. By combining the estimates (3.2.5) and the following in the proof of Theorem 3.2.17 we see that for t big enough

$$\left\| \operatorname{tr}\left(\widetilde{\mathbb{P}}_t\left(\exp\left(-f_t^*\widetilde{\mathbb{A}}_t^2\right)\right) - \exp\left(-\left(y + \widetilde{\nabla}^{\ker}\right)^2\right)\right)\right\|_{\mathcal{C}^0(B_0)} \leq \frac{C}{\sqrt{t}}e^{-y^2/2}. \tag{3.2.6}$$

Now we know by [BGV04, Lemma 10.31] or by a straight forward calculation that

$$\operatorname{tr}^{\mathrm{odd}}\left(\exp\left(-\widetilde{\mathbb{A}}_t^2\right)\right)\Big|_{s=1} = \operatorname{tr}^{\mathrm{odd}}\left(\exp\left(-\mathbb{A}_t^2\right)\right) - t\operatorname{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt}\exp\left(-\mathbb{A}_t^2\right)\right)ds$$

and $\widetilde{\nabla}^{\ker}$ is just a pullback from B_0 and therefore its curvature $\left(\widetilde{\nabla}^{\ker}\right)^2$ does not involve ds. So equation (3.2.6) tells us

$$\left\| f_t^* \operatorname{tr}^{\mathrm{ev}}\left(\mathbb{P}_t\left(\frac{d\mathbb{A}_t}{dt}\exp\left(-\mathbb{A}_t^2\right)\right)\right)\right\|_{\mathcal{C}^0(B_0)} \leq \frac{C}{t^{3/2}}e^{-y^2/2}.$$

Using the estimate of Proposition 3.2.9 for the projection $1 - \mathbb{P}_t$ we see that

$$\left\| f_t^* \operatorname{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt}\exp\left(-\mathbb{A}_t^2\right)\right)\right\|_{\mathcal{C}^0(B_0)} \leq \frac{C}{t^{3/2}}e^{-y^2/2}.$$

This proves that $f_t^* \operatorname{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt}\exp\left(-\mathbb{A}_t^2\right)\right)$ is integrable on $B_0 \times \left(-a\sqrt{t}, a\sqrt{t}\right) \times (0,\infty)$. By the transformation theorem $\operatorname{tr}^{\mathrm{ev}}\left(\frac{d\mathbb{A}_t}{dt}\exp\left(-\mathbb{A}_t^2\right)\right)$ is integrable on $B_0 \times (-a,a) \times (0,\infty)$ and therefore on all of $B \times (0,\infty)$.

$\square$

3.2.20 Definition. We define

$$\hat{\eta} := \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{tr}^{\mathrm{ev}} \left(\frac{d\mathbb{A}_t}{dt} \exp\left(-\mathbb{A}_t^2\right) \right) dt \in L^1\left(B, \Lambda^{\mathrm{ev}} T^* B \otimes \mathbb{C}\right),$$

which is a well-defined differential form on B with coefficients in $L^1(B)$ by Proposition 3.2.19 and the Fubini theorem. We define $\tilde{\eta}$ by

$$\tilde{\eta} = \sum_k (2\pi i)^{-k}\, \hat{\eta}_{[2k]} \in L^1\left(B, \Lambda^{\mathrm{ev}} T^* B\right).$$

We can see $\tilde{\eta}$ as a current

$$\begin{aligned} \tilde{\eta} \colon \Omega^\bullet(B) &\to \mathbb{R}, \\ \omega &\mapsto \int_B \tilde{\eta} \wedge \omega \end{aligned}$$

and define its differential as a current

$$d\tilde{\eta}(\omega) = -\tilde{\eta}(d\omega).$$

3.2.21 Remark. We know even more about the coefficients of $\tilde{\eta}$ than just being integrable. Since we can prove that $\tilde{\eta}$ is smooth outside the tubular neighbourhood N_a of B_0 for all $a > 0$, it is smooth if restricted to $B \backslash B_0$. But since our estimates where in the $\mathcal{C}^\ell$-norm on B_0, we also know that $i^*\tilde{\eta} \in \Omega^\bullet(B_0)$ is smooth (dominated convergence theorem). Therefore the only singularity occurs if one crosses B_0.

3.2.22 Theorem. *We assume that TX admits a spin structure and denote by ΣX the corresponding spinor bundle. If the Dirac bundle V is of the form $\Sigma X \otimes L$ then*

$$d\tilde{\eta} = \int_{M/B} \hat{A}\left(TX, \nabla^X\right) \operatorname{ch}\left(L, \nabla^L\right) + \delta_{B_0} \operatorname{ch}\left(\ker D \to B_0, \nabla^{\ker}\right), \tag{3.2.7}$$

where δ_{B_0} is the current of integration over the hypersurface B_0.

Proof. Equation (3.2.7) follows from the transgression formula (2.1.2)

$$d \int_s^T \operatorname{tr}^{\mathrm{ev}} \left(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2} \right) = \operatorname{tr}^{\mathrm{odd}}\left(e^{-A_s^2}\right) - \operatorname{tr}^{\mathrm{odd}}\left(e^{-\mathbb{A}_T^2}\right)$$

since we know by [BF86, Theorem 2.10] that for $n = \dim M_b$

$$\begin{aligned} &\lim_{s \to 0} \frac{1}{\sqrt{\pi}} \operatorname{tr}^{\mathrm{odd}}\left(e^{-\mathbb{A}_s^2}\right) \\ &= (2\pi i)^{-(n+1)/2} \int_{M/B} \det\left(\frac{R^X/2}{\sinh\left(R^X/2\right)} \right)^{1/2} \operatorname{tr}\left(\exp\left(-\left(\nabla^L\right)^2\right)\right) \end{aligned}$$

and by Theorem 3.2.17 that

$$\lim_{T\to\infty} \frac{1}{\sqrt{\pi}} \operatorname{tr}^{\mathrm{odd}}\left(e^{-\mathbb{A}_T^2}\right) = -\delta_{B_0} \operatorname{tr}\left(\exp\left(-\left(\nabla^{\ker}\right)^2\right)\right).$$

If we define the $2\pi i$-scaling as above the resulting formula is

$$\begin{aligned} d\widetilde{\eta} = & \int\limits_{M/B} \det\left(\frac{R^X/4\pi i}{\sinh\left(R^X/4\pi i\right)}\right)^{1/2} \operatorname{tr}\left(\exp\left(-\left(\nabla^L\right)^2/2\pi i\right)\right) \\ & + \delta_{B_0} \operatorname{tr}\left(\exp\left(-\left(\nabla^{\ker}\right)^2/2\pi i\right)\right) \\ = & \int\limits_{M/B} \hat{A}\left(TX, \nabla^X\right) \operatorname{ch}\left(L, \nabla^L\right) + \delta_{B_0} \operatorname{ch}\left(\ker D \to B_0, \nabla^{\ker}\right). \end{aligned}$$

□

4 Large time contribution for $\eta\left(D_{M,\varepsilon}\right)$

In Section 4.1 we will define and study the projections q_ε onto the subspace of $L^2\left(M, V\otimes\pi^*\Sigma B\right)$ which will model eigensections of $D_{M,\varepsilon}$ corresponding to eigenvalues that decay at least as ε as $\varepsilon\to 0$. We will write $D_{M,\varepsilon}$ with respect to the decomposition coming from q_ε and $1-q_\varepsilon$ as

$$\begin{pmatrix} D_{M,\varepsilon,1} & D_{M,\varepsilon,2} \\ D_{M,\varepsilon,3} & D_{M,\varepsilon,4} \end{pmatrix}$$

and give operator estimates on the matrix entries in Section 4.2. We will see that $\varepsilon^{-1}D_{M,\varepsilon,1}$ converges to the twisted Dirac operator D_{B_0} on $\ker D_X\otimes\Sigma B_0\to B_0$. Then we will prove that the off-diagonals $\varepsilon^{-1}D_{M,\varepsilon,2}$ and $\varepsilon^{-1}D_{M,\varepsilon,3}$ are small enough and the operator $\varepsilon^{-1}D_{M,\varepsilon,4}$ is bounded below.
In the last Section 4.3 of this chapter we will first use the approximations of Section 4.2 to get estimates of the resolvent of $\varepsilon^{-1}D_{M,\varepsilon}$ using the Schur complement method. Finally we argue with holomorphic functional calculus that there exists an $0<\alpha<1$ such that

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^{\alpha-2}}^{\infty}\operatorname{tr}\left(D_{M,\varepsilon}e^{-tD^2_{M,\varepsilon}}\right)\frac{dt}{\sqrt{t}} = -\eta\left(D_{B_0}\right)+\sum_{\nu=1}^{\dim\ker D_{B_0}}\operatorname{sign}\left(\lambda_\nu(\varepsilon)\right),$$

where $\lambda_\nu(\varepsilon)$ are the finitely many eigenvalues of $D_{M,\varepsilon}$ that decay faster than ε.
The main ideas for the projections and estimates in this chapter are inspired by [BL91, Section VIII, IX]. We replace the twisting bundle ξ of [BL91] by π_*V so we have to be very careful by which norms we estimate. We will point out where we follow and where we deviate from [BL91]. If a statement is very similar but not the same as one in [BL91] we will cite it as „cf. [BL91, Lemma XY]“.

Let us recall the assumptions we made in the preliminaries.

4.0.23 Assumption. We assume that we can find a covering $\{U_i\}_{1\leq i\leq k}$ for B such that on each U_i either $\left(D_X\right)_b$ is invertible or we have a smooth function $f_i: U_i\to(-K,K)$ which has 0 as a regular value, such that $\operatorname{spec}\left(D_X\right)_b\cap(-K-\delta,K+\delta)=\{f_i(b)\}$ and $f_i(b)$ is of multiplicity 1.
Furthermore we assume that the metric g_B on B is such that in a tubular neighbourhood $\exp\colon B_0\times(-a,a)\xrightarrow{\sim}N_a$ the small eigenvalue is given by $f_i(x,y)=y$.

4.1 Modelling the eigensections for small eigenvalues of $D_{M,\varepsilon}$

We know by [BC89] and also by [Dai91] that eigenvalues $\lambda(\varepsilon)$ of $D_{M,\varepsilon}$ which are bounded below for small ε are negligible for large times. As $\varepsilon \to 0$ just eigenvalues that decay at least as ε can be seen in the integral

$$\int_{\varepsilon^{\alpha-2}}^{\infty} \operatorname{tr}\left(D_{M,\varepsilon} e^{-tD^2_{M,\varepsilon}}\right) \frac{dt}{\sqrt{t}}.$$

We first want to motivate and try to explain heuristically what happens in the limit $\varepsilon \to 0$ to $\varepsilon^{-1} D_{M,\varepsilon}$ and which parts of our operators give rise to small eigenvalues.
Since the vertical Dirac operators D_X are bounded below outside a tubular neighbourhood of B_0, eigensections corresponding to small eigenvalues are approximately supported in the neighbourhood $N_a \cong B_0 \times (-a,a)$ of B_0, where D_X has a kernel. Locally the bundle looks like

$$\pi_* V \otimes \Sigma B|_{B_0 \times (-a,a)} \cong \left((\ker D_X \oplus \operatorname{im} D_X) \otimes (\Sigma B_0 \oplus \Sigma B_0)\right) \times (-a,a).$$

The part in $\operatorname{im} D_X$ should be 0 if we want the eigenvalue to be small. So we are considering sections of $((\ker D_X \otimes \Sigma B_0) \oplus (\ker D_X \otimes \Sigma B_0)) \times (-a,a) \to B_0 \times (-a,a)$. If we write the operator $D_{M,\varepsilon}$ restricted to that space we get as an approximation (up to an error of $O(\varepsilon^2)$) the operator

$$D_{M,\varepsilon} \sim A_\varepsilon = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} D_{B_0} & 0 \\ 0 & -D_{B_0} \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & \partial_y \\ -\partial_y & 0 \end{pmatrix}$$

where D_{B_0} denotes the twisted Dirac operator on $\ker D_X \otimes \Sigma B_0 \to B_0$.
We see that if $\sigma_\lambda \in \Gamma\left(B_0, \ker D_X \otimes \Sigma B_0\right)$ is an eigensection of D_{B_0} corresponding to an eigenvalue λ, the section

$$\begin{pmatrix} 0 \\ \exp\left(-\frac{y^2}{2\varepsilon}\right) \sigma_\lambda \end{pmatrix}$$

is an eigensection of A_ε with eigenvalue $-\varepsilon\lambda$.
We also observe that in contrast to [Dai91, Theorem 1.5] we might have eigenvalues that decay as $\sqrt{\varepsilon}$. If D_{B_0} has a kernel, A_ε has eigenvalues $\pm\sqrt{\varepsilon 2n}$ for all $n \in \mathbb{N}$, since these are the eigenvalues of

$$\begin{pmatrix} 0 & y + \varepsilon\partial_y \\ y - \varepsilon\partial_y & 0 \end{pmatrix}.$$

We will not be concerned with these $\sqrt{\varepsilon}$-eigenvalues in this chapter though, since we are just interested in large times $t \geq \varepsilon^{\alpha-2}$. However if one wants to calculate the value of the whole integral for the adiabatic limit of $\eta(D_{M,\varepsilon})$, these eigenvalues could cause another summand for $\varepsilon^{\beta-1} \leq t \leq \varepsilon^{\alpha-2}$ in the splitting

$$\begin{aligned}\eta\left(D_{M,\varepsilon}\right) =& \frac{1}{\sqrt{\pi}} \int_0^{\varepsilon^{\beta-1}} \operatorname{tr}\left(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2}\right) \frac{dt}{\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_{\varepsilon^{\beta-1}}^{\varepsilon^{\alpha-2}} \operatorname{tr}\left(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2}\right) \frac{dt}{\sqrt{t}} \\ &+ \frac{1}{\sqrt{\pi}} \int_{\varepsilon^{\alpha-2}}^{\infty} \operatorname{tr}\left(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2}\right) \frac{dt}{\sqrt{t}},\end{aligned}$$

for $0 < \alpha, \beta < 1$.

4.1.1 Remark. Recall Remark 2.3.2 of the preliminaries. The normal bundle $\nu B_0 \to B_0$ is trivial, $\nu B_0 \cong B_0 \times \mathbb{R}$, since the small eigenvalue of the vertical Dirac operators comes with a distinguished sign. We choose $a > 0$ small enough such that

$$\begin{aligned}\exp\colon B_0 \times (-a, a) &\to N_a \\ (b, y) &\mapsto \exp_b(y)\end{aligned}$$

is a diffeomorphism onto a tubular neighbourhood N_a of B_0 in B. On $B_0 \times (-a, a)$ we consider the metric $\exp^* g_B$, such that exp becomes an isometry. The Gauß lemma, see for example [Kli82, Lemma 1.9.1], shows that with respect to $T(B_0 \times (-a, a)) = TB_0 \oplus \mathbb{R}$

$$\exp^* g_B = \begin{pmatrix} \tilde{g}_{B_0} & 0 \\ 0 & 1 \end{pmatrix}$$

where $\tilde{g}_{B_0}|_{(b,0)} = g_{B_0}|_b$, but in general $\tilde{g}_{B_0}$ depends on $y \in (-a, a)$.
We trivialize the vector bundle $\pi_* V \otimes \Sigma B$ as in Lemma 3.2.4 of Section 3.2 by parallel transport along normal geodesics with respect to the connection

$$\nabla^{\pi_* V \otimes \Sigma B, 0} = \nabla^{\pi_* V} \otimes 1 + 1 \otimes \nabla^{\Sigma B} = \left(\nabla^V - \frac{1}{2} k\right) \otimes 1 + 1 \otimes \nabla^{\Sigma B}.$$

So if we denote the projection onto the first component by $g\colon B_0 \times (-a, a) \to B_0$ the bundles $\exp^*\left(\pi_* V \otimes \Sigma B|_{N_a}\right) \cong g^*\left(\pi_* V \otimes \Sigma B|_{B_0}\right)$ are isomorphic. We will switch between the two pictures without actually mentioning it.

4.1.2 Definition ([BL91, Equation (8.21)])**.** The two differential forms $d\operatorname{vol}_{B_0} \wedge dy$ and $d\operatorname{vol}_{\exp^* g_B}$ both define volume forms on $B_0 \times (-a, a)$ so at each point they agree up to a positive constant. We define $h\colon B_0 \times (-a, a) \to \mathbb{R}$ by

$$d\operatorname{vol}_{\exp^* g_B}|_{(x,y)} = h(x, y)\, d\operatorname{vol}_{B_0}|_x\, dy.$$

By the discussion of Remark 4.1.1 it is obvious that $h(x, 0) = 1$ for all $x \in B_0$ and we choose a small enough such that there exist $C, c > 0$ such that $c \leq h(x, y) \leq C$ for all $|y| \leq a$ and $x \in B_0$.

Since $B_0 \subset B$ is oriented and of codimension one we have the *scalar second fundamental form* $A\colon \Gamma\left(B_0, TB_0\right) \times \Gamma\left(B_0, TB_0\right) \to \mathcal{C}^\infty(B_0)$ which is defined by

$$A(X,Y) = g_B\left(\nabla^B_X Y, \frac{\partial}{\partial y}\right),$$

for $X, Y \in \Gamma\left(B_0, TB_0\right)$ and $\frac{\partial}{\partial y} \in \Gamma\left(B_0, \nu B_0\right)$ the unique positive oriented, orthonormal section of the normal bundle.

4.1.3 Lemma. *For a local orthonormal frame $f_1, ..., f_{m-1}$ of TB_0 the following equation holds*

$$\frac{\partial h}{\partial y}(x,0) = -\sum_{\alpha=1}^{m-1} A(f_\alpha, f_\alpha)_x = -\operatorname{tr} A_x.$$

Proof. The statement follows by a proposition of Gauß, see [MS86, Appendix I, Section 1.4]. □

4.1.4 Definition. We define L^2-products $\langle\langle\cdot,\cdot\rangle\rangle_{L^2,L^2}$ and $\langle\langle\cdot,\cdot\rangle\rangle_{L^2,W^1}$ on $\Gamma\left(B, \pi_* V \otimes \Sigma B\right)$ by using the fibrewise L^2-metric $g_{\pi_* V}$ or Sobolev-1-product $g_{\pi_* V,1}$ respectively. We define a Sobolev-1-product $\langle\langle\cdot,\cdot\rangle\rangle_{W^1,L^2}$ on $\Gamma\left(B, \pi_* V \otimes \Sigma B\right)$ the same way by using the fibrewise L^2-metric and the connection $\nabla^{\pi_* V\otimes\Sigma B,0}$ to differentiate into the base directions. We denote the corresponding norms by $\|\cdot\|_{L^2,L^2}$, $\|\cdot\|_{L^2,W^1}$ and $\|\cdot\|_{W^1,L^2}$.
If we have a section of $\pi_* V \otimes \Sigma B$ whose support is contained in N_a we can integrate with respect to the volume form $d\operatorname{vol}_{B_0} \wedge dy$. We will denote the corresponding products by $\langle\cdot,\cdot\rangle_{L^2,L^2}$, $\langle\cdot,\cdot\rangle_{L^2,W^1}$ and $\langle\cdot,\cdot\rangle_{W^1,L^2}$ and the corresponding norms by $|\cdot|_{L^2,L^2}$, $|\cdot|_{L^2,W^1}$ and $|\cdot|_{W^1,L^2}$. We again use the connection $\nabla^{\pi_* V\otimes\Sigma B,0}$ for $|\cdot|_{W^1,L^2}$.

4.1.5 Remark. Note that for the norms defined above changing the connection on ΣB gives an equivalent norm, but changing the connection on $\pi_* V$ does in general not.
We also check that under the identification $\Gamma\left(B, \pi_* V \otimes \Sigma B\right) \cong \Gamma\left(M, V \otimes \pi^* \Sigma B\right)$ the norm $\|\cdot\|_{L^2,L^2}$ equals $\|\cdot\|_{L^2(M,V\otimes\pi^*\Sigma B)}$. For the Sobolev-1-norms we see that there exist constants $c, C > 0$ such that

$$c\,\|\cdot\|_{W^1(M,V\otimes\pi^*\Sigma V)} \le \|\cdot\|_{W^1,L^2} + \|\cdot\|_{L^2,W^1} \le C\,\|\cdot\|_{W^1(M,V\otimes\pi^*\Sigma B)}.$$

4.1.6 Lemma. *For a section $\tau \in \Gamma\left(B, \pi_* V \otimes \Sigma B\right)$ whose support is included in N_a the following equation holds true*

$$\left|h^{1/2}\tau\right|_{L^2,L^2} = \|\tau\|_{L^2,L^2}.$$

If we denote the connection $g^*\nabla^{(\ker\oplus\operatorname{im})\otimes(\Sigma B_0\oplus\Sigma B_0)}$ by ∇, where $g\colon B_0 \times (-a,a) \to B_0$ is the projection onto the first component,

$$\begin{aligned}\left|\nabla^{\pi_* V\otimes\Sigma B,0}\big|_{TB_0}\,\tau\right|&_{L^2(B_0\times\{y\},T^*B_0\otimes\pi_* V\otimes\Sigma B)}\\ &\le C\left(1+|y|\right)\left|\nabla|_{TB_0}\,\tau\right|_{L^2(B_0\times\{y\},T^*B_0\otimes\pi_* V\otimes\Sigma B)}\end{aligned}$$

Proof. The first statement follows by the Definition 4.1.2 of the function h.
Since $\Gamma(B, \pi_* V \otimes \Sigma B) \cong \Gamma(M, V \otimes \pi^* \Sigma B)$ we consider for the second statement the situation on $\pi^{-1}(N_a) \cong \pi^{-1}(B_0) \times (-a,a)$ as in Lemma 3.2.4 and denote the projection by $\tilde{g}\colon \pi^{-1}(B_0) \times (-a,a) \to \pi^{-1}(B_0)$. We replace the connection $\nabla^{\pi_* V \otimes \Sigma B, 0} = \nabla^{V \otimes \pi^* \Sigma B} - \frac{1}{2}k$ by $\tilde{g}^* \left(\nabla^{V \otimes \pi^* \Sigma B}\big|_{\pi^{-1}(B_0)} - \frac{1}{2}k \right)$. This difference is an element

$$\omega \in \Omega^1 \left(\pi^{-1}(B_0) \times (-a,a), \mathrm{End}\,(V \otimes \pi^* \Sigma B) \right)$$

which is zero restricted to $\pi^{-1}(B_0)$. Therefore, since the fibres are compact, it follows (by locally Taylor expanding and using that B_0 is compact) that

$$\|\omega\|_{\mathcal{C}^\ell(B_0 \times \{y\})} \leq C(\ell)\,|y|$$

and hence

$$\begin{aligned} &\left| \nabla^{\pi_* V \otimes \Sigma B, 0}\big|_{TB_0} \tau \right|_{L^2(B_0 \times \{y\}, T^* B_0 \otimes \pi_* V \otimes \Sigma B)} \\ &\qquad \leq C\,(1 + |y|) \left| g^* \nabla^{\pi_* V \otimes \Sigma B, 0}\big|_{B_0} \tau \right|_{L^2(B_0 \times \{y\}, T^* B_0 \otimes \pi_* V \otimes \Sigma B)} . \end{aligned}$$

Now we want to replace $g^* \nabla^{\pi_* V \otimes \Sigma B, 0}$ by $\nabla = g^* \nabla^{(\ker \oplus \mathrm{im}) \otimes (\Sigma B_0 \oplus \Sigma B_0)}$. Changing $\nabla^{\Sigma B}$ into $\nabla^{\Sigma B_0 \oplus \Sigma B_0}$ gives an equivalent norm. So let us consider the difference $\nabla^{\pi_* V}\big|_{B_0} - \nabla^{\ker \oplus \mathrm{im}}$. We know by Proposition 3.2.14 that for $Y \in TB_0$

$$\nabla_Y^{\pi_* V} - \nabla_Y^{\ker \oplus \mathrm{im}} = \begin{pmatrix} 0 & P_0 \nabla_Y^{\pi_* V}(D_X) Q_0 \left(D_X^+\right)^{-1} Q_0 \\ -Q_0 \left(D_X^+\right)^{-1} Q_0 \nabla_Y^{\pi_* V}(D_X) P_0 & 0 \end{pmatrix},$$

where P and Q denote the projections onto L and W respectively as in Proposition and Definition 3.2.3 of Chapter 3, so P_0 and Q_0 are the projections onto $\ker D_X$ and $\mathrm{im}\, D_X$ on B_0.
One can check by a straight-forward calculation that $\nabla_Y^{\pi_* V}(D_X)$ is the sum of a fibre-wise differential operator of order one and an endomorphism of V. Therefore $\nabla_Y^{\pi_* V} - \nabla_Y^{\ker \oplus \mathrm{im}} \in \mathrm{End}\, \pi_* V$ does not involve differentiation and the following estimate holds true

$$\left\| \left(\nabla_Y^{\pi_* V} - \nabla_Y^{\ker \oplus \mathrm{im}} \right) \tau \right\|_{g_{\pi_* V}} \leq C \, \|\tau\|_{g_{\pi_* V}} .$$

Therefore

$$\left| \nabla^{\pi_* V \otimes \Sigma B, 0}\big|_{B_0} \tau \right|_{L^2(B_0, T^* B_0 \otimes \pi_* V \otimes \Sigma B)} \leq C \left| \nabla|_{B_0} \tau \right|_{L^2(B_0, T^* B_0 \otimes \pi_* V \otimes \Sigma B)}$$

which proves the second part of the lemma.

$\square$

4.1.7 Lemma. *For all $X \in \Gamma(B_0 \times (-a,a), TB_0)$ there exists a constant $C > 0$ such that for all $s \in \Gamma(B_0, \ker D_X)$ the following estimate holds true*

$$\left| g^* \nabla_X^{\ker \oplus \mathrm{im}} \iota(s) - \iota\left(\nabla_X^{\ker} s \right) \right|_{L^2(B_0 \times \{y\}, \pi_* V)} \leq C\,|y|\,\|s\|_{L^2(B_0, \ker D_X)} ,$$

where $\iota\colon \ker D_X \times (-a,a) \xrightarrow{\sim} L$ *is given by parallel transport along normal geodesics with respect to* $\nabla^L = P\nabla^{\pi_* V}P$.

Proof. Let us define

$$\psi(y) = \left| g^*\nabla_X^{\ker\oplus\mathrm{im}}\iota(s) - \iota\left(\nabla_X^{\ker} s\right)\right|_{L^2(B_0\times\{y\},\pi_* V)}.$$

Then we see that $\psi(0) = 0$ and the statement follows by Taylor approximation.

□

4.1.8 Remark. We want to analyze the spin structure of ΣB when being restricted to B_0. Note that since B is even-dimensional and B_0 is a hypersurface, B_0 is odd-dimensional and $TB|_{B_0} = TB_0 \oplus \mathbb{R}$. Therefore $\Sigma B|_{B_0} = \Sigma B_0 \oplus \Sigma B_0$. For a local orthonormal frame $f_1, ..., f_{m-1}, \frac{\partial}{\partial y}$ of $TB_0 \oplus \mathbb{R} \to B_0$, Clifford multiplication and $\mathbb{Z}_2$-grading ω can be written as

$$\begin{aligned}
\omega &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
c_B(f_\alpha) &= \begin{pmatrix} c_{B_0}(f_\alpha) & 0 \\ 0 & -c_{B_0}(f_\alpha) \end{pmatrix} \\
c_B\left(\frac{\partial}{\partial y}\right) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}$$

Now if by abuse of notation we denote by $f_1, ..., f_{m-1}, \frac{\partial}{\partial y}$ also the parallel transport of this basis along normal geodesics we get a local orthonormal frame of $(TB_0 \oplus \mathbb{R}) \times (-a,a) \to B_0 \times (-a,a)$. We know that Clifford multiplication is compatible with the connection $\nabla^{\Sigma B}$, therefore

$$\nabla^{\Sigma B}_{\frac{\partial}{\partial y}}\left(c_B(f_\alpha)\right) = 0$$

as well as

$$\nabla^{\Sigma B}_{\frac{\partial}{\partial y}}\left(c_B\left(\frac{\partial}{\partial y}\right)\right) = 0.$$

But since we used the connection $\nabla^{\Sigma B}$ to trivialize ΣB along normal geodesics this implies that Clifford multiplication is constant in y in this trivialization.

4.1.9 Definition ([BL91, Section IX.a)]).

1. For a constant $a_1 \in (0, a/2]$, which will be specified in Proposition 4.2.9, let $\rho\colon \mathbb{R} \to [0,1]$ be a smooth cut-off function such that
$$\rho(y) = \begin{cases} 1, & \text{for all } |y| \le a_1/2, \\ 0, & \text{for all } |y| \ge a_1. \end{cases}$$

2. We define the constant α_ε by
$$C\sqrt{\varepsilon} \le \alpha_\varepsilon = \int_{\mathbb{R}} e^{-y^2/\varepsilon}\rho^2(y)dy \le \sqrt{\pi}\sqrt{\varepsilon}.$$

3. The isometry (with respect to the L^2-metric on the left and $\langle\cdot,\cdot\rangle_{L^2,L^2}$ on the right)

$$I_\varepsilon\colon L^2\left(B_0, \ker D_X \otimes \Sigma B_0\right) \to L^2\left(N_a, (\pi_* V \otimes \Sigma B)|_{N_a}\right)$$

is defined by

$$(I_\varepsilon s)(b,y) = \alpha_\varepsilon^{-1/2} \rho(y) e^{-y^2/2\varepsilon} \left(\iota_{(b,y)} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)(s(b))$$

where $\iota\colon (\ker D_X) \times (-a,a) \xrightarrow{\sim} L \subset \pi_* V$ is given by parallel transport along normal geodesics with respect to the connection $\nabla^L = P\nabla^{\pi_* V} P$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is with respect to the decomposition $\Sigma B|_{B_0 \times (-a,a)} = (\Sigma B_0 \oplus \Sigma B_0) \times (-a,a)$.

4. We define the isometry (with respect to the L^2-metric on the left and $\langle\langle\cdot,\cdot\rangle\rangle_{L^2,L^2}$ on the right)

$$J_\varepsilon\colon L^2\left(B_0, \ker D_X \otimes \Sigma B_0\right) \to L^2\left(B, \pi_* V \otimes \Sigma B\right)$$

by

$$J_\varepsilon(s) = h^{-1/2} I_\varepsilon(s).$$

5. Let

$$p_\varepsilon\colon L^2\left(N_a, \left(\pi_*^{L^2} V \otimes \Sigma B\right)\Big|_{N_a}\right) \to L^2\left(N_a, \left(\pi_*^{L^2} V \otimes \Sigma B\right)\Big|_{N_a}\right)$$

be the orthogonal projection (with respect to $|\cdot|_{L^2,L^2}$) onto $\operatorname{im} I_\varepsilon$, where $\pi_*^{L^2} V$ denotes the fibrewise L^2-sections, and

$$q_\varepsilon\colon L^2\left(B, \pi_*^{L^2} V \otimes \Sigma B\right) \cong L^2\left(M, V \otimes \pi^* \Sigma B\right) \to L^2\left(M, V \otimes \pi^* \Sigma B\right)$$

the orthogonal projection (with respect to $\|\cdot\|_{L^2,L^2}$) onto $\operatorname{im} J_\varepsilon$.

4.1.10 Remark. We will see in the next chapter that sections $\tau \in W^1\left(M, V \otimes \pi^* \Sigma B\right)$ which lie in the image of J_ε model the small eigenvalues of $D_{M,\varepsilon}$ which behave at least as ε as $\varepsilon \to 0$ and that the operator $\frac{1}{\varepsilon} J_\varepsilon^{-1} q_\varepsilon D_{M,\varepsilon} J_\varepsilon$ converges to $-D_{B_0}$, where D_{B_0} is the twisted Dirac operator on $\ker D_X \otimes \Sigma B_0 \to B_0$.

4.1.11 Lemma (cf. [BL91, Proposition 9.2, 9.5])**.** *Let*

$$\tau = \tau_1 \otimes \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix} \in L^2\left(N_a, \left(\pi_*^{L^2} V \otimes \Sigma B\right)\Big|_{N_a}\right)$$

with respect to the decomposition $\Sigma B|_{N_a} \cong (\Sigma B_0 \oplus \Sigma B_0) \times (-a,a)$ *and let* $\{U_i\}$ *a covering of* B_0 *with a subordinate partition of unity* ψ_i *such that for each* i *there is a local orthonormal frame* $\sigma_i\colon U_i \to \ker D_X$ *of* $\ker D_X$*. Then the projection* p_ε *is given by*

$$\begin{aligned} p_\varepsilon(\tau)(x,y) =& \alpha_\varepsilon^{-1} \rho(y) e^{-y^2/2\varepsilon} \sum_i \psi_i(x) \\ & \cdot \int_{\mathbb{R}} \rho(\tilde{y}) e^{-\tilde{y}^2/2\varepsilon} g_{\pi_* V}\left(\iota_{(x,\tilde{y})}(\sigma_i), \tau_1(x,\tilde{y})\right) \iota_{(x,y)}(\sigma_i) \otimes \begin{pmatrix} 0 \\ \tau_3(x,\tilde{y}) \end{pmatrix} d\tilde{y}. \end{aligned}$$

The projection q_ε is given by

$$q_\varepsilon = h^{-1/2} p_\varepsilon h^{1/2}.$$

Proof. One can check the formula by a straightforward calculation. See also the very similar formulas in [BL91, Proposition 9.2, 9.5]. It is also easy to see that the formula is independent of the choice of U_i, ψ_i and σ_i. □

4.1.12 Lemma ([BL91, Proposition 9.3])**.** *1. For any $\gamma \in \mathbb{N}_{>0}$ we find a constant $C = C(\gamma) > 0$ such that for all $\tau \in L^2\left(N_a, \left(\pi_* V \otimes \Sigma B_0\right)|_{N_a}\right)$*

$$|p_\varepsilon\left(y^\gamma \tau\right)|_{L^2,L^2} \leq C\varepsilon^{\gamma/2} \left|\tau\right|_{L^2,L^2}. \tag{4.1.1}$$

2. There exist constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all sections $\tau \in W^1\left(N_a, \left(\pi_* V \otimes \Sigma B\right)|_{N_a}\right)$

$$|p_\varepsilon \tau|_{W^1,L^2} \leq C\left(|\tau|_{W^1,L^2} + \frac{1}{\sqrt{\varepsilon}}\left|\tau\right|_{L^2,L^2}\right). \tag{4.1.2}$$

3. We find a constant $C > 0$ and an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all sections $s \in W^1\left(B_0, \ker D_X \otimes \Sigma B_0\right)$

$$|I_\varepsilon(s)|_{W^1,L^2} \leq C\left(\|s\|_{W^1(B_0,\ker D_X \otimes \Sigma B_0)} + \frac{1}{\sqrt{\varepsilon}}\|s\|_{L^2(B_0,\ker D_X\otimes\Sigma B_0)}\right). \tag{4.1.3}$$

Proof. For the first estimate (4.1.1) one checks that

$$\int\limits_{\mathbb{R}} e^{-y^2/2\varepsilon}\left|y\right| dy = C\varepsilon \tag{4.1.4}$$

and $\alpha_\varepsilon^{-1} \leq c\sqrt{\varepsilon}^{-1}$. Together with the previous Lemma 4.1.11 this proves that

$$|p_\varepsilon\left(y\sigma\right)|_{L^2,L^2} \leq C\sqrt{\varepsilon}\left|\sigma\right|_{L^2,L^2}$$

and by induction equation (4.1.1) follows.
For the second estimate (4.1.2) we assume that

$$\tau = \tau_1 \otimes \begin{pmatrix}\tau_2\\ \tau_3\end{pmatrix}$$

and write

$$
\begin{aligned}
&|p_\varepsilon(\tau)|_{W^1,L^2} \\
&\leq C\,|\tau|_{L^2,L^2} \\
&\quad + \Bigg|\alpha_\varepsilon^{-1}\left(\frac{\partial\rho(y)}{\partial y} - \rho(y)\frac{y}{\varepsilon}\right) e^{-y^2/2\varepsilon}\sum_i \psi_i(x) \\
&\quad\quad \cdot \int_{\mathbb{R}} \rho(\tilde{y})\, e^{-\tilde{y}^2/2\varepsilon} g_{\pi_*V}\left(\iota_{(x,\tilde{y})}(\sigma_i), \tau_1(x,\tilde{y})\right) \iota_{(x,y)}(\sigma_i) \otimes \begin{pmatrix} 0 \\ \tau_3(x,\tilde{y}) \end{pmatrix} d\tilde{y}\Bigg|_{L^2,L^2} \\
&\quad + \Bigg|\alpha_\varepsilon^{-1}\rho(y)e^{-y^2/2\varepsilon}\sum_i \psi_i(x)\int_{\mathbb{R}} \rho(\tilde{y})\, e^{-\tilde{y}^2/2\varepsilon} \\
&\quad\quad \cdot\ \nabla^{\pi_*V\otimes\Sigma B,0}\big|_{TB_0}\left(g_{\pi_*V}\left(\iota_{(x,\tilde{y})}(\sigma_i), \tau_1(x,\tilde{y})\right) \iota_{(x,y)}(\sigma_i) \otimes \begin{pmatrix} 0 \\ \tau_3(x,\tilde{y}) \end{pmatrix}\right) d\tilde{y}\Bigg|_{L^2,L^2}.
\end{aligned}
$$

A straightforward calculation shows that

$$
\begin{aligned}
&\Bigg|\alpha_\varepsilon^{-1}\left(\frac{\partial\rho(y)}{\partial y} - \rho(y)\frac{y}{\varepsilon}\right) e^{-y^2/2\varepsilon}\int_{\mathbb{R}} \rho(\tilde{y})\, e^{-\tilde{y}^2/2\varepsilon} \\
&\quad g_{\pi_*V}\left(\iota_{(x,\tilde{y})}(\sigma_i), \tau_1(x,\tilde{y})\right) \iota_{(x,y)}(\sigma_i) \otimes \begin{pmatrix} 0 \\ \tau_3(x,\tilde{y}) \end{pmatrix} d\tilde{y}\Bigg|_{L^2,L^2} \leq \left(C + \frac{C'}{\sqrt{\varepsilon}}\right)|\tau|_{L^2,L^2}.
\end{aligned}
$$

Using this equation and Lemma 4.1.6, Lemma 4.1.7 and equation (4.1.4) we see that

$$
|p_\varepsilon(\tau)|_{W^1,L^2} \leq C\,|\tau|_{L^2,L^2} + \frac{C}{\sqrt{\varepsilon}}\,|\tau|_{L^2,L^2} + C\,|\tau|_{W^1,L^2}.
$$

For the last estimate (4.1.3) we see that

$$
\begin{aligned}
|I_\varepsilon(s)|_{W^1,L^2} = |I_\varepsilon(s)|_{L^2,L^2} &\\
+ &\left|\alpha_\varepsilon^{-1/2}\left(\frac{\partial\rho(y)}{\partial y} - \rho(y)\frac{y}{\varepsilon}\right) e^{-y^2/2\varepsilon}\left(\iota\otimes\begin{pmatrix}0\\1\end{pmatrix}\right)(s)\right|_{L^2,L^2} \\
+ &\left|\alpha_\varepsilon^{-1/2}\rho(y)e^{-y^2/2\varepsilon}\nabla^{\pi_*V\otimes\Sigma B,0}\left(\iota\otimes\begin{pmatrix}0\\1\end{pmatrix}\right)(s)\right|_{L^2,L^2}.
\end{aligned}
$$

Since I_ε is an isometry with respect to $|.|_{L^2,L^2}$ we know that

$$
|I_\varepsilon(s)|_{L^2,L^2} = \|s\|_{L^2(B_0,\ker D_X\otimes\Sigma B_0)}.
$$

As above one checks that

$$
\begin{aligned}
&\left|\alpha_\varepsilon^{-1/2}\left(\frac{\partial\rho(y)}{\partial y} - \rho(y)\frac{y}{\varepsilon}\right) e^{-y^2/2\varepsilon}\left(\iota\otimes\begin{pmatrix}0\\1\end{pmatrix}\right)(s)\right|_{L^2,L^2} \\
&\leq \left(\frac{C}{\sqrt{\varepsilon}} + C'\right)\|s\|_{L^2(B_0,\ker D_X\otimes\Sigma B_0)}.
\end{aligned}
$$

For the last summand we use again Lemma 4.1.6 and Lemma 4.1.7 to check that

$$\begin{aligned}
&\left|\alpha_\varepsilon^{-1/2}\rho(y)e^{-y^2/2\varepsilon}\nabla^{\pi_*V\otimes\Sigma B,0}\left(\iota\otimes\begin{pmatrix}0\\1\end{pmatrix}\right)(s)\right|_{L^2,L^2}\\
&\quad\leq C\left|\alpha_\varepsilon^{-1/2}\rho(y)e^{-y^2/2\varepsilon}\nabla^{\ker\otimes\Sigma B_0}s\right|_{L^2,L^2}+C\left|\alpha_\varepsilon^{-1/2}\rho(y)e^{-y^2/2\varepsilon}\left|y\right|s\right|_{L^2,L^2}\\
&\quad\leq C\left\|s\right\|_{W^1(B_0,\ker D_X\otimes\Sigma B_0)}+C\varepsilon^{1/2}\left\|s\right\|_{L^2(B_0,\ker D_X\otimes\Sigma B_0)}.
\end{aligned}$$

Therefore there exists an ε_0 small enough such that for all $\varepsilon\in(0,\varepsilon_0)$

$$\left|I_\varepsilon(s)\right|_{W^1,L^2}\leq C\left\|s\right\|_{W^1(B_0,\ker D_X\otimes\Sigma B_0)}+\frac{C}{\sqrt{\varepsilon}}\left\|s\right\|_{L^2(B_0,\ker D_X\otimes\Sigma B_0)}.$$

□

4.2 Limiting behaviour of $\varepsilon^{-1}D_{M,\varepsilon}$ as $\varepsilon\to 0$

4.2.1 Technical preliminary work

Remember that for a local orthonormal frame $f_1,\dots,f_m$ of T_HM (see [BC89, Equation (4.26)] or [Goe14, Equation (2.4)] where in our case the twist part is independent of ε)

$$\begin{aligned}
D_{M,\varepsilon}&=\tilde{D}_X+\varepsilon D_{B,\varepsilon}\\
&=\tilde{D}_X+\varepsilon\sum_{\alpha=1}^m c(f_\alpha)\left(\nabla_{f_\alpha}^{V\otimes\pi^*\Sigma B}-\frac{1}{2}k(f_\alpha)\right)-\frac{\varepsilon^2}{4}\sum_{\alpha<\beta}c(f_\alpha)c(f_\beta)c\left(T(f_\alpha,f_\beta)\right)\\
&=\tilde{D}_X+\varepsilon\tilde{D}_B+\varepsilon^2\tilde{T},
\end{aligned}$$

where $\tilde{D}_X=D_X\otimes\omega$ on $\pi_*V\otimes\Sigma B$. We use the isometry J_ε of Section 4.1 to decompose $L^2\left(M,V\otimes\pi^*\Sigma B\right)$ into the image of J_ε and its orthogonal complement. So

$$L^2\left(B,\pi_*^{L^2}V\otimes\Sigma B\right)\cong L^2\left(M,V\otimes\pi^*\Sigma B\right)=\operatorname{im}q_\varepsilon\oplus\operatorname{im}q_\varepsilon^\perp$$

By taking the intersection of these two subspaces with smooth (or at least once weakly differentiable) sections, we can also decompose the total Dirac operator as

$$D_{M,\varepsilon}=\begin{pmatrix}q_\varepsilon D_{M,\varepsilon}q_\varepsilon & q_\varepsilon D_{M,\varepsilon}q_\varepsilon^\perp\\ q_\varepsilon^\perp D_{M,\varepsilon}q_\varepsilon & q_\varepsilon^\perp D_{M,\varepsilon}q_\varepsilon^\perp\end{pmatrix}=\begin{pmatrix}D_{M,\varepsilon,1} & D_{M,\varepsilon,2}\\ D_{M,\varepsilon,3} & D_{M,\varepsilon,4}\end{pmatrix}.$$

4.2.1 Lemma ([Goe14, Lemma 2.1])**.** *The anticommutator of $\tilde{D}_X$ and $D_{B,\varepsilon}$ is the sum of a fibrewise differential operator of order one and an endomorphism of $V\otimes\pi^*\Sigma B$. Therefore*

$$\left\|[\tilde{D}_X,D_{B,\varepsilon}]\tau\right\|_{L^2,L^2}\leq(C_1+C_2\varepsilon)\left\|\tau\right\|_{L^2,W^1},$$

for all $\tau\in W^1\left(M,V\otimes\pi^*\Sigma B\right)$.

4.2.2 Lemma. *The operator $\tilde{D}_B$ is symmetric, that means*

$$\left\langle\left\langle \tilde{D}_B\sigma, \tau \right\rangle\right\rangle_{L^2,L^2} = \left\langle\left\langle \sigma, \tilde{D}_B\tau \right\rangle\right\rangle_{L^2,L^2}$$

for all $\sigma, \tau \in \Gamma(B, \pi_* V \otimes \Sigma B)$. Furthermore it fulfills an elliptic estimate

$$\|\tau\|^2_{W^1,L^2} \leq \left\|\tilde{D}_B\sigma\right\|^2_{L^2,L^2} + C_1 \|\tau\|^2_{L^2,L^2} + C_2 \|\tau\|^2_{L^2,W^1}.$$

Proof. The first part follows as the usual symmetry of Dirac operators, using that $\nabla^{\pi_* V\otimes\Sigma B,0}$ is compatible with $g_{\pi_* V\otimes\Sigma B}$. For the second statement we see by a direct calculation that

$$\begin{aligned}\tilde{D}_B^2 &= \frac{1}{2}\sum_{\alpha,\beta=1}^{m} c(f_\alpha)c(f_\beta)F^{\pi_* V\otimes\Sigma B,0}_{f_\alpha,f_\beta} + \Delta^{\pi_* V\otimes\Sigma B,0} \\ &= \frac{1}{2}\sum_{\alpha,\beta} c(f_\alpha)c(f_\beta)\left(F^{V\otimes\pi^*\Sigma B,0}_{f_\alpha^H,f_\beta^H} + \frac{1}{2}d_M k\left(f_\alpha^H, f_\beta^H\right) - \nabla^{V\otimes\pi^*\Sigma B}_{T(f_\alpha,f_\beta)}\right) + \Delta^{\pi_* V\otimes\Sigma B,0},\end{aligned}$$

where $\Delta^{\pi_* V\otimes\Sigma B,0}$ denotes the Laplacian of the connection $\nabla^{\pi_* V\otimes\Sigma B,0}$. This equality leads to the elliptic estimate by observing that $\nabla^{V\otimes\pi^*\Sigma B,0}_{T(f_\alpha,f_\beta)}$ differentiates into vertical directions, therefore we need the $\|\tau\|_{L^2,W^1}$-part.

□

4.2.3 Definition. As before we denote the covariant derivative

$$g^*\left(\left(\nabla^{\ker D_X} \oplus \nabla^{\operatorname{im} D_X}\right)\otimes 1 + 1\otimes\left(\nabla^{\Sigma B_0}\oplus\nabla^{\Sigma B_0}\right)\right)$$

by ∇, where $g\colon B_0\times(-a,a)\to B_0$ is the projection onto the first component. For a local orthonormal frame $f_1,...,f_{m-1}$ of $TB_0\to B_0$, $\frac{\partial}{\partial y}$ the oriented section of $\nu B_0\to B_0$ of length one, we define the differential operator $D_H\colon \Gamma\left(N_a, \pi_* V\otimes \Sigma B|_{N_a}\right)\to \Gamma\left(N_a, \pi_* V\otimes\Sigma B|_{N_a}\right)$ by

$$D_H = \sum_{\alpha=1}^{m-1} c(f_\alpha)\nabla_{f_\alpha} = c|_{B_0}\circ g_{B_0}^{-1}\circ\nabla|_{TB_0},$$

the operator $D_N\colon \Gamma\left(N_a, \pi_* V\otimes\Sigma B|_{N_a}\right)\to\Gamma\left(N_a, \pi_* V\otimes\Sigma B|_{N_a}\right)$ by

$$D_N = c\left(\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}$$

the operator $G\colon \Gamma\left(N_a, \pi_* V\otimes\Sigma B|_{N_a}\right)\to\Gamma\left(N_a, \pi_* V\otimes\Sigma B|_{N_a}\right)$ by

$$G = \sum_{\alpha=1}^{m-1} c(f_\alpha)\left(g^*\nabla^{\pi_* V}_{f_\alpha} - g^*\nabla^{\ker\oplus\operatorname{im}}_{f_\alpha}\right)\otimes 1$$

and last $R_\varepsilon\colon \Gamma\left(N_a, \pi_* V\otimes\Sigma B|_{N_a}\right)\to\Gamma\left(N_a, \pi_* V\otimes\Sigma B|_{N_a}\right)$ by

$$R_\varepsilon = h^{1/2}\tilde{D}_B h^{-1/2} + \varepsilon\tilde{T} - D_H - D_N - G.$$

4.2.4 Lemma. *The operator $D_H + D_N$ is symmetric for sections with compact support with respect to $\langle\cdot,\cdot\rangle_{L^2,L^2}$, that means for all $\sigma,\tau \in W^1_c\left(N_a, \pi_* V \otimes \Sigma B|_{N_a}\right)$*

$$\langle (D_H + D_N)\,\sigma, \tau\rangle_{L^2,L^2} = \langle \sigma, (D_H + D_N)\,\tau\rangle_{L^2,L^2}\,.$$

Furthermore it satisfies an elliptic estimate for compactly supported $\tau \in W^1_c\left(N_a, \pi_* V \otimes \Sigma B|_{N_a}\right)$

$$|\sigma|^2_{W^1,L^2} \leq |(D_H + D_N)\,\tau|^2_{L^2,L^2} + C_1\,|\tau|^2_{L^2,L^2} + C_2\,|\tau|_{L^2,W^1}\,.$$

Proof. The first statement follows again as in Lemma 4.2.2 by the same arguments as the symmetry for usual Dirac operators on compact manifolds. Since we have sections with compact support there are no boundary terms when we apply Stokes' theorem.
For the elliptic estimate we check that

$$(D_H + D_N)^2 = \Delta^{\nabla} + \frac{1}{2}\sum_{\alpha\neq\beta} c\left(f_\alpha\right) c\left(f_\beta\right) \nabla^2_{f_\alpha,f_\beta},$$

where Δ^{∇} denotes the Laplacian of the connection $\nabla = g^*\nabla^{(\ker\oplus\,\mathrm{im})\otimes(\Sigma B_0\oplus\Sigma B_0)}$.
Since we have a formula for the off-diagonals of $\nabla^{\pi_* V}$ on B_0 by Proposition 3.2.14, we know as in the proof of Lemma 4.1.6 that

$$\left|\left(\nabla^2 - g^*\left(\nabla^{\pi_* V\otimes\Sigma B,0}|_{B_0}\right)^2\right)\tau\right|_{L^2,L^2} \leq C\,|\tau|_{L^2,L^2}\,.$$

We use again that

$$\left(\nabla^{\pi_* V\otimes\Sigma B,0}\right)^2_{f_\alpha,f_\beta} = F^{V\otimes\pi^*\Sigma B}_{f^H_\alpha,f^H_\beta} + dk\left(f^H_\alpha, f^H_\beta\right) - \nabla^{V\otimes\pi^*\Sigma B}_{T(f_\alpha,f_\beta)}$$

and $T(f_\alpha, f_\beta)$ is vertical, therefore

$$\left|\left\langle \frac{1}{2}\sum_{\alpha\neq\beta} c\left(f_\alpha\right) c\left(f_\beta\right) \nabla^2_{f_\alpha,f_\beta}\tau, \tau\right\rangle_{L^2,L^2}\right| \leq C_1\,|\tau|^2_{L^2,L^2} + C_2\,|\tau|^2_{L^2,W^1}\,.$$

Furthermore again by Lemma 4.1.6 and the fact that $|y| \leq a$ we have

$$\left\langle \Delta^{\nabla}\tau, \tau\right\rangle_{L^2,L^2} = \langle \nabla\tau, \nabla\tau\rangle_{L^2,L^2} \leq C\left\langle \nabla^{\pi_* V\otimes\Sigma B,0}\tau, \nabla^{\pi_* V\otimes\Sigma B,0}\tau\right\rangle_{L^2,L^2}\,.$$

Using that $D_H + D_N$ is symmetric and putting together these estimates it follows that

$$\left|(D_H + D_N)^2\,\tau\right|^2_{L^2,L^2} \geq |\tau|^2_{W^1,L^2} - C_1\,|\tau|^2_{L^2,L^2} - C_2\,|\tau|^2_{L^2,W^1}\,.$$

$\square$

4.2.5 Proposition (cf. [BL91, Proposition 9.7]). *The remainder term R_ε is $\mathcal{C}^\infty(-a,a)$-linear and fulfills the following estimate for all $\tau \in W^1\left(N_a, \pi_*V \otimes \Sigma B|_{N_a}\right)$*

$$\begin{aligned}\|R_\varepsilon\tau\|_{L^2(B_0\times\{y\},\pi_*V\otimes\Sigma B)} \leq & C_1\,|y|\,\|\tau\|_{W^1(B_0\times\{y\},\pi_*V\otimes\Sigma B)} \\ & + C_2\left(|y|+\varepsilon\right)\|\tau\|_{L^2(B_0\times\{y\},\pi_*V\otimes\Sigma B)},\end{aligned}$$

where we use the connection $\nabla^{\pi_*V\otimes\Sigma B,0}$ for the Sobolev-1-norm.

Proof. First recall that

$$R_\varepsilon = h^{1/2}\tilde{D}_B h^{-1/2} + \varepsilon\tilde{T} - D_H - D_N - G.$$

We know that

$$h^{1/2}\tilde{D}_B h^{-1/2} = -\frac{1}{2h}c\left(\operatorname{grad} h\right) + \tilde{D}_B.$$

(1) First summand
By the definition of the function h we know that $h|_{B_0} \equiv 1$ and therefore for an orthonormal vector f_α of TB_0

$$\left|f_\alpha(h)_{(x,y)}\right| \leq C\,|y|\,.$$

For the oriented normal vector we know by Lemma 4.1.3 that

$$\left.\frac{\partial h}{\partial y}\right|_{B_0} = -\operatorname{tr}(A)$$

Since we chose a small enough such that $|h| \geq C$ on $B_0 \times (-a,a)$ we conclude

$$\left\|-\frac{1}{2h}c\left(\operatorname{grad} h\right) - \frac{1}{2}\operatorname{tr}(A)c\left(\frac{\partial}{\partial y}\right)\right\|_{g_{\pi_*V\otimes\Sigma B}} \leq C\,|y|\,.$$

(2) Second summand
In our chosen trivialization and by the considerations of Remark 4.1.1 we already know that

$$\tilde{D}_B = D_N + c\circ\tilde{g}_{B_0}^{-1}\circ\nabla^{\pi_*V\otimes\Sigma B,0}\big|_{TB_0}.$$

From now on we will (by abuse of notation) write ∇^* instead of $\nabla^*|_{TB_0}$. We want to compare the operators $c\circ\tilde{g}_{B_0}^{-1}\circ\nabla^{\pi_*V\otimes\Sigma B,0}$ and $c|_{B_0}\circ g_{B_0}^{-1}\circ g^*\nabla^{\pi_*V\otimes\Sigma B,0}$ since

$$\begin{aligned}c|_{B_0}\circ g_{B_0}^{-1}\circ g^*\nabla^{\pi_*V\otimes\Sigma B,0} &= D_H + G + c\circ g_{B_0}^{-1}\left(1\otimes g^*\left(\nabla^{\Sigma B} - \nabla^{\Sigma B_0\oplus\Sigma B_0}\right)\right) \\ &= D_H + G - \frac{1}{2}\operatorname{tr}(A)c\left(\frac{\partial}{\partial y}\right).\end{aligned}$$

So consider the difference

$$\begin{aligned}&\left\|\left(c\tilde{g}_{B_0}^{-1}\nabla^{\pi_*V\otimes\Sigma B,0} - c|_{B_0}g_{B_0}^{-1}g^*\nabla^{\pi_*V\otimes\Sigma B,0}\right)\tau\right\|_{L^2(B_0\times\{y\},\pi_*V\otimes\Sigma B)} \\ &\leq \left\|\left(c\tilde{g}_{B_0}^{-1} - c|_{B_0}g_{B_0}^{-1}\right)\nabla^{\pi_*V\otimes\Sigma B,0}\tau\right\|_{L^2(B_0\times\{y\},\pi_*V\otimes\Sigma B)} \\ &\quad + \left\|c|_{B_0}g_{B_0}^{-1}\left(\nabla^{\pi_*V\otimes\Sigma B,0} - g^*\nabla^{\pi_*V\otimes\Sigma B,0}\right)\tau\right\|_{L^2(B_0\times\{y\},\pi_*V\otimes\Sigma B)} \\ &\leq C\,|y|\,\|\tau\|_{W^1(B_0\times\{y\},\pi_*V\otimes\Sigma B)} + C\,|y|\,\|\tau\|_{L^2(B_0\times\{y\},\pi_*V\otimes\Sigma B)}\end{aligned}$$

by the same considerations as in the proof of Lemma 4.1.6.
Combining the above estimates with

$$\left\|\varepsilon\tilde{T}\tau\right\|_{g_{\pi_* V\otimes\Sigma B}} \leq C\varepsilon\|\tau\|_{g_{\pi_* V\otimes\Sigma B}},$$

we see that

$$\begin{aligned}
&\|R_\varepsilon\tau\|_{L^2(B_0\times\{y\},\pi_* V\otimes\Sigma B)}\\
&\leq \left\|\left(-\frac{1}{2h}c(\operatorname{grad} h)-\frac{1}{2}\operatorname{tr}(A)c\left(\frac{\partial}{\partial y}\right)\right)\tau\right\|_{L^2(B_0\times\{y\},\pi_* V\otimes\Sigma B)}\\
&\quad+\left\|\left(c\tilde{g}_{B_0}^{-1}\nabla^{\pi_* V\otimes\Sigma B,0}-c|_{B_0}\,g_{B_0}^{-1}g^*\nabla^{\pi_* V\otimes\Sigma B,0}\right)\tau\right\|_{L^2(B_0\times\{y\},\pi_* V\otimes\Sigma B)}\\
&\quad+\left\|\varepsilon\tilde{T}\tau\right\|_{L^2(B_0\times\{y\},\pi_* V\otimes\Sigma B)}\\
&\leq C(|y|+\varepsilon)\|\tau\|_{L^2(B_0\times\{y\},\pi_* V\otimes\Sigma B)}+C|y|\|\tau\|_{W^1(B_0\times\{y\},\pi_* V\otimes\Sigma B)}
\end{aligned}$$

□

4.2.6 Lemma. *The following identities of operators are fulfilled*

$$p_\varepsilon\tilde{D}_X p_\varepsilon = 0,$$

and also

$$p_\varepsilon D_N p_\varepsilon = 0.$$

Proof. The fibrewise Dirac operators D_X on $\pi_* V$ of Section 3.2 act on $\pi_* V\otimes\Sigma B$ as

$$\tilde{D}_X = D_X\otimes\omega$$

where $\omega\in\operatorname{End}(\Sigma B)$ is the $\mathbb{Z}_2$-grading of ΣB. On N_a

$$\omega=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

with respect to the decomposition $\Sigma B|_{N_a}\cong(\Sigma B_0\oplus\Sigma B_0)\times(-a,a)$. Now the claim follows by our formula for p_ε in Lemma 4.1.12 since

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}=0.$$

The same argument proves $p_\varepsilon D_N p_\varepsilon = 0$ since

$$c\left(\frac{\partial}{\partial y}\right)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

□

4.2.7 Proposition (cf. [BL91, Proposition 9.9])**.** *There exist constants $C_1, C_2, C_3, C_4 > 0$ and an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in W^1\left(N_a, \pi_* V \otimes \Sigma B|_{N_a}\right)$ the following estimates hold*

$$\left|D_H\left(p_\varepsilon(\tau)\right) - p_\varepsilon\left(D_H(\tau)\right)\right|_{L^2,L^2} \le C_1\sqrt{\varepsilon}\,|\tau|_{L^2,L^2} \tag{4.2.1}$$

$$\left|p_\varepsilon\left(D_N + \frac{1}{\varepsilon}\tilde{D}_X\right)\tau\right|_{L^2,L^2} \le C_2\sqrt{\varepsilon}\,|\tau|_{L^2,L^2} \tag{4.2.2}$$

$$|p_\varepsilon R\tau|_{L^2,L^2} \le C_3\sqrt{\varepsilon}\,|\tau|_{W^1,L^2} \tag{4.2.3}$$

$$\|G\tau\|_{L^2,L^2} \le C_4\,\|\tau\|_{L^2,L^2}\,. \tag{4.2.4}$$

Proof. For the first estimate (4.2.1) one checks that for $\tau = \tau_1 \otimes \begin{pmatrix}\tau_2\\ \tau_3\end{pmatrix}$

$$\begin{aligned}
&D_H\left(p_\varepsilon(\tau)\right) - p_\varepsilon\left(D_H(\tau)\right)\\
&= \alpha_\varepsilon^{-1}\rho(y)e^{-y^2/2\varepsilon}\sum_{i,\alpha}\psi_i(x)\int_{\mathbb{R}}\rho\left(\tilde{y}\right)e^{-\tilde{y}^2/2\varepsilon}\Big[g_{\pi_*V}\left(g^*\nabla^{\ker\oplus\operatorname{im}}_{f_\alpha}\iota_{(x,\tilde{y})}(\sigma_i), \tau_1(x,\tilde{y})\right)\iota_{(x,y)}(\sigma_i)\\
&\quad + g_{\pi_*V}\left(\iota_{(x,\tilde{y})}(\sigma_i), \tau_1(x,\tilde{y})\right)g^*\nabla^{\ker\oplus\operatorname{im}}_{f_\alpha}\iota_{(x,y)}(\sigma_i)\Big] \otimes \begin{pmatrix}0\\ -c_{B_0}\left(f_\alpha\right)\tau_3(x,\tilde{y})\end{pmatrix}d\tilde{y}.
\end{aligned}$$

We verify that the integrand vanishes for $\tilde{y} = y = 0$ since on B_0

$$\begin{aligned}
&g_{\pi_*V}\left(\sigma_i, \nabla^{\ker\oplus\operatorname{im}}_X\tau\right)\sigma_i = P\nabla^{\ker\oplus\operatorname{im}}_X\tau = \nabla^{\ker\oplus\operatorname{im}}_X P\tau\\
&= g_{\pi_*V}\left(\nabla^{\ker\oplus\operatorname{im}}_X\sigma_i, \tau\right)\sigma_i + g_{\pi_*V}\left(\sigma_i, \nabla^{\ker\oplus\operatorname{im}}_X\tau\right)\sigma_i + g_{\pi_*V}\left(\sigma_i, \tau\right)\nabla^{\ker\oplus\operatorname{im}}_X\sigma_i.
\end{aligned}$$

Therefore for any other $y, \tilde{y} \in (-a, a)$

$$\begin{aligned}
&\Big\|g_{\pi_*V}\left(g^*\nabla^{\ker\oplus\operatorname{im}}_{f_\alpha}\iota_{(x,\tilde{y})}(\sigma_i), \tau_1(x,\tilde{y})\right)\iota_{(x,y)}(\sigma_i)\\
&\quad + g_{\pi_*V}\left(\iota_{(x,\tilde{y})}(\sigma_i), \tau_1(x,\tilde{y})\right)g^*\nabla^{\ker\oplus\operatorname{im}}_{f_\alpha}\iota_{(x,y)}(\sigma_i)\Big\|^2_{g_{\pi_*V}} \le C\left\|\tau_1(x,\tilde{y})\right\|^2_{g_{\pi_*V}}\left(y^2 + \tilde{y}^2\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\left|D_H\left(p_\varepsilon(\tau)\right) - p_\varepsilon\left(D_H(\tau)\right)\right|^2_{L^2,L^2}\\
&\le C\alpha_\varepsilon^{-2}\int_{B_0\times\mathbb{R}}\rho^2(y)e^{-y^2/\varepsilon}\sum_{i,\alpha}\psi_i^2(x)\int_{\mathbb{R}}\rho^2\left(\tilde{y}\right)e^{-\tilde{y}^2/\varepsilon}\left(y^2 + \tilde{y}^2\right)\\
&\qquad\qquad \cdot \left\|\tau_1(x,\tilde{y})\right\|^2_{g_{\pi_*V}}\left\|\begin{pmatrix}0\\ -c_{B_0}\left(f_\alpha\right)\tau_3(x,\tilde{y})\end{pmatrix}\right\|^2_{g_{\Sigma B}} d\tilde{y}\,\mathrm{dvol}_{B_0}\,dy\\
&\le C\varepsilon\,|\tau|^2_{L^2,L^2}\,.
\end{aligned}$$

Here we used

$$C\sqrt{\varepsilon} \le \alpha_\varepsilon = \int_{\mathbb{R}}\rho^2(y)e^{-y^2/\varepsilon}dy \le C'\sqrt{\varepsilon}$$

and
$$\int_{\mathbb{R}} \rho^2(y) e^{-y^2/\varepsilon} y^2 dy \leq C'' \varepsilon^{3/2}.$$
The second equation (4.2.2) follows as the corresponding statement in [BL91, Proposition 9.9].
Also equation (4.2.3) follows as in [BL91, Proposition 9.9] by equation (4.1.1) in Lemma 4.1.12 and by the estimate of the remainder term R_ε in Proposition 4.2.5.
For the last estimate (4.2.4) remember that
$$G = \sum_{\alpha=1}^{m-1} c\left(f_\alpha\right)\left(g^* \nabla_{f_\alpha}^{\pi_* V} - g^* \nabla_{f_\alpha}^{\ker \oplus \mathrm{im}}\right) \otimes 1.$$
We already proved in Lemma 4.1.6 that on B_0
$$\left\|\left(\nabla_{f_\alpha}^{\pi_* V} - \nabla_{f_\alpha}^{\ker \oplus \mathrm{im}}\right)\tau_1\right\|_{g_{\pi_* V}} \leq C \left\|\tau_1\right\|_{g_{\pi_* V}}$$
by which estimate (4.2.4) follows. $\square$

4.2.8 Lemma (cf. [BL91, Proposition 9.13])**.** *There exist constants $C_1, C_2, C_3, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\tau \in W^1\left(B, \pi_* V \otimes \Sigma B\right)$ with $\operatorname{supp}\tau \subset B \setminus N_{\frac{a_1}{2}}$*
$$\left\|\frac{1}{\varepsilon} D_{M,\varepsilon}\tau\right\|_{L^2,L^2}^2 \geq \frac{C_1}{\varepsilon^2}\|\tau\|_{L^2,L^2}^2 + \frac{C_2}{\varepsilon^2}\|\tau\|_{L^2,W^1}^2 + C_3\|\tau\|_{W^1,L^2}^2.$$

Proof. Since $D_{M,\varepsilon}$ is self-adjoint
$$\begin{aligned}
\left\|\frac{1}{\varepsilon} D_{M,\varepsilon}\tau\right\|_{L^2,L^2}^2 &= \frac{1}{\varepsilon^2}\left\langle\left\langle D_{M,\varepsilon}^2\tau, \tau\right\rangle\right\rangle_{L^2,L^2} \\
&= \left\langle\left\langle\left(\frac{1}{\varepsilon^2}\tilde{D}_X^2 + \frac{1}{\varepsilon}[\tilde{D}_X, D_{B,\varepsilon}] + D_{B,\varepsilon}^2\right)\tau, \tau\right\rangle\right\rangle_{L^2,L^2} \\
&\geq \frac{1}{\varepsilon^2}\left\|\tilde{D}_X\tau\right\|_{L^2,L^2}^2 - \frac{1}{\varepsilon}\left|\left\langle\left\langle[\tilde{D}_X, D_{B,\varepsilon}]\tau, \tau\right\rangle\right\rangle_{L^2,L^2}\right| + \left\langle\left\langle D_{B,\varepsilon}^2\tau, \tau\right\rangle\right\rangle_{L^2,L^2}.
\end{aligned}$$
For the fibrewise Dirac operator D_X we have two different estimates. First we use that D_X is a fibrewise elliptic operator
$$\|\tau\|_{L^2,W^1}^2 \leq \left\|\tilde{D}_X\tau\right\|_{L^2,L^2}^2 + C'\|\tau\|_{L^2,L^2}^2,$$
then we use that the support of τ is included in $B \setminus N_{\frac{a_1}{2}}$ and therefore
$$\left\|\tilde{D}_X\tau\right\|_{L^2,L^2} \geq C\|\tau\|_{L^2,L^2}.$$

Combining the two estimates we know that for all $s \in [0,1]$

$$\left\|\tilde{D}_X\tau\right\|^2_{L^2,L^2} \geq s\,\|\tau\|^2_{L^2,W^1} + \big(C(1-s) - C's\big)\,\|\tau\|^2_{L^2,L^2}\,.$$

By choosing $s \in \left(0, \frac{C}{C+C'}\right)$ the constant $C_2 = C(1-s) - C's$ fulfills $C_2 > 0$ and

$$\left\|\tilde{D}_X\tau\right\|^2_{L^2,L^2} \geq C_1\,\|\tau\|^2_{L^2,W^1} + C_2\,\|\tau\|^2_{L^2,L^2}\,. \tag{4.2.5}$$

By Lemma 4.2.1, the Cauchy inequality and the fact that $\|\cdot\|_{L^2,L^2} \leq \|\cdot\|_{L^2,W^1}$ we know that

$$\left|\left\langle\left\langle\left[\tilde{D}_X, D_{B,\varepsilon}\right]\tau, \tau\right\rangle\right\rangle_{L^2,L^2}\right| \leq (C_3 + C_4\varepsilon)\,\|\tau\|^2_{L^2,W^1}\,. \tag{4.2.6}$$

So we still need to estimate $D^2_{B,\varepsilon} = \tilde{D}^2_B + \varepsilon\left[\tilde{D}_B, \tilde{T}\right] + \varepsilon^2\tilde{T}^2$. Since the operator $\tilde{T}$ is an endomorphism of $V \otimes \pi^*\Sigma B$ and $\left[\tilde{D}_B, \tilde{T}\right]$ is a differential operator of order one of $\pi_*V \otimes \Sigma B \to B$ we get, by using the elliptic estimate of Lemma 4.2.2,

$$\left\langle\left\langle D^2_{B,\varepsilon}\tau, \tau\right\rangle\right\rangle_{L^2,L^2} \geq (1 - \varepsilon C_5)\,\|\tau\|^2_{W^1,L^2} - \left(C_6 + \varepsilon^2 C_7\right)\|\tau\|^2_{L^2,L^2} - C_8\,\|\tau\|^2_{L^2,W^1}\,. \tag{4.2.7}$$

Combining the estimates (4.2.5), (4.2.6) and (4.2.7) we finally get

$$\begin{aligned}\left\|\frac{1}{\varepsilon}D_{M,\varepsilon}\tau\right\|^2_{L^2,L^2} \geq{}& \left(\frac{C_2}{\varepsilon^2} - C_6 - \varepsilon^2 C_7\right)\|\tau\|^2_{L^2,L^2} + \left(\frac{C_1}{\varepsilon^2} - \frac{C_3}{\varepsilon} - C_4 - C_8\right)\|\tau\|^2_{L^2,W^1}\\ &+ (1 - \varepsilon C_5)\,\|\tau\|^2_{W^1,L^2}\,,\end{aligned}$$

which proves the claimed estimate by choosing ε small enough such that all constants are positive. $\square$

4.2.9 Proposition (cf. [BL91, Proposition 9.12])**.** *There exists a constant* $0 < a_1 \leq a/2$ *and constants* $C_1, C_2, C_3 > 0$*, such that for all* $\tau \in \operatorname{im} q_\varepsilon^\perp \cap W^1(M, V \otimes \pi^*\Sigma B)$ *whose support is included in* N_{a_1}

$$\left\|\frac{1}{\varepsilon}D_{M,\varepsilon}\tau\right\|^2_{L^2,L^2} \geq \frac{C_1}{\varepsilon}\,\|\tau\|^2_{L^2,L^2} + \frac{C_2}{\varepsilon^2}\,\|\tau\|^2_{L^2,W^1} + C_3\,\|\tau\|^2_{W^1,L^2}\,.$$

Proof. We follow the ideas of the proof of [BL91, Proposition 9.12].
Since the support of τ is included in $B_0 \times (-a_1, a_1)$ we can define the section

$$\overline{\tau} = h^{1/2}\tau$$

and since $\tau \in \operatorname{im} q_\varepsilon^\perp$ we know that $\overline{\tau} \in \operatorname{im} p_\varepsilon^\perp$ because of Lemma 4.1.11. By definition of our norms

$$\left\|\frac{1}{\varepsilon}D_{M,\varepsilon}\tau\right\|^2_{L^2,L^2} = \left|h^{1/2}\frac{1}{\varepsilon}D_{M,\varepsilon}h^{-1/2}\overline{\tau}\right|^2_{L^2,L^2}\,.$$

Again because we are now working locally around B_0, we can write our operator with the help of the operators of Definition 4.2.3

$$\begin{aligned} h^{1/2}\frac{1}{\varepsilon}D_{M,\varepsilon}h^{-1/2} &= \frac{1}{\varepsilon}\tilde{D}_X + D_N + D_H + G + R_\varepsilon \\ &= L_\varepsilon + R_\varepsilon + G \end{aligned}$$

where $L_\varepsilon = \frac{1}{\varepsilon}\tilde{D}_X + D_N + D_H$. Recall that on N_a the bundle $\pi_* V|_{N_a}$ splits orthogonally into the line bundle $L = \operatorname{im} P$ and the infinite part $W = \operatorname{im} Q$. We decompose the section $\overline{\tau}$ with respect to this decomposition into $\overline{\tau} = \overline{\tau}^L + \overline{\tau}^W = h^{1/2}\left(\tau^L + \tau^W\right)$ where $\overline{\tau}^L$ or τ^L is the part which is in the line bundle and therefore corresponds to the small eigenvalue of D_X.

Estimate of $(R_\varepsilon + G)\,\overline{\tau}$
By Proposition 4.2.5 and since $|y| \leq a_1$

$$|R_\varepsilon \tau|^2_{L^2,L^2} \leq C a_1^2 \,|\tau|^2_{W^1,L^2} + C\left(a_1^2 + \varepsilon^2\right)|\tau|^2_{L^2,L^2}$$

and by equation (4.2.4) in Proposition 4.2.7

$$|G\overline{\tau}|^2_{L^2,L^2} \leq C\,|\overline{\tau}|^2_{L^2,L^2}\,.$$

Alltogether we get the following estimate for the remainder $R_\varepsilon + G$

$$|(R_\varepsilon + G)\,\overline{\tau}|^2_{L^2,L^2} \leq C(a_1^2 + \varepsilon^2)\,|\overline{\tau}|^2_{L^2,L^2} + C a_1^2\,|\overline{\tau}|^2_{W^1,L^2}\,. \tag{4.2.8}$$

Estimate of $\langle L_\varepsilon \overline{\tau}^L, L_\varepsilon \overline{\tau}^W \rangle_{L^2,L^2}$
Since D_X is obviously diagonal with respect to $L \oplus W$ and $D_H + D_N$ is symmetric for sections with compact support, see Lemma 4.2.4, we know that

$$\langle L_\varepsilon \overline{\tau}^L, L_\varepsilon \overline{\tau}^W \rangle_{L^2,L^2} = \langle (D_H + D_N)^2 \overline{\tau}^L, \overline{\tau}^W \rangle_{L^2,L^2}.$$

Now we want to prove that $(D_H + D_N)^2\overline{\tau}^L$ is 'almost' in L, so the product becomes comparibly small. This follows since the decomposition $L \oplus W$ equals $\ker D_X \oplus \operatorname{im} D_X$ on B_0 and we use Lemma 4.1.7 together with the fact that $|y| \leq a_1$ to see that

$$\left|\left\langle (D_H + D_N)^2\,\overline{\tau}^L, \overline{\tau}^W \right\rangle_{L^2,L^2}\right| \leq C a_1 \,\|\tau\|_{W^1,L^2}\,\|\tau\|_{L^2,L^2} \leq C a_1\,\|\tau\|^2_{W^1,L^2}\,. \tag{4.2.9}$$

Estimate of $L_\varepsilon \overline{\tau}^W$
By Lemma 4.2.4 we know that $D_H + D_N$ has the same properties as $\tilde{D}_B$ and since for $\overline{\tau}^W$ the estimate

$$\left|\tilde{D}_X \overline{\tau}^W\right|_{L^2,L^2} \geq C\,\left|\overline{\tau}^W\right|_{L^2,L^2}$$

holds true, the same proof as the one given for Lemma 4.2.8 leads to

$$\begin{aligned} \left|L_\varepsilon \overline{\tau}^W\right|^2_{L^2,L^2} &\geq \frac{C_1}{\varepsilon^2}\left|\overline{\tau}^W\right|^2_{L^2,L^2} + \frac{C_2}{\varepsilon^2}\left|\overline{\tau}^W\right|^2_{L^2,W^1} + C_3\left|\overline{\tau}^W\right|^2_{W^1,L^2} \\ &\geq \frac{C_1}{\varepsilon^2}\left\|\tau^W\right\|^2_{L^2,L^2} + \frac{C_2}{\varepsilon^2}\left\|\tau^W\right\|^2_{L^2,W^1} + C_3\left\|\tau^W\right\|^2_{W^1,L^2}\,. \end{aligned} \tag{4.2.10}$$

Estimate of $L_\varepsilon \overline{\tau}^L$

By definition the operators D_H and D_N anticommute (since Clifford anticommutes and differentiation with the pulled back connection commutes) and since $P\tilde{D}_X = yP \otimes \omega$ and D_H just differentiates into B_0-directions, $P\tilde{D}_X$ and D_H also anticommute. Therefore

$$\begin{aligned} \left|L_\varepsilon \overline{\tau}^L\right|^2_{L^2,L^2} &= \left|D_H \overline{\tau}^L\right|^2_{L^2,L^2} + \left|\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)\overline{\tau}^L\right|^2_{L^2,L^2} \\ &\geq \left|D_H \overline{\tau}^L\right|^2_{L^2,L^2} + \left|\left(\frac{1}{\varepsilon}\tilde{D}_X + PD_N\right)\overline{\tau}^L\right|^2_{L^2,L^2} - C\left|(1+y)\overline{\tau}^L\right|^2_{L^2,L^2}, \end{aligned} \tag{4.2.11}$$

where the estimate comes from the fact that on B_0 we can estimate the off-diagonals with respect to the decomposition $\ker D_X \oplus \operatorname{im} D_X$ of $\nabla^{\pi_* V}_{\frac{\partial}{\partial y}} = \frac{\partial}{\partial y}$ because of Proposition 3.2.14, see Lemma 4.1.7 and we know that P is parallel along normal geodesics with respect to ∇^L.

We will first investigate the spectral properties of the operator $P\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)P$. We see that it acts on $W^1\left(B_0 \times (-a,a), L \otimes \Sigma B\right)$ as

$$P \otimes \begin{pmatrix} 0 & \frac{y}{\varepsilon} \\ \frac{y}{\varepsilon} & 0 \end{pmatrix} + P \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y}$$

where the matrix decomposition is again with respect to $\Sigma B = \Sigma B_0 \oplus \Sigma B_0$. So if we choose a local basis of $\ker D_X \otimes \Sigma B_0 \oplus \Sigma B_0 \to B_0$ and transport it parallel along $(-a,a)$ we see that we need to calculate the eigenvalues of

$$\begin{pmatrix} 0 & \frac{y}{\varepsilon} - \frac{\partial}{\partial y} \\ \frac{y}{\varepsilon} + \frac{\partial}{\partial y} & 0 \end{pmatrix} : W^1\left(\mathbb{R}, \mathbb{R}^2\right) \to L^2\left(\mathbb{R}, \mathbb{R}^2\right),$$

which are $\pm\sqrt{\frac{2m}{\varepsilon}}$, $m \in \mathbb{Z}$ and each with multiplicity one (we see this using Getzler scaling and Hermite polynomials).

Therefore, if p^*_ε denotes the projection onto the kernel of $P\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)P$ we see that as in [BL91, Eq. (9.75)]

$$\left|P\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)\overline{\tau}^L\right|^2_{L^2,L^2} \geq \frac{C}{\varepsilon}\left|\overline{\tau}^L - p^*_\varepsilon \overline{\tau}^L\right|^2_{L^2,L^2}. \tag{4.2.12}$$

The formula for the projection p^*_ε again looks a bit different as in [BL91]. For a section $s = s_1 \otimes (s_2, s_3) \in \Gamma\left(B_0 \times (-a,a), \pi_* V \otimes \Sigma B_0 \oplus \Sigma B_0\right)$

$$\begin{aligned} \left(p^*_\varepsilon s\right)(x,y) =& \beta_\varepsilon^{-1} e^{-y^2/2\varepsilon} \sum_i \psi_i(x) \\ &\cdot \int_{-a}^{a} e^{-\tilde{y}^2/2\varepsilon} g_{\pi_* V}\left(\iota_{(x,\tilde{y})}(\sigma_i(x)), s_1(x,\tilde{y})\right) \iota_{(x,y)}(\sigma_i(x)) \otimes \begin{pmatrix} 0 \\ s_3(x,\tilde{y}) \end{pmatrix} d\tilde{y}, \end{aligned}$$

where as in Lemma 4.1.11 $\{U_i\}_i$ is a covering of B_0 with subordinate partition of unity ψ_i, such that there exist local orthonormal frames $\sigma_i\colon U_i \to \ker D_X$ and

$$\beta_\varepsilon = \int_{-a}^{a} e^{-y^2/2\varepsilon} dy.$$

By assumption $0 = p_\varepsilon \overline{\tau} = p_\varepsilon \overline{\tau}^L$ and therefore as in [BL91, Eq. (9.80)] and by Lemma 4.1.11

$$\begin{aligned}\left(p_\varepsilon^* \overline{\tau}^L\right)(x,y) =& \beta_\varepsilon^{-1} e^{-y^2/2\varepsilon} \sum_i \psi_i(x) \int_{-a}^{a} (1-\rho(\tilde{y})) e^{-\tilde{y}^2/2\varepsilon} \\ & \cdot g_{\pi_* V}\left(\iota_{(x,\tilde{y})}(\sigma_i(x)), \overline{\tau}_1^L(x,\tilde{y})\right) \iota_{(x,y)}(\sigma_i(x)) \otimes \begin{pmatrix} 0 \\ \overline{\tau}_3^L\,(x,\tilde{y}) \end{pmatrix} d\tilde{y}\end{aligned}$$

and since $1-\rho(\tilde{y}) = 0$ for $|\tilde{y}| \leq a_1/2$

$$\left|p_\varepsilon^* \overline{\tau}^L\right|_{L^2,L^2}^2 \leq C(a_1)\sqrt{\varepsilon}\left|\overline{\tau}^L\right|_{L^2,L^2}^2.$$

Together with equation (4.2.12) and (4.2.11) this implies that

$$\left|\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)\overline{\tau}^L\right|_{L^2,L^2}^2 \geq \left(\frac{C_1}{\varepsilon} - \frac{C(a_1)}{\sqrt{\varepsilon}} - C_2\right)\left|\overline{\tau}^L\right|_{L^2,L^2}^2 - C_3\left|y\overline{\tau}^L\right|_{L^2,L^2}^2. \tag{4.2.13}$$

But we just want to use half of that estimate, we want another estimate for $\left|P\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)\overline{\tau}^L\right|_{L^2,L^2}^2$ which we want to combine with $\left|D_H \overline{\tau}^L\right|_{L^2,L^2}^2$ for an elliptic estimate. As in [BL91, Eq. (9.77)] we know that

$$\begin{aligned}\left|\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)\overline{\tau}^L\right|_{L^2,L^2}^2 &= \left|\frac{1}{\varepsilon}\tilde{D}_X\overline{\tau}^L\right|_{L^2,L^2}^2 + \left|D_N\overline{\tau}^L\right|_{L^2,L^2}^2 + 2\operatorname{Re}\left\langle\frac{1}{\varepsilon}\tilde{D}_X\overline{\tau}^L, D_N\overline{\tau}^L\right\rangle_{L^2,L^2} \\ &= \frac{\alpha}{2}\left(\left|\frac{1}{\varepsilon}\tilde{D}_X\overline{\tau}^L\right|_{L^2,L^2}^2 + \left|D_N\overline{\tau}^L\right|_{L^2,L^2}^2\right) \\ &\quad + 2\operatorname{Re}\left\langle\frac{1}{\varepsilon}\tilde{D}_X\overline{\tau}^L, D_N\overline{\tau}^L\right\rangle_{L^2,L^2} \\ &\quad + \left(1-\frac{\alpha}{2}\right)\left(\left|\frac{1}{\varepsilon}\tilde{D}_X\overline{\tau}^L\right|_{L^2,L^2}^2 + \left|D_N\overline{\tau}^L\right|_{L^2,L^2}^2\right) \\ &\geq \frac{\alpha}{2}\left(\left|\frac{1}{\varepsilon}y\overline{\tau}^L\right|_{L^2,L^2}^2 + \left\langle -\frac{\partial^2}{\partial y^2}\overline{\tau}^L, \overline{\tau}^L\right\rangle_{L^2,L^2}\right) \\ &\quad - \frac{\alpha}{2}\left|\left\langle\frac{1}{\varepsilon}\left(\tilde{D}_X D_N + D_N\tilde{D}_X\right)\overline{\tau}^L, \overline{\tau}^L\right\rangle_{L^2,L^2}\right|\end{aligned}$$

for all $\alpha \in (0,1]$ (since $|A|^2 + |B|^2 \geq -2\,\mathrm{Re}\langle A, B\rangle$). Therefore using half of this estimate and half of estimate (4.2.13) and the fact that $P\left(\tilde{D}_X D_N + D_N \tilde{D}_X\right) = P \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we see that

$$\left|\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)\overline{\tau}^L\right|^2_{L^2,L^2} \geq \frac{\alpha}{4}\left\langle -\frac{\partial^2}{\partial y^2}\overline{\tau}^L, \overline{\tau}^L\right\rangle_{L^2,L^2} + \left(\frac{\alpha}{4\varepsilon^2} - \frac{C_3}{2}\right)\left|y\overline{\tau}^L\right|^2_{L^2,L^2} + \left(\frac{C_1}{2\varepsilon} - \frac{\alpha}{4\varepsilon} - \frac{C(a_1)}{2\sqrt{\varepsilon}} - \frac{C_2}{2}\right)\left|\overline{\tau}^L\right|^2_{L^2,L^2}.$$

Now we choose $\alpha \in (0,1]$ such that $C_1 - \frac{\alpha}{2} \geq \frac{C_1}{2}$ and therefore

$$\left|\left(\frac{1}{\varepsilon}\tilde{D}_X + D_N\right)\overline{\tau}^L\right|^2_{L^2,L^2} \geq \frac{\alpha}{4}\left\langle -\frac{\partial^2}{\partial y^2}\overline{\tau}^L, \overline{\tau}^L\right\rangle_{L^2,L^2} + \left(\frac{\alpha}{4\varepsilon^2} - \frac{C_3}{2}\right)\left|y\overline{\tau}^L\right|^2_{L^2,L^2} + \left(\frac{C_1}{4\varepsilon} - \frac{C(a_1)}{2\sqrt{\varepsilon}} - \frac{C_2}{2}\right)\left|\overline{\tau}^L\right|^2_{L^2,L^2}. \tag{4.2.14}$$

By using the ellipitic estimate of Lemma 4.2.4 for the operator $D_N + D_H$ we see that

$$\frac{\alpha}{4}\left\langle -\frac{\partial^2}{\partial y^2}\overline{\tau}^L, \overline{\tau}^L\right\rangle_{L^2,L^2} + \left|D_H\overline{\tau}^L\right|^2_{L^2,L^2} \geq C\left|\overline{\tau}^L\right|^2_{W^1,L^2} - C\left|\overline{\tau}^L\right|^2_{L^2,L^2} - C\left|\overline{\tau}^L\right|^2_{L^2,W^1}. \tag{4.2.15}$$

Finally if we put together the estimates (4.2.14) and (4.2.15) we get

$$\begin{aligned}\left|L_\varepsilon\overline{\tau}^L\right|^2_{L^2,L^2} \geq & C\left|\overline{\tau}^L\right|^2_{W^1,L^2} + \left(\frac{C_1}{\varepsilon^2} - C_2\right)\left|y\overline{\tau}^L\right|^2_{L^2,L^2} \\ & + \left(\frac{C_3}{\varepsilon} - \frac{C_4(a_1)}{\sqrt{\varepsilon}} - C_5\right)\left|\overline{\tau}^L\right|^2_{L^2,L^2} - C_5\left|\overline{\tau}^L\right|^2_{L^2,W^1}.\end{aligned} \tag{4.2.16}$$

Combining the estimates

We put together the estimates of the cases above, namely estimates (4.2.8), (4.2.9), (4.2.10) and (4.2.16) to see that

$$\begin{aligned}
\left\|\frac{1}{\varepsilon}D_{M,\varepsilon}\tau\right\|^2_{L^2,L^2} &= \left|h^{1/2}\frac{1}{\varepsilon}D_{M,\varepsilon}h^{-1/2}\overline{\tau}\right|^2_{L^2,L^2}\\
&\geq \frac{1}{2}\left|L_\varepsilon\overline{\tau}\right|^2_{L^2,L^2} - \left|\left(R_\varepsilon+G\right)\overline{\tau}\right|^2_{L^2,L^2}\\
&\geq \frac{1}{2}\left|L_\varepsilon\overline{\tau}^L\right|^2_{L^2,L^2} + \frac{1}{2}\left|L_\varepsilon\overline{\tau}^W\right|^2_{L^2,L^2} - \left|\left\langle L_\varepsilon\overline{\tau}^L, L_\varepsilon\overline{\tau}^W\right\rangle_{L^2,L^2}\right| - \left|\left(R_\varepsilon+G\right)\overline{\tau}\right|^2_{L^2,L^2}\\
&\geq C_1\left|\overline{\tau}^L\right|^2_{W^1,L^2} + C_2\left|\overline{\tau}^W\right|^2_{W^1,L^2} - \left(C_3a_1^2+C_4a_1\right)\left|\overline{\tau}\right|^2_{W^1,L^2}\\
&\quad+\left(\frac{C_5}{\varepsilon}-\frac{C_6(a_1)}{\sqrt{\varepsilon}}-C_7-C_8\varepsilon^2-C_9a_1^2\right)\left|\overline{\tau}^L\right|^2_{L^2,L^2}\\
&\quad+\left(\frac{C_{10}}{\varepsilon^2}-C_{11}\varepsilon^2-C_{12}a_1^2\right)\left|\overline{\tau}^W\right|^2_{L^2,L^2}\\
&\quad-C_{13}\left|\overline{\tau}^L\right|^2_{L^2,W^1}+\frac{C_{14}}{\varepsilon^2}\left|\overline{\tau}^W\right|^2_{L^2,W^1}\\
&\quad+\left(\frac{C_{15}}{\varepsilon^2}-C_{16}\right)\left|y\overline{\tau}^L\right|^2_{L^2,L^2}.
\end{aligned}$$

We know by definition that $|\overline{\tau}|_{L^2,L^2} = \|\tau\|_{L^2,L^2}$ and

$$|\overline{\tau}|^2_{W^1,L^2} = \|\tau\|^2_{L^2,L^2} + \left\|h^{-1/2}\nabla^{\pi_*V\otimes\Sigma B,0}h^{1/2}\tau\right\|^2_{L^2,L^2} \leq C\,\|\tau\|^2_{W^1,L^2}$$

since we chose a small enough such that h and its derivatives are bounded.
Therefore we see that if we choose a_1 small enough, there exist constants $C_1, C_2, C_3 > 0$ and an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\tau \in \Gamma\left(B, \pi_*V \otimes \Sigma B\right)$ whose support is in N_{a_1} and $q_\varepsilon\sigma = 0$ the follwing holds true

$$\left\|\frac{1}{\varepsilon}D_{M,\varepsilon}\tau\right\|^2_{L^2,L^2} \geq C_1\|\tau\|^2_{W^1,L^2} + \frac{C_2}{\varepsilon}\|\tau\|^2_{L^2,L^2} + \frac{C_3}{\varepsilon}\|\tau\|^2_{L^2,W^1}.$$

For the τ^L-part we estimated some of the $\|\tau\|_{L^2,L^2}$-norm by the $\|\tau\|_{L^2,W^1}$-norm. But L is a finite-dimensional bundle so these norms are equivalent. □

4.2.10 Theorem (cf. [BL91, Theorem 9.11])**.** *Let $a_1 \leq a/2$ as in Proposition 4.2.9. Then there exist constants $C_1, C_2, C_3 > 0$ such that for ε small enough and for all $\tau \in \operatorname{im} q_\varepsilon^\perp \cap W^1\left(M, V \otimes \pi^*\Sigma B\right)$*

$$\left\|\frac{1}{\varepsilon}D_{M,\varepsilon}\tau\right\|^2_{L^2,L^2} \geq \frac{C_1}{\varepsilon}\|\tau\|^2_{L^2,L^2} + \frac{C_2}{\varepsilon}\|\tau\|^2_{L^2,W^1} + C_3\|\tau\|^2_{W^1,L^2}.$$

Proof. We combine the estimates of Lemma 4.2.8 and Proposition 4.2.9 as in the 3rd step of the proof of Theorem 9.11 [BL91, pp. 115]. □

4.2.2 Main estimates

4.2.11 Theorem (cf. [BL91, Theorem 9.8]). *There exist constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $s \in W^1(B_0, \ker D_X \otimes \Sigma B_0)$ the following estimate holds true*

$$\left\| \left(J_\varepsilon^{-1} \frac{1}{\varepsilon} D_{M,\varepsilon,1} J_\varepsilon + D_{B_0} \right) s \right\|_{L^2(B_0, \ker D_X \otimes \Sigma B_0)} \leq C\sqrt{\varepsilon}\, \|s\|_{W^1(B_0,\ker D_X \otimes \Sigma B_0)},$$

where D_{B_0} denotes the twisted Dirac operator on $\ker D_X \otimes \Sigma B_0$.
Note that $J_\varepsilon^{-1}\frac{1}{\varepsilon}D_{M,\varepsilon,1}J_\varepsilon$ converges to $-D_{B_0}$ which is explained in the considerations for the Clifford actions in Remark 4.1.8.

Proof. We follow the ideas of the proof of [BL91, Theorem 9.8] and adopt carefully the estimates to our situation.
By Lemma 4.1.11 and Definition 4.2.3 we know that

$$\begin{aligned} J_\varepsilon^{-1}\frac{1}{\varepsilon}D_{M,\varepsilon,1}J_\varepsilon &= I_\varepsilon^{-1} p_\varepsilon \left(h^{1/2} \frac{1}{\varepsilon} D_{M,\varepsilon} h^{-1/2} \right) I_\varepsilon \\ &= I_\varepsilon^{-1} p_\varepsilon \left(\frac{1}{\varepsilon}\tilde{D}_X + D_N + D_H + G + R_\varepsilon \right) I_\varepsilon. \end{aligned}$$

So by Lemma 4.2.6

$$J_\varepsilon^{-1}\frac{1}{\varepsilon}D_{M,\varepsilon,1}J_\varepsilon = I_\varepsilon^{-1} p_\varepsilon \left(D_H + G + R_\varepsilon \right) I_\varepsilon.$$

(1) Let us first consider the term $p_\varepsilon D_H I_\varepsilon$. We will see that this operator converges in the right sense to the twisted Dirac operator

$$\begin{aligned} -D_{B_0} &= -\sum_{\alpha=1}^{m-1} \left(\nabla_{f_\alpha}^{\ker} \otimes c_{B_0}(f_\alpha) + 1 \otimes c_{B_0}(f_\alpha) \nabla_{f_\alpha}^{\Sigma B_0} \right) \\ &= \sum_{\alpha=1}^{m-1} \left(\nabla_{f_\alpha}^{\ker} \otimes (-c_{B_0}(f_\alpha)) + 1 \otimes (-c_{B_0}(f_\alpha))\, \nabla_{f_\alpha}^{\Sigma B_0} \right) \end{aligned}$$

or rather $-I_\varepsilon D_{B_0}$ and then we will make use of the fact that I_ε is an isometry with respect to $|\cdot|_{L^2,L^2}$ and the L^2-norm on $\Gamma(B_0, \ker D_X \otimes \Sigma B_0)$.
By using the definitions and Lemma 4.1.11 one can check that for a section $s_1 \otimes s_2 \in \Gamma(B_0, \ker D_X \otimes \Sigma B_0)$

$$\begin{aligned} p_\varepsilon D_H I_\varepsilon(s_1 \otimes s_2) &= \alpha_\varepsilon^{-1/2} \rho(y) e^{-y^2/2\varepsilon} \iota(s_1) \otimes \begin{pmatrix} 0 \\ \sum_{\alpha=1}^{m-1} -c_{B_0}(f_\alpha) \nabla_{f_\alpha}^{\Sigma B_0} s_2 \end{pmatrix} \\ &+ p_\varepsilon \left(\alpha_\varepsilon^{-1/2} \rho(\tilde{y}) e^{-\tilde{y}^2/2\varepsilon} \sum_{\alpha=1}^{m-1} g^* \nabla_{f_\alpha}^{\ker \oplus \mathrm{im}} \iota(s_1) \otimes \begin{pmatrix} 0 \\ -c_{B_0}(f_\alpha) s_2 \end{pmatrix} \right). \end{aligned}$$

Now we know by Lemma 4.1.7 that

$$\left| g^* \nabla_{f_\alpha}^{\ker \oplus \mathrm{im}} \iota(s_1) - \iota\left(\nabla_{f_\alpha}^{\ker} s_1 \right) \right|_{L^2(B_0 \times \{\tilde{y}\}, \pi_* V)} \leq C\, |\tilde{y}|\, \|s_1\|_{L^2(B_0, \ker D_X)}. \tag{4.2.17}$$

Using Proposition 4.1.12 this proves that

$$\left| p_\varepsilon \left(\alpha_\varepsilon^{-1/2} \rho(\tilde{y}) e^{-\tilde{y}^2/2\varepsilon} \sum_{\alpha=1}^{m-1} g^* \nabla^{\ker \oplus \mathrm{im}}_{f_\alpha} \iota(s_1) \otimes \begin{pmatrix} 0 \\ -c_{B_0}(f_\alpha) s_2 \end{pmatrix} \right) \right.$$
$$\left. - p_\varepsilon \left(I_\varepsilon \left(\sum_{\alpha=1}^{m-1} \nabla^{\ker}_{f_\alpha} s_1 \otimes \left(-c_{B_0}(f_\alpha) s_2\right) \right) \right) \right|_{L^2,L^2} \leq C\sqrt{\varepsilon} \left\| s_1 \otimes s_2 \right\|_{L^2(B_0, \ker D_X)}.$$

Combining the above estimates we conclude that

$$\begin{aligned} &\left\| \left(I_\varepsilon^{-1} p_\varepsilon D_X I_\varepsilon + D_{B_0} \right) (s_1 \otimes s_2) \right\|_{L^2(B_0, \ker D_X \otimes \Sigma B_0)} \\ &= \left| \left(p_\varepsilon D_H I_\varepsilon + I_\varepsilon D_{B_0} \right) (s_1 \otimes s_2) \right|_{L^2,L^2} \\ &\leq C\sqrt{\varepsilon} \left\| s_1 \otimes s_2 \right\|_{L^2(B_0, \ker D_X \otimes \Sigma B_0)}. \end{aligned}$$

(2) Now we will prove that the G-part is small

$$\left\| I_\varepsilon^{-1} p_\varepsilon G I_\varepsilon (s_1 \otimes s_2) \right\|_{L^2(B_0, \ker D_X \otimes \Sigma B_0)} \leq C\sqrt{\varepsilon} \left\| s_1 \otimes s_2 \right\|_{L^2(B_0, \ker D_X \otimes \Sigma B_0)}.$$

For this we consider the function

$$\gamma_i = g_{\pi_* V} \left(\left(g^* \nabla^{\pi_* V}_{f_\alpha} - g^* \nabla^{\ker \oplus \mathrm{im}}_{f_\alpha} \right) \iota(s_1), \iota(\sigma_i) \right) : U_i \times (-a, a) \to \mathbb{R},$$

where U_i is a covering of B_0 and $\sigma_i \colon U_i \to \ker D_X$ a local orthonormal frame as in Lemma 4.1.11.
At $\tilde{y} = 0$ we know that $\iota_{x,0}(s_1), \iota_{x,0}(\sigma_i) \in \ker D_{X,x}$, but $\nabla^{\pi_* V} - \nabla^{\ker \oplus \mathrm{im}}$ interchanges $\ker D_X$ and $\mathrm{im}\, D_X$ and therefore

$$\gamma_i|_{U_i} \equiv 0.$$

Therefore we can check that

$$|\gamma_i(x, \tilde{y})| \leq C\, |\tilde{y}|\, \|s_1\|_{g_{\ker D_X}}.$$

Writing down the explicit formula for $p_\varepsilon G I_\varepsilon (s_1 \otimes s_2)$, we see by the above estimate and by using Proposition 4.1.12 that

$$\begin{aligned} &\left\| I_\varepsilon^{-1} p_\varepsilon G I_\varepsilon (s_1 \otimes s_2) \right\|_{L^2(B_0, \ker D_X \otimes \Sigma B_0)} \\ &= \left| p_\varepsilon G I_\varepsilon (s_1 \otimes s_2) \right|_{L^2,L^2} \\ &\leq C\sqrt{\varepsilon} \left\| s_1 \otimes s_2 \right\|_{L^2(B_0, \ker D_X \otimes \Sigma B_0)}. \end{aligned}$$

(3) In the last part of the proof we show that the remainder term R_ε is small. This follows by Proposition 4.2.5, Proposition 4.1.12 and the same considerations as in part (2) of this proof

$$\left| p_\varepsilon R_\varepsilon I_\varepsilon (s_1 \otimes s_2) \right|_{L^2,L^2} \leq C\sqrt{\varepsilon} \left\| s_1 \otimes s_2 \right\|_{W^1(B_0, \ker D_X \otimes \Sigma B_0)}.$$

$\square$

4.2.12 Theorem ([BL91, Theorem 9.10])**.** *There exist constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in \operatorname{im} q_\varepsilon^\perp \cap W^1(M, V \otimes \pi^*\Sigma B)$*

$$\left\| \frac{1}{\varepsilon} D_{M,\varepsilon,2}\tau \right\|_{L^2,L^2} \leq C\left(\sqrt{\varepsilon}\, \|\tau\|_{W^1,L^2} + \|\tau\|_{L^2,L^2}\right)$$

and for all $\tau \in \operatorname{im} q_\varepsilon \cap W^1(M, V \otimes \Sigma B)$

$$\left\| \frac{1}{\varepsilon} D_{M,\varepsilon,3}\tau \right\|_{L^2,L^2} \leq C\left(\sqrt{\varepsilon}\, \|\tau\|_{W^1,L^2} + \|\tau\|_{L^2,L^2}\right).$$

Proof. The proof follows exactly as the proof of [BL91, Theorem 9.10] by using Proposition 4.2.7 instead of [BL91, Proposition 9.9]. □

4.2.13 Theorem (cf. [BL91, Theorem 9.14])**.** *There exist constants $C_1, C_2, C_3, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all sections $\tau \in \operatorname{im} q_\varepsilon^\perp \cap W^1(M, V \otimes \pi^*\Sigma B)$*

$$\left\| \frac{1}{\varepsilon} D_{M,\varepsilon,4}\tau \right\|_{L^2,L^2} \geq \frac{C_1}{\sqrt{\varepsilon}} \|\tau\|_{L^2,L^2} + \frac{C_2}{\sqrt{\varepsilon}} \|\tau\|_{L^2,W^1} + C_3 \|\tau\|_{W^1,L^2}.$$

Proof. As in the proof of [BL91, Theorem 9.14] we know that

$$D_{M,\varepsilon}\tau = D_{M,\varepsilon,2}\tau + D_{M,\varepsilon,4}\tau.$$

By Theorem 4.2.10 we know that

$$\left\| \frac{1}{\varepsilon} D_{M,\varepsilon}\tau \right\|_{L^2,L^2} \geq \frac{C_1}{\sqrt{\varepsilon}} \|\tau\|_{L^2,L^2} + \frac{C_2}{\varepsilon} \|\tau\|_{L^2,W^1} + C_3 \|\tau\|_{W^1,L^2}.$$

We know by Theorem 4.2.12 that

$$\left\| \frac{1}{\varepsilon} D_{M,\varepsilon,2}\tau \right\|_{L^2,L^2} \leq C \|\tau\|_{L^2,L^2} + C\sqrt{\varepsilon}\, \|\tau\|_{W^1,L^2}.$$

If we combine these two estimates the claimed estimate holds true for all ε small enough. □

4.3 Calculation of the integral for large times

We want to use the results of the previous Section 4.2 to obtain estimates of the resolvent of $D_{M,\varepsilon}$. As in [BL91, Section IX. e)] and [Goe14, Section 2] we use the Schur complement method. Then we use holomorphic functional calculus as in Section 3.2 to compute the large time contribution

$$\frac{1}{\sqrt{\pi}} \int_{\varepsilon^{\alpha-2}}^{\infty} \operatorname{tr}\left(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2}\right) \frac{dt}{\sqrt{t}},$$

as $\varepsilon \to 0$ for a certain choice of $0 < \alpha < 1$.

4.3.1 Resolvent estimates

4.3.1 Proposition ([BL91, Proposition 9.16])**.** *There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the operator*

$$\overline{D_{M,\varepsilon}} = \begin{pmatrix} D_{M,\varepsilon,1} & 0 \\ 0 & D_{M,\varepsilon,4} \end{pmatrix}$$

is self-adjoint with domain $W^1\left(M, V \otimes \pi^*\Sigma B\right)$.

Proof. As the proof of [BL91, Proposition 9.16] we want to make use of the Kato-Rellich theorem [RS75, Theorem X.12]. We just have to adapt our estimates. We already know that $D_{M,\varepsilon}$ is self-adjoint with domain $W^1\left(M, V \otimes \pi^*\Sigma B\right)$ and $\overline{D_{M,\varepsilon}} - D_{M,\varepsilon}$ is symmetric. It remains to show that $\overline{D_{M,\varepsilon}} - D_{M,\varepsilon}$ is $D_{M,\varepsilon}$-bounded with relative bound $C < 1$ for all ε small enough, i.e., that

$$\left\|\left(\overline{D_{M,\varepsilon}} - D_{M,\varepsilon}\right)\tau\right\|_{L^2,L^2} \leq C\left\|D_{M,\varepsilon}\tau\right\|_{L^2,L^2} + \|\tau\|_{L^2,L^2}.$$

We know that

$$D_{M,\varepsilon}^2 = \tilde{D}_X^2 + \varepsilon[\tilde{D}_X, D_{B,\varepsilon}] + \varepsilon^2 D_{B,\varepsilon}^2,$$

$D_{B,\varepsilon} = \tilde{D}_B + \varepsilon\tilde{T}$, and use the elliptic estimate for D_X and $\tilde{D}_B$, see Lemma 4.2.2, and the fact that $[\tilde{D}_X, D_{B,\varepsilon}] = [\tilde{D}_X, \tilde{D}_B + \varepsilon\tilde{T}]$ is a fibrewise differential operator of order one plus an endomorphism, see Lemma 4.2.1. Therefore

$$\begin{aligned} \left\|D_{M,\varepsilon}\tau\right\|_{L^2,L^2}^2 \geq {} & \left(-C_1 - C_2\varepsilon^2 + C_3\varepsilon^4\right)\|\tau\|_{L^2,L^2}^2 \\ & + \left(C_4 - C_5\varepsilon - C_6\varepsilon^2\right)\|\tau\|_{L^2,W^1}^2 + \left(C_7\varepsilon^2 - C_8\varepsilon^3\right)\|\tau\|_{W^1,L^2}^2 \end{aligned}$$

and we see that there exists an $\varepsilon_0 > 0$ and constants $K_1, K_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\varepsilon\|\tau\|_{W^1,L^2} \leq K_1\left\|D_{M,\varepsilon}\tau\right\|_{L^2,L^2} + K_2\|\tau\|_{L^2,L^2}.$$

The remaining part of the proof follows as the proof of [BL91, Proposition 9.16]. Combining the estimates of Theorem 4.2.12 and Lemma 4.1.12 leads to

$$\begin{aligned} \left\|\left(\overline{D_{M,\varepsilon}} - D_{M,\varepsilon}\right)\tau\right\|_{L^2,L^2} & \leq C_2\varepsilon^{3/2}\|\tau\|_{W^1,L^2} + C_2\varepsilon\|\tau\|_{L^2,L^2} \\ & \leq C_1K_1\sqrt{\varepsilon}\left\|D_{M,\varepsilon}\tau\right\|_{L^2,L^2} + \left(K_2\sqrt{\varepsilon} + C_2\varepsilon\right)\|\tau\|_{L^2,L^2}. \end{aligned}$$

Now it is clear that that the difference $\overline{D_{M,\varepsilon}} - D_{M,\varepsilon}$ is $D_{M,\varepsilon}$-bounded with relative bound $C < 1$ for all ε small enough and therefore the Kato-Rellich theorem proves the statement. □

4.3.2 Remark. The fact that $\overline{D_{M,\varepsilon}}$ is self-adjoint implies that the spectrum of $D_{M,\varepsilon,4}$ is contained in $\mathbb{R}$ and therefore its resolvent

$$R_\varepsilon(z) = \left(z - \varepsilon^{-1}D_{M,\varepsilon,4}\right)^{-1} : \operatorname{im} q_\varepsilon^\perp \to \operatorname{im} q_\varepsilon^\perp$$

exists as a bounded linear operator on $L^2\left(M, V \otimes \pi^*\Sigma B\right)$ for all $z \in \mathbb{C}\backslash\mathbb{R}$. In particular if $|\operatorname{Im} z| \geq C$

$$\left\|R_\varepsilon(z)\tau\right\|_{L^2(M,V\otimes\pi^*\Sigma B)} \leq \frac{1}{C}\|\tau\|_{L^2(M,V\otimes\pi^*\Sigma B)}.$$

4.3.3 Definition. As in Remark 4.3.2 we define for $z \notin \operatorname{spec} \varepsilon^{-1} D_{M,\varepsilon,4}$ the *resolvent*

$$R_\varepsilon(z) = \left(z - \varepsilon^{-1} D_{M,\varepsilon,4}\right)^{-1} : \operatorname{im} q_\varepsilon^\perp \to \operatorname{im} q_\varepsilon^\perp$$

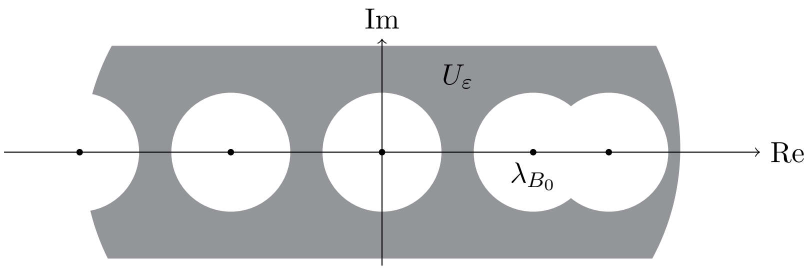
and the *Schur complement*

$$M_\varepsilon(z) = z - \varepsilon^{-1} D_{M,\varepsilon,1} - \varepsilon^{-2} D_{M,\varepsilon,2} R_\varepsilon(z) D_{M,\varepsilon,3}.$$

Furthermore we define the subset $U_\varepsilon \subset \mathbb{C}$ by

$$U_\varepsilon = \left\{ z \in \mathbb{C} \;\middle|\; |z| \leq \frac{G_1}{\sqrt{\varepsilon}} \text{ and } \inf_{\mu \in \operatorname{spec} D_{B_0}} |z - \mu| \geq G_2 \right\},$$

where $0 < G_1 < \frac{C_1}{2}$ with C_1 from Theorem 4.2.13 will be fixed in the proof of Proposition 4.3.6 and $0 < G_2 \leq 1$ is chosen, such that $G_2 < \lambda_{B_0}/2$ where λ_{B_0} is the smallest absolut value of an eigenvalue of D_{B_0}.



4.3.4 Remark. We want to use the resolvent $R_\varepsilon(z)$ of $\varepsilon^{-1} D_{M,\varepsilon,4}$ and its Schur complement $M_\varepsilon(z)$ to compute the full resolvent of $D_{M,\varepsilon}$ and also to get the desired estimates. We will prove that $R_\varepsilon(z)$ exists and $M_\varepsilon(z)$ is invertible for all $z \in U_\varepsilon$. Therefore $z \notin \operatorname{spec} \varepsilon^{-1} D_{M,\varepsilon}$ since

$$\begin{aligned}&\left(z - \varepsilon^{-1} D_{M,\varepsilon}\right)^{-1} \\ &\quad = \begin{pmatrix} M_\varepsilon(z)^{-1} & \varepsilon^{-1} M_\varepsilon(z)^{-1} D_{M,\varepsilon,2} R_\varepsilon(z) \\ \varepsilon^{-1} R_\varepsilon(z) D_{M,\varepsilon,3} M_\varepsilon(z)^{-1} & R_\varepsilon(z)\left(1 + \varepsilon^{-2} D_{M,\varepsilon,3} M_\varepsilon(z)^{-1} D_{M,\varepsilon,2} R_\varepsilon(z)\right) \end{pmatrix}.\end{aligned}$$

4.3.5 Lemma ([BL91, Proposition 9.18])**.** *There exists an $\varepsilon_0 > 0$ and a constant $C > 0$ such that for all $z \in \mathbb{C}$, $|z| \leq \frac{C_1}{2\sqrt{\varepsilon}}$, $\varepsilon \in (0, \varepsilon_0)$ and $p \geq 2 \dim M + 1$*

$$\|R_\varepsilon(z)\|_{0,0} \leq C\sqrt{\varepsilon}\,|z|\,, \tag{4.3.1}$$

$$\|R_\varepsilon(z)\|_{0,1} \leq C\,|z|\,, \tag{4.3.2}$$

$$\|R_\varepsilon(z)\|_p \leq C\,|z|\,, \tag{4.3.3}$$

where $\|.\|_{0,0}$ *and* $\|.\|_{0,1}$ *denote the operator norm for operators*

$$L^2\left(M, V\otimes\pi^*\Sigma B\right)\to L^2\left(M, V\otimes\pi^*\Sigma B\right) \text{ or } L^2\left(M, V\otimes\pi^*\Sigma B\right)\to W^1\left(M, V\otimes\pi^*\Sigma B\right)$$

respectively and $\|.\|_p$ *denotes the p-Schatten norm on* $L^2\left(M, V\otimes\pi^*\Sigma B\right)$.

Proof. The first inequality (4.3.1) follows directly by Theorem 4.2.13, since it states in particular that

$$\left\|\varepsilon^{-1}D_{M,\varepsilon,4}\tau\right\|_{L^2(M,V\otimes\pi^*\Sigma B)}\geq\frac{C_1}{\sqrt{\varepsilon}}\left\|\tau\right\|_{L^2(M,V\otimes\Sigma B)}$$

and we chose $|z|\leq\frac{C_1}{2\sqrt{\varepsilon}}$.
For estimate (4.3.2) we first prove an elliptic estimate for $\varepsilon^{-1}D_{M,\varepsilon,4}$. For that we recall that under the identification $\Gamma\left(B,\pi_*V\otimes\Sigma B\right)\cong\Gamma\left(M,V\otimes\pi^*\Sigma B\right)$ the norms $\|.\|_{L^2,L^2}$ and $\|.\|_{L^2(M,V\otimes\pi^*\Sigma B)}$ correspond and $\|.\|_{W^1,L^2}+\|.\|_{L^2,W^1}$ and $\|.\|_{W^1(M,V\otimes\pi^*\Sigma B)}$ are equivalent. Hence the estimate of Theorem 4.2.13 leads to

$$\|\tau\|_{W^1(M,V\otimes\pi^*\Sigma B)}\leq C\left\|\varepsilon^{-1}D_{M,\varepsilon,4}\tau\right\|_{L^2(M,V\otimes\pi^*\Sigma B)}+C\left\|\tau\right\|_{L^2(M,V\otimes\pi^*\Sigma B)}. \tag{4.3.4}$$

Using this we see that

$$\begin{aligned}&\|R_\varepsilon(i)\tau\|_{W^1(M,V\otimes\pi^*\Sigma B)}\\ &\leq C\left\|\varepsilon^{-1}D_{M,\varepsilon,4}R_\varepsilon(i)\tau\right\|_{L^2(M,V\otimes\pi^*\Sigma B)}+C\left\|R_\varepsilon(i)\tau\right\|_{L^2(M,V\otimes\pi^*\Sigma B)}\\ &\leq C\left\|\tau\right\|_{L^2(M,V\otimes\pi^*\Sigma B)},\end{aligned}$$

where we used (4.3.1) in the last step, since for ε small enough $|i|=1\leq\frac{C_1}{2\sqrt{\varepsilon}}$. Using

$$\|R_\varepsilon(z)\|_{0,1}\leq\|R_\varepsilon(i)\|_{0,1}+\|R_\varepsilon(i)\|_{0,1}\left|(i-z)\right|\|R_\varepsilon(z)\|_{0,0}$$

proves (4.3.2).
The last inequality (4.3.3) follows from inequality (4.3.2) and the fact that for $p\geq 2\dim M+1$

$$\left\|(i-D_M)^{-1}\right\|_p\leq C,$$

since

$$\|R_\varepsilon(z)\|_p\leq\left\|(i-D_M)^{-1}\right\|_p\|i-D_M\|_{1,0}\|R_\varepsilon(z)\|_{0,1}\leq C\left|z\right|.$$

□

4.3.6 Proposition ([BL91, Theorem 9.21])**.** *There exists an* ε_0 *such that for all* $\varepsilon\in(0,\varepsilon_0)$ *and* $z\in U_\varepsilon$ *the operator*

$$M_\varepsilon(z)=z-\varepsilon^{-1}D_{M,\varepsilon,1}-\varepsilon^{-2}D_{M,\varepsilon,2}R_\varepsilon(z)D_{M,\varepsilon,3}$$

is invertible. Furthermore there exists a constant $C > 0$ such that

$$\begin{aligned}\left\|M_\varepsilon(z)^{-1}\right\|_{0,0} &\leq C,\\ \|M_\varepsilon(z)\|_p &\leq C\,|z|\,,\\ \left\|\varepsilon^{-1}D_{M,\varepsilon,3}M_\varepsilon(z)^{-1}\right\|_{0,0} &\leq C,\end{aligned}$$

where $p \geq 2\dim B_0 + 1$.

Proof. The statement follows just as [BL91, Theorem 9.21] by writing

$$M_\varepsilon(z) = m_\varepsilon(z)\left(z + J_\varepsilon D_{B_0} J_\varepsilon^{-1}\right),$$

where

$$\begin{aligned}m_\varepsilon(z) = 1 -& \left(\varepsilon^{-1}D_{M,\varepsilon,1}J_\varepsilon + J_\varepsilon D_{B_0}\right)\left(z + D_{B_0}\right)^{-1}J_\varepsilon^{-1}\\ &- \varepsilon^{-2}D_{M,\varepsilon,2}R_\varepsilon(z)D_{M,\varepsilon,3}\left(z + J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}.\end{aligned}$$

Then we use Theorem 4.2.11 and the fact that D_{B_0} is a Dirac operator to see that

$$\left\|\left(\varepsilon^{-1}D_{M,\varepsilon,1}J_\varepsilon + J_\varepsilon D_{B_0}\right)\left(z + D_{B_0}\right)^{-1}J_\varepsilon^{-1}\right\|_{0,0} \leq C\sqrt{\varepsilon}\,|z|\,,$$

Theorem 4.2.12, equation (4.3.1) and (4.3.2) of Lemma 4.3.5 to see that

$$\left\|\varepsilon^{-1}D_{M,\varepsilon,2}R_\varepsilon(z)\right\|_{0,0} \leq C\sqrt{\varepsilon}\,|z|$$

and finally we see by using Theorem 4.2.12, Lemma 4.1.12 and the fact that D_{B_0} is a Dirac operator as in [BL91, Proposition 9.19] that

$$\left\|\varepsilon^{-1}D_{M,\varepsilon,3}\left(z + D_{B_0}\right)^{-1}\right\|_{0,0} \leq C.$$

Combining the above estimates proves that for ε small enough

$$\begin{aligned}&\Big\|\left(\varepsilon^{-1}D_{M,\varepsilon,1}J_\varepsilon + J_\varepsilon D_{B_0}\right)\left(z + D_{B_0}\right)^{-1}J_\varepsilon^{-1}\\ &\quad -\varepsilon^{-2}D_{M,\varepsilon,2}R_\varepsilon(z)D_{M,\varepsilon,3}\left(z + J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\Big\|_{0,0} \leq C\sqrt{\varepsilon}\,|z|\,.\end{aligned} \tag{4.3.5}$$

Therefore we define the constant $G_1 = \frac{1}{2C}$ in the definition of U_ε and hence all $z \in U_\varepsilon$ satisfy $|z| \leq \frac{1}{2C\sqrt{\varepsilon}}$,

$$\|m_\varepsilon(z) - 1\|_{0,0} \leq \frac{1}{2}$$

for all $z \in U_\varepsilon$. This proves that $m_\varepsilon(z)$ is invertible for all ε small enough and all $z \in U_\varepsilon$ with

$$\left\|m_\varepsilon(z)^{-1}\right\|_{0,0} \leq C'. \tag{4.3.6}$$

On the other hand we know that $z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}$ is also invertible for all $z\in U_\varepsilon$ since by the choice of U_ε, $\inf_{\nu\in\operatorname{spec}D_{B_0}}|z-\nu|\geq G_2$ and

$$\left\|\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right\|_{0,0}\leq\frac{1}{G_2},$$

which implies that for ε small enough and all $z\in U_\varepsilon$ the Schur complement $M_\varepsilon(z)$ is invertible and

$$\left\|M_\varepsilon(z)^{-1}\right\|_{0,0}\leq\left\|\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right\|_{0,0}\left\|m_\varepsilon(z)^{-1}\right\|_{0,0}\leq C.$$

The inequality for the p-Schatten norm follows by the inequality

$$\left\|\left(z+D_{B_0}\right)^{-1}\right\|_p\leq C\left|z\right|,$$

for $p\geq 2\dim B_0+1$, since D_{B_0} is a Dirac operator on B_0, and equation (4.3.6)

$$\left\|m_\varepsilon(z)^{-1}\right\|_{0,0}\leq C$$

by

$$\left\|M_\varepsilon(z)^{-1}\right\|_p\leq\left\|\left(z-D_{B_0}\right)^{-1}\right\|_p\left\|m_\varepsilon(z)^{-1}\right\|_{0,0}\leq C_2\left|z\right|.$$

For the last estimate we prove just as in [BL91, Proposition 9.19] that

$$\left\|\varepsilon^{-1}D_{M,\varepsilon,3}\left(z-J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right\|_{0,0}\leq C$$

by using the estimates of Lemma 4.1.12 and Theorem 4.2.12. Then the statement follows for all $z\in U_\varepsilon$ by equation (4.3.6) since

$$\left\|\varepsilon^{-1}D_{M,\varepsilon,3}M_\varepsilon(z)^{-1}\right\|_{0,0}\leq\left\|\varepsilon^{-1}D_{M,\varepsilon,3}\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right\|_{0,0}\left\|m_\varepsilon(z)^{-1}\right\|_{0,0}\leq C.$$

□

4.3.7 Proposition. *There exist constants $C_1,C_2,\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, $z\in U_\varepsilon$ and $p\geq 2\dim M+1$ the following inequalities hold true*

$$\left\|\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}-\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right\|_{0,0}\leq C_1\sqrt{\varepsilon}\left|z\right|, \tag{4.3.7}$$

$$\left\|\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}\right\|_p\leq C_2\left|z\right|. \tag{4.3.8}$$

Proof. For the first estimate (4.3.7) we know that

$$\begin{aligned}&\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}-\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\\&=\begin{pmatrix}M_\varepsilon(z)^{-1}-\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1} & \varepsilon^{-1}M_\varepsilon(z)^{-1}D_{M,\varepsilon,2}R_\varepsilon(z)\\ \varepsilon^{-1}R_\varepsilon(z)D_{M,\varepsilon,3}M_\varepsilon(z)^{-1} & R_\varepsilon(z)\left(1+\varepsilon^{-2}D_{M,\varepsilon,3}M_\varepsilon(z)^{-1}D_{M,\varepsilon,2}R_\varepsilon(z)\right)\end{pmatrix}\end{aligned}$$

So let us start with the left upper corner. We know by equations (4.3.5) and (4.3.6) in the proof of Proposition 4.3.6 and by the choice of U_ε that for ε small enough and for all $z \in U_\varepsilon$

$$\begin{aligned}\left\| M_\varepsilon(z)^{-1} - \left(z + J_\varepsilon D_{B_0} J_\varepsilon^{-1}\right)^{-1} \right\|_{0,0} &\leq \left\| \left(z + J_\varepsilon D_{B_0} J_\varepsilon^{-1}\right)^{-1} \right\|_{0,0} \left\| m_\varepsilon(z) - 1 \right\|_{0,0} \\ &\leq C\sqrt{\varepsilon}\,|z|\,.\end{aligned}$$

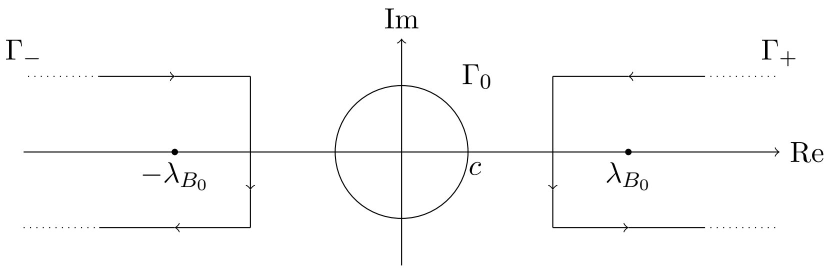
For the remaining three entries we use the inequalities of Proposition 4.3.6, Theorem 4.2.12 and the estimates (4.3.1) and (4.3.2) of Lemma 4.3.5 to see that they are all bounded in the $\|.\|_{0,0}$-norm by $C\sqrt{\varepsilon}\,|z|$.
For the inequality (4.3.8) the same estimates as above combined with estimate (4.3.3) of Lemma 4.3.5 and the fact that the p-Schatten norm of $M_\varepsilon(z)$ is bounded for $z \in U_\varepsilon$ lead to the statement that

$$\left\| \left(z - \varepsilon^{-1} D_{M,\varepsilon}\right)^{-1} \right\|_p \leq C\,|z|$$

for all ε small enough and $z \in U_\varepsilon$. □

4.3.2 Main Theorems

4.3.8 Definition. We define the contour $\Gamma = \Gamma_- \,\dot\cup\, \Gamma_0 \,\dot\cup\, \Gamma_+$ as in [Goe14, Section 2.d]. We denote the smallest absolut value of a non-zero eigenvalue of D_{B_0} by λ_{B_0}. Let $c > 0$ be a constant such that $c < \lambda_{B_0}/2$.



4.3.9 Remark. Since $D_{M,\varepsilon}$ is a self-adjoint operator its spectrum has to be real. Therefore we know by Proposition 4.3.7 that there exists an ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the contour Γ encloses not just the eigenvalues of D_{B_0} but also the eigenvalues of $\varepsilon^{-1} D_{M,\varepsilon}$.

4.3.10 Definition. We choose $\varepsilon_0 > 0$ small enough such that all statements in the previous sections are fulfilled. Then we define projections P_ε and $Q_\varepsilon = 1 - P_\varepsilon$ on $L^2\left(M, V \otimes \pi^*\Sigma B\right)$ for all $\varepsilon \in (0, \varepsilon_0)$ by

$$P_\varepsilon = \frac{1}{2\pi i} \int\limits_{\Gamma_0} \left(z - \varepsilon^{-1} D_{M,\varepsilon}\right)^{-1} dz,$$

$$Q_\varepsilon = 1 - P_\varepsilon = \frac{1}{2\pi i} \int\limits_{\Gamma_+ \cup \Gamma_-} \left(z - \varepsilon^{-1} D_{M,\varepsilon}\right)^{-1} dz.$$

4.3.11 Theorem. *There exists an $0<\alpha<1$ such that*

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^{\alpha-2}}^{\infty}\operatorname{tr}\left(\left(1-P_\varepsilon\right)\circ D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}\circ\left(1-P_\varepsilon\right)\right)\frac{dt}{\sqrt{t}}=-\eta\left(D_{B_0}\right),$$

where again D_{B_0} denotes the twisted Dirac operator on $\Sigma B_0\otimes\ker D_X\to B_0$.

Proof. We write

$$\begin{aligned}&\int\limits_{\varepsilon^{\alpha-2}}^{\infty}\operatorname{tr}\left(\left(1-P_\varepsilon\right)\circ D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}\circ\left(1-P_\varepsilon\right)\right)\frac{dt}{\sqrt{t}}\\&=\int\limits_{\varepsilon^{\alpha}}^{\infty}\operatorname{tr}\left(\left(1-P_\varepsilon\right)\circ \varepsilon^{-1}D_{M,\varepsilon}e^{-t\varepsilon^{-2}D_{M,\varepsilon}^2}\circ\left(1-P_\varepsilon\right)\right)\frac{dt}{\sqrt{t}}\\&=\int\limits_{\varepsilon^{\alpha}}^{\infty}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{\Gamma^+\cup\Gamma^-}ze^{-tz^2}\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}dz\right)\frac{dt}{\sqrt{t}}\end{aligned}$$

as well as

$$-\eta\left(D_{B_0}\right)=\eta\left(-D_{B_0}\right)=\frac{1}{\sqrt{\pi}}\int\limits_{0}^{\infty}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{\Gamma^+\cup\Gamma^-}ze^{-tz^2}\left(z+D_{B_0}\right)^{-1}dz\right)\frac{dt}{\sqrt{t}}.$$

Now

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^{\alpha}}^{\infty}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{\Gamma_\pm\cap U_\varepsilon}ze^{-tz^2}\left(\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}-\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right)\right)\frac{dt}{\sqrt{t}}=0$$

follows in the very same way as the proof using holomorphic functional calculus of

$$\lim_{\varepsilon\to o}\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^{\alpha-2}}^{\infty}\operatorname{tr}\left(\left(1-P_\varepsilon\right)\circ D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}\circ\left(1-P_\varepsilon\right)\right)\frac{dt}{\sqrt{t}}=\eta\left(D_B\otimes\ker D_X\right)$$

for constant kernel dimension of D_X in [Goe14, Proposition 2.10]. By defining holomorphic functions $F_{k,t}^{\pm}\colon\mathbb{C}\to\mathbb{C}$ for all $k\in\mathbb{N}_{>0}$ which satisfy

$$\frac{d^k}{dz^k}F_{k,t}^{\pm}(z)=ze^{-tz^2}\text{ and }\lim_{z\to\pm\infty}F_{k,t}^{\pm}(z)=0,$$

one proves with the resolvent estimates of Proposition 4.3.7 that

$$\begin{aligned}
&\left\|(1-P_\varepsilon)\circ\left(\varepsilon^{-1}D_{M,\varepsilon}e^{-t\varepsilon^{-2}D^2_{M,\varepsilon}}\right)\circ(1-P_\varepsilon)-J_\varepsilon\left(D_{B_0}e^{-tD^2_{B_0}}\right)J_\varepsilon^{-1}\right\|_1\\
&\leq C\int\limits_{\Gamma_\pm\cap U_\varepsilon}\left|F^\pm_{p,t}(z)\right|\sum_{j=0}^{p}\left\|\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}\right\|_p^j\\
&\qquad\cdot\left\|\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}-\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right\|_{0,0}\left\|\left(z+D_{B_0}\right)^{-1}\right\|_p^{p-j}dz\\
&\leq C\sqrt{\varepsilon}\int\limits_{\Gamma_\pm\cap U_\varepsilon}\left|F^\pm_{p,t}z^{p+1}\right|dz\\
&\leq C\sqrt{\varepsilon}t^{-p-3/2}e^{-ct}.
\end{aligned}$$

Now by choosing $0<\alpha<\frac{1}{2p+4}$ we see that

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^\alpha}^{\infty}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{\Gamma_\pm\cap U_\varepsilon}ze^{-tz^2}\left(\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}-\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right)\right)\frac{dt}{\sqrt{t}}=0.$$

Since

$$\left\|\left(z+D_{B_0}\right)^{-1}\right\|_p\leq C\left|z\right|,$$

it is easy to see that the remaining part of $\eta\left(-D_{B_0}\right)$ converges to 0

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int\limits_{0}^{\varepsilon^\alpha}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{\Gamma_+\cup\Gamma_-}ze^{-tz^2}\left(z+D_{B_0}\right)^{-1}dz\right)\frac{dt}{\sqrt{t}}\\
&\quad+\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^\alpha}^{\infty}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{(\Gamma_+\cup\Gamma_-)\cap U_\varepsilon^c}ze^{-tz^2}\left(z+D_{B_0}\right)^{-1}dz\right)\frac{dt}{\sqrt{t}}=0.
\end{aligned}$$

It remains to show that

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^\alpha}^{\infty}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{(\Gamma_+\cup\Gamma_-)\cap U_\varepsilon^c}ze^{-tz^2}\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}dz\right)\frac{dt}{\sqrt{t}}=0.$$

Therefore we check that for ε small enough and all $z\in\Gamma$, so in particular for $z\in\Gamma\cap U_\varepsilon^c$,

$$\begin{aligned}
&\left\|\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}\right\|_p\\
&\leq\left\|\left(i-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}\right\|_p+\left|i-z\right|\left\|\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}\right\|_{0,0}\left\|\left(i-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}\right\|_p\\
&\leq C\left|z\right|
\end{aligned}$$

by equation (4.3.8) in Proposition 4.3.7 and the choice of our contour Γ. Therefore there exist constants $c, C > 0$ such that for all ε small enough

$$\left|\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{(\Gamma_+\cup\Gamma_-)\cap U_\varepsilon^c} ze^{-tz^2}\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}dz\right)\right| \le ce^{-Ct\varepsilon^{-1}}$$

since $|z| \ge \frac{G_2}{\sqrt{\varepsilon}}$ for $z \in U_\varepsilon^c$. But this leeds to

$$\begin{aligned}
&\left|\int\limits_{\varepsilon^\alpha}^{\infty}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{(\Gamma_+\cup\Gamma_-)\cap U_\varepsilon^c} ze^{-tz^2}\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}dz\right)\frac{dt}{\sqrt{t}}\right| \\
&\le c\int\limits_{\varepsilon^\alpha}^{\infty} e^{-Ct\varepsilon^{-1}}\frac{dt}{\sqrt{t}} \\
&= c\sqrt{\varepsilon}\int\limits_{\varepsilon^{\alpha-1}}^{\infty} e^{-Ct}\frac{dt}{\sqrt{t}}
\end{aligned}$$

and since we chose $\alpha < 1$

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^\alpha}^{\infty}\operatorname{tr}\left(\frac{1}{2\pi i}\int\limits_{(\Gamma_+\cup\Gamma_-)\cap U_\varepsilon^c} ze^{-tz^2}\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}dz\right)\frac{dt}{\sqrt{t}} = 0$$

and the theorem follows. □

4.3.12 Theorem. *Let $0 < \alpha < 1$ be chosen as in Theorem 4.3.11 and let us assume that there exists an $\varepsilon_0 > 0$ such that $\dim\ker D_{M,\varepsilon}$ is constant for all $\varepsilon \in (0, \varepsilon_0)$. Then*

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int\limits_{\varepsilon^{\alpha-2}}^{\infty}\operatorname{tr}\left(P_\varepsilon\circ D_{M,\varepsilon}e^{-tD_{M,\varepsilon}^2}\circ P_\varepsilon\right)\frac{dt}{\sqrt{t}} = \sum_{\nu=1}^{\dim\ker D_{B_0}}\operatorname{sign}\left(\lambda_\nu(\varepsilon)\right),$$

where $\lambda_\nu(\varepsilon)$ are the finitely many eigenvalues of $D_{M,\varepsilon}$ that decay faster than ε as $\varepsilon \to 0$.

Proof. By the resolvent estimates of Propostition 4.3.7 we can show by the same methods as in the proof of [Goe14, Proposition 2.10] that

$$\lim_{\varepsilon\to 0}\left\|\frac{1}{2\pi i}\int\limits_{\Gamma_0}\left(\left(z-\varepsilon^{-1}D_{M,\varepsilon}\right)^{-1}-\left(z+J_\varepsilon D_{B_0}J_\varepsilon^{-1}\right)^{-1}\right)dz\right\|_1 = 0$$

and therefore there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\operatorname{rk} P_\varepsilon = \dim\ker D_{B_0} < \infty.$$

But this means that there are just finitely many eigenvalues $\left\{\varepsilon^{-1}\lambda_\nu(\varepsilon)\right\}_{\nu=1,\ldots,\dim\ker D_{B_0}}$ of $\varepsilon^{-1}D_{M,\varepsilon}$ converging to 0 as $\varepsilon \to 0$. Therefore and by our assumption we find a constant $\varepsilon_0 > 0$ small enough such that the signs of $\lambda_\nu(\varepsilon)$ are constant for all $\varepsilon \in (0, \varepsilon_0)$. By using the Mellin formula we see that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\sqrt{\pi}} \int_{\varepsilon^{\alpha-2}}^{\infty} \operatorname{tr}\left(P_\varepsilon \circ D_{M,\varepsilon} e^{-tD^2_{M,\varepsilon}} \circ P_\varepsilon\right) \frac{dt}{\sqrt{t}} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\pi}} \int_{\varepsilon^{\alpha}}^{\infty} \operatorname{tr}\left(P_\varepsilon \circ \varepsilon^{-1} D_{M,\varepsilon} e^{-t\varepsilon^{-2}D^2_{M,\varepsilon}} \circ P_\varepsilon\right) \frac{dt}{\sqrt{t}} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\pi}} \sum_{\nu=1}^{\dim\ker D_{B_0}} \operatorname{sign}\left(\varepsilon^{-1}\lambda_\nu(\varepsilon)\right) \int_{\varepsilon^{\alpha}}^{\infty} e^{-u} \frac{du}{\sqrt{u}} \\
&= \sum_{\nu=1}^{\dim\ker D_{B_0}} \operatorname{sign}\left(\lambda_\nu(\varepsilon)\right)
\end{aligned}$$

for ε small enough as explained above. $\square$

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