Comparison of semimartingales
and Lévy processes with
applications to financial
mathematics

Dissertation
zur Erlangung des Doktorgrades
der Fakultät für Mathematik und Physik
der Albert-Ludwigs-Universität
Freiburg im Breisgau

vorgelegt von

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Juli 2005
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Datum der Promotion: 07. Oktober 2005
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Introduction

This thesis studies univariate and multivariate stochastic and convex type orderings of semimartingales and of finite-dimensional distributions of Lévy processes with several applications, especially in financial mathematics. The orderings are with respect to one of the following order generating function classes \( \mathcal{F} \):

\[
\begin{align*}
\mathcal{F}_{st} &:= \{ f : \mathbb{R}^d \to \mathbb{R}, f \text{ is increasing} \}, \\
\mathcal{F}_{cx} &:= \{ f : \mathbb{R}^d \to \mathbb{R}, f \text{ is convex} \}, \\
\mathcal{F}_{dcx} &:= \{ f : \mathbb{R}^d \to \mathbb{R}, f \text{ is directionally convex} \}, \\
\mathcal{F}_{sm} &:= \{ f : \mathbb{R}^d \to \mathbb{R}, f \text{ is supermodular} \}, \\
\mathcal{F}_{icx} &:= \mathcal{F}_{cx} \cap \mathcal{F}_{st}, \\
\mathcal{F}_{icx} &:= \mathcal{F}_{cx} \cap \mathcal{F}_{st}, \\
\mathcal{F}_{ism} &:= \mathcal{F}_{sm} \cap \mathcal{F}_{st}.
\end{align*}
\]

In the literature, various ordering results for univariate and multivariate random variables are given; we refer to Marshall and Olkin (1979), Tong (1980), Shaked and Shanthikumar (1994), Szekli (1995), and Müller and Stoyan (2002) for results and references. There is also an extended theory for ordering of discrete time processes (like queuing sequences, renewal sequences, Markov chains) or related point processes with a wide variety of applications. Stochastic ordering results w.r.t. \( \mathcal{F}_{st} \) have also been established under various conditions for diffusion type processes (cp. Yamada (1973), Ikeda and Watanabe (1977), O’Brien (1980), Gal’čuk (1982), Gal’čuk and Davis (1982)) and for Markov processes (cp. Massey (1987)). These results are parallel to classical comparison theorems for solutions of differential equations. Several convex comparison results for one-dimensional (exponential) stochastic models have been developed in recent papers in financial mathematics. The main aim in these papers is to derive sharp upper or lower bounds for option prices in incomplete markets or to derive comparison of option prices that result from different martingale measures. The methods used in these papers are based on stochastic calculus (Itô formula) and the propagation of convexity property (see El Karoui, Jeanblanc-Picqué, and Shreve (1998), Frey and Sin (1999), Bellamy and Jeanblanc (2000), Gushchin and Mordecki (2002), Henderson, Hobson, Howison, and Kluge
(2003)), as well as on the coupling method (see Hobson (1998b), Henderson and Hobson (2003), Möller (2004), Henderson (2005)), and also on time change for Brownian motions, (see Eriksson (2004, 2005)). The ordering results are of the following type. The price of a European option with convex payoff function $g$ w.r.t. a stochastic volatility model $S$ with (positive) volatility process $\sigma_t$ is bounded above by the price of this option (with the same convex payoff function $g$) w.r.t. a generalized Black–Scholes model $S^*$ with volatility $\sigma^*(t, S^*_t)$, if the volatilities are ordered in a suitable sense, namely $\sigma_t(\omega) \leq \sigma^*(t, S_t(\omega))$ almost surely. As $\sigma_t$ and $\sigma^*(t, S^*_t)$ are the roots of the differential Gaussian characteristics of the stochastic logarithms $X = \log(S)$, $X^* = \log(S^*)$, the ordering result translates as follows in terms of semimartingale characteristics and convex type orders: Ordering of the differential characteristics of the stochastic logarithms of $S$, $S^*$ in a suitable sense implies convex ordering $S_T \leq_{\text{ex}} S^*_T$ in this case.

In Chapter 1 we extend the stochastic calculus approach of El Karoui, Jeanblanc-Picqué, and Shreve (1998), Frey and Sin (1999), Bellamy and Jeanblanc (2000), and Gushchin and Mordecki (2002) to obtain orderings of terminal values of $d$-dimensional semimartingales $S, S^*$, where $S^*$ is assumed to be Markovian. The ordering is derived in terms of the differential characteristics of the stochastic logarithms $X = \log(S)$, $X^* = \log(S^*)$ and also in terms of the differential characteristics of $S$ and $S^*$. The argument strongly relies on the Markov property of $S^*$ and on the propagation of order property of the backward function $G(t, s) = E^*(g(S^*_T) \mid S^*_t = s)$. We develop a new approach to establish this property for several univariate and multivariate models in Section 1.2.

In Chapter 2 we derive orderings of finite-dimensional distributions of univariate and multivariate Lévy processes in terms of their Lévy characteristics by a different approach. In the first step we establish that ordering of all time marginals of two Markov processes w.r.t. $\mathcal{F}$, where $\mathcal{F}$ is one of the order generating function classes in (1), implies ordering of the finite-dimensional distributions w.r.t. $\mathcal{F}$, if a $\leq_F$-monotone separating transition kernel exists. In Section 2.1 we derive orderings of marginals of compound Poisson processes by a coupling argument, which relies on a random sum representation. We establish that the transition kernels of Lévy processes are $\leq_F$-monotone for all $\mathcal{F}$ in (1), hence ordering of the finite-dimensional distributions follows. Then we establish several cut and domination criteria for the Lévy measures of one-dimensional compound Poisson processes that imply stochastic and (increasing) convex ordering of the processes. In Section 2.2 we extend these results to Lévy processes with infinite Lévy measures. We truncate the Lévy measures around the origin, establish finite-dimensional ordering of the re-
resulting compound Poisson processes, and then obtain ordering of the limits. Again, we derive several cut and domination criteria for univariate Lévy processes in terms of the corresponding Lévy measures. In Section 2.3 we extend these results to processes with independent increments.

In Chapter 3 we give several applications of the ordering results of Chapters 1 and 2, mainly to the field of financial mathematics. In Section 3.1 we obtain non-trivial bounds for European option prices in several univariate and multivariate incomplete market models. These include stochastic volatility models with and without jumps, exponential and stochastic exponential Lévy models and Lévy driven diffusions. Section 3.2 deals with comparison of martingale measures in incomplete market models. We consider several well established martingale measures in a diffusion with jumps model of stochastic exponential type, in a compound Poisson model, in a PII model and in a stochastic volatility model, and obtain orderings of European option prices w.r.t. these measures. In Section 3.3 we derive several ordering results for prices of path-dependent options, some of which use ordering results of Chapters 1 and 2, others extend these results and are derived in a similar fashion. We consider lookback options, Asian options with continuous averaging, American options, and single-barrier options. The ordering results are parallel to the results of Chapters 1 and 2. Ordering the differential characteristics of the underlyings implies ordering of the options prices of the path-dependent options. Finally, in Section 3.4 we obtain ordering results for finite-dimensional distributions of $\alpha$-stable processes, $\alpha \in (1, 2)$, and of NIG processes in the parameters of the models by two different approaches. The first approach is an application of the cut and domination criteria of Chapter 2, the second approach makes use of mixing type representations of generalized hyperbolic (GH) distributions.

I thank my advisor Ludger Rüschendorf for encouraging me to this project and for numerous suggestions and comments. In particular, Chapter 1 is based on Bergenthum and Rüschendorf (2004a) and the first part of Bergenthum and Rüschendorf (2004b), and Chapter 2 is partly based on Bergenthum and Rüschendorf (2004b). I thank my colleagues from Abteilung für Mathematische Stochastik for the good time I had during my PhD studies, and especially I thank Monika Hattenbach for helping with the Latex problems and checking through the manuscript. Finally, I thank my family and friends, and, last but not least, very special thanks go to Charlotte Bergen.
Chapter 1

Comparison of terminal values of semimartingales

In this chapter we derive several comparison results for terminal values of multivariate special semimartingales in terms of differential characteristics (of their stochastic logarithms). This is an extension to the multivariate case of the stochastic calculus approach given in El Karoui, Jeanblanc-Picqué, and Shreve (1998), Frey and Sin (1999), Bellamy and Jeanblanc (2000) and Gushchin and Mordecki (2002). El Karoui, Jeanblanc-Picqué, and Shreve (1998) establish that European option prices w.r.t. stochastic volatility models $S$ are smaller than the prices that come from a diffusion model $S^*$, if the diffusion coefficient $\sigma^*$ dominates the stochastic volatility $\sigma$ in a suitable sense, see also the example in the introduction. The reversed comparison result also holds true. Frey and Sin (1999) consider the special case where the bounding diffusion $S^*$ is given by a geometric Brownian motion with constant volatility. Bellamy and Jeanblanc (2000) establish that European option prices w.r.t. a diffusion with jumps model $S$ are bounded below by the European option price w.r.t. a diffusion $S^*$ that has the same diffusion coefficient as $S$. Gushchin and Mordecki (2002) extend these results to the comparison of European option prices w.r.t. a special semimartingale model $S$ with the corresponding price w.r.t. a Markovian special semimartingale $S^*$.

In all these comparison results the comparison process $S^*$ is Markovian. This assumption seems to be necessary, as the following comparison example for two stochastic volatility models shows. This example was communicated to us by J. Kallsen.

Example 1 (Larger volatility does not imply a larger price in SV models). Let $\bar{\sigma} \in \mathbb{R}_+$ and $W$ be a one-dimensional Brownian motion on a
stochastic basis \((\Omega, A, (A_t)_{t \in [0, T]}, P)\). Let \(\bar{S}\) be the solution of
\[
d\bar{S}_t = \bar{\sigma} \bar{S}_t dW_t, \quad \bar{S}_0 = 1.
\]
For \(K > 1\) we define an \((A_t)\)-adapted process \(\sigma_t^{*}\) as
\[
\sigma_t^{*} := \begin{cases} 
\bar{\sigma}, & \max_{u \leq t} \bar{S}_u \leq K, \\
0, & \text{else}.
\end{cases}
\]
Let \(S^*\) be a solution of \(dS^*_t = \sigma^*_t S^*_t dW_t, \quad S^*_0 = 1\). Let \(0 < \tilde{\sigma} < \bar{\sigma}\) and \(T_0 \in [0, T]\) be fixed. Define an \((A_t)\)-adapted process \(\sigma_t\) as
\[
\sigma_t := \begin{cases} 
\tilde{\sigma}, & \max_{u \leq t} \bar{S}_u \leq K, 0 \leq t \leq T_0, \\
\bar{\sigma}, & \max_{u \leq t} \bar{S}_u \leq K, T_0 < t \leq T, \\
0, & \text{otherwise},
\end{cases}
\]
and let \(S\) be a solution of \(dS_t = \sigma_t S_t dW_t, S_0 = 1\). Then \(\sigma_t \leq \sigma_t^{*}\) and \(S, S^*\) are not Markovian as their volatilities are path-dependent.

The price of a European call option with strike \(K\) is zero under the \(S^*\) model. But as there is a positive probability for paths with \(S_T > S^*_T\), hence \(P(S_T > K) > 0\), it follows that the price of that call option is positive with respect to the \(S\) model, although the volatility \(\sigma\) of \(S\) is smaller than the volatility \(\sigma^*\) of \(S^*\).

We derive stochastic and convex type comparison results for terminal values of \(d\)-dimensional special semimartingales \(S, S^*\) with finite time horizon \(T\) that are defined on stochastic bases \((\Omega, A, (A_t)_{t \in [0, T]}, P)\) and \((\Omega^*, A^*, (A^*_t)_{t \in [0, T]}, P^*)\), respectively. We assume that the characteristics \(B(h), C, \nu\) of \(S\) are absolutely continuous w.r.t. the Lebesgue measure. In this case, \(B(h), C, \nu\) have representation
\[
B^i_t(h) = \int_0^t b^i_s(h) ds, \quad C^{ij}_t = \int_0^t c^{ij}_s ds, \quad \nu(\omega; dt, dx) = dt K_{\omega,t}(dx),
\]
with differential characteristics \(b(h), c\) and \(K\), where \(b(h) = (b^i(h))_{i \leq d}\) is a \(d\)-dimensional predictable process with associated truncation function \(h\), \(c = (c^{ij})_{i,j \leq d}\) is a predictable process with values in \(M_+(d, \mathbb{R})\), the set of all symmetric, positive semidefinite \(d \times d\)-matrices with entries in \(\mathbb{R}\), and \(K_{\omega,t}(dx)\) is a transition kernel from \((\Omega \times [0, T], P)\) into \((\mathbb{R}^d, \mathcal{B}^d)\) that satisfies some conditions, cp. Jacod and Shiryaev (2003, Proposition II.2.9). Similarly, we assume that the characteristics of the comparison process \(S^*\)
are absolutely continuous w.r.t. the Lebesgue measure. \( S \) and \( S^* \) are quasi-leftcontinuous in this case, hence discrete time models are excluded from our considerations. By

\[ S \sim (b(h), c, K)_h \]

we denote that the semimartingale \( S \) has differential characteristics \( b_t(\omega)(h), c_t(\omega), K_{t,\omega} \) associated with a truncation function \( h \). As \( S \) is assumed to be special, \( h = \text{id} \) may serve as “truncation function”, and we write \( S \sim (b(\text{id}), c, K)_\text{id} \) or \( S \sim (b, c, K), b := b(\text{id}) \), in this case.

For the comparison process \( S^* \) we additionally require a Markovian structure, cp. the discussion and example above. We assume that \( S^* \) is a diffusion with jumps in the sense of Jacod and Shiryaev (2003, Definition III.2.18) with differential characteristics that are deterministic functions of time and current state of \( S^* \), i.e.

\[
\begin{align*}
b^*_t(\omega^*)(h) &= b^*(t, S^*_t(\omega^*))(h), \\
c^*_t(\omega^*) &= c^*(t, S^*_t(\omega^*)), \\
K^*_{t,\omega^*}(\cdot) &= K^*(t, S^*_t(\omega^*); \cdot).
\end{align*}
\]

As we assume that the comparison process \( S^* \) is a special semimartingale, we may take \( h = \text{id} \) as “truncation function” also for \( S^* \), and we write

\[ S^* \sim (b^*(t, S^*_t), c^*(t, S^*_t), K^*(t, S^*_t; \cdot)) \]

to indicate that \( S^* \) has differential characteristics (1.1) associated with \( h = \text{id} \). The differential characteristics of the stochastic logarithm \( X^* = \Log(S^*) \), where \( \Log(S^*) = (\Log(S^*_1), \ldots, \Log(S^*_d)) \) is defined componentwise, are of the form

\[ X^* \sim (b^{X^*}(t, S^*_t), c^{X^*}(t, S^*_t), K^{X^*}(t, S^*_t; \cdot)) \]

cp. Lemma A.1.2. Typical examples for comparison processes \( S^* \) are (stochastic exponentials of) Lévy processes, or more generally processes with independent increments that are semimartingales with deterministic characteristics (PII), and solutions of diffusion with jumps type stochastic differential equations. For notation and results on semimartingales we refer to Jacod and Shiryaev (2003).

In Section 1.1 we establish multivariate stochastic and convex type comparison results for the terminal values \( S_T, S^*_T \) in terms of differential characteristics of the stochastic logarithms \( X = \Log(S), X^* = \Log(S^*) \) (Subsection 1.1.1), or in terms of the differential characteristics of \( S, S^* \) (Subsection 1.1.2). A crucial condition in these theorems is the propagation of order
property. This property is known in financial mathematics as *propagation of convexity*. In Section 1.2 we give references for results on the propagation of convexity in univariate models. We establish propagation of (monotone) convexity for some classes of multivariate diffusions (with drift) and multivariate diffusions with jumps (and drift), respectively, in Subsections 1.2.1 and 1.2.2. For diffusions with jumps models the results are also new in the univariate case. In Subsection 1.2.3 we derive the propagation of order property for PII for all order generating function classes in (1).

### 1.1 Multivariate comparison results

#### 1.1.1 Comparison in terms of local characteristics of the stochastic logarithms

One motivation to derive orderings in terms of the differential characteristics of the stochastic logarithms \( X = \log(S) \) and \( X^* = \log(S^*) \) is given in the example in the introduction. As the volatility of a continuous model \( S \) is given by the differential quadratic characteristic of the stochastic logarithm \( X = \log(S) \), the claim “a higher volatility yields higher option prices” translates into ordering of the differential characteristics of \( X \) and \( X^* \). We define the stochastic logarithm \( X = \log(S) \) of a \( d \)-dimensional semimartingale \( S \) componentwise by

\[
\log(S) = (\log(S_1), \ldots, \log(S_d)).
\]

In turn, the multivariate stochastic exponential is also defined componentwise. Throughout this subsection we make the following assumption.

**Assumption MG** Assume that \( S \sim (0, c_S, K_S) \) and \( S^* \sim (0, c^{S^*}(t, S^*_t), K^{S^*}(t, S^*_t)) \) \((S^* \text{ Markovian})\) are positive \( d \)-dimensional martingales under measures \( P \) and \( P^* \), respectively, and let the differential characteristics of their stochastic logarithms \( X, X^* \) be given by

\[
X_t(\omega) \sim (0, c_t(\omega), K_{\omega,t}(\cdot)), \quad X_0 = 0,
\]

\[
X^*_t(\omega^*) \sim (0, c^*(t, S^*_t(\omega^*)), K^*(t, S^*_{t-}(\omega^*); \cdot)), \quad X^*_0 = 0.
\]

The stochastic calculus approach that we use to derive orderings of \( S_T, S^*_T \) relies on a partial-integro-differential equation (PIDE), the Kolmogorov backward equation, for the backward function

\[
\mathcal{G}(t, s) = E^*(g(S^*_T) \mid S^*_t = s),
\]

where \( E^* \) denotes the expectation with respect to \( P^* \). In the literature, several results concerning the validity of the Kolmogorov backward equation
are given. For example Skorohod (1991, §3.4, Theorem 25) obtains this PIDE for processes with independent increments with absolutely continuous characteristics. Cont and Voltchkova (2005) obtain Kolmogorov backward equations in exponential Lévy models. We give a general version of this PIDE in the next lemma. For $\mathcal{H} \in C^{1,2}([0,T] \times \mathbb{R}^d)$ we define the functional $T_{\log}$ in terms of the differential characteristics of the stochastic logarithm $X^* = \log(S^*)$ by

$$T_{\log}\mathcal{H}(t,s) := D_t\mathcal{H}(t,s) + \frac{1}{2} \sum_{i,j \leq d} D_{ij}^2 \mathcal{H}(t,s)s^i s^j c^{*ij}(t,s)$$

$$+ \int_{(-\infty)} \mathcal{H}(t,s,x) K^*(t,s; dx),$$

where $(\Lambda_{\log}\mathcal{H})(t,s,x) := \mathcal{H}(t,s(1+x)) - \mathcal{H}(t,s) - \sum_{i \leq d} D_i \mathcal{H}s^i x^i$, and $c^*(t,s)$ and $K^*(t,s)$ are coefficient functions that come from the differential Gaussian and jump characteristic, respectively, of the stochastic logarithm $X^* = \log(S^*)$. For $W^*(\omega^*,t,s) := \mathcal{H}(t,S^*_t(\omega^*) + s) - \mathcal{H}(t,S^*_t(\omega^*)) - \sum_{i \leq d} D_i \mathcal{H}(t,S^*_t(\omega^*))s^i$ and the random measure $\mu^{S^*}$ that is associated to the jumps of $S^*$, we denote by $W^* \ast \mu^{S^*}_t$ the integral process

$$W^* \ast \mu^{S^*}_t(\omega^*) := \int_{[0,t] \times \mathbb{R}^d} W^*(\omega^*,u,s) \mu^{S^*}(\omega^*; du, ds). \quad (1.2)$$

**Lemma 1.1.1 (Kolmogorov backward equation).** Assume that $\mathcal{H} \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and that $|W^*| \ast \mu^{S^*} \in \mathcal{A}^{+}_{\text{loc}}$. If $\mathcal{H}(t,S^*_t)$ is a local $(\mathcal{A}^+_t)$-martingale under $P^*$, then there is a $P^*$-null set $N^*$, s.th.

$$T_{\log}\mathcal{H}(t,S^*_t(\omega^*)) = 0, \quad \forall t \in [0,T], \forall \omega^* \in N^*.c.$$  

If $\mathcal{H}(t,\cdot) \in \mathcal{F}_{\text{cx}}$, for all $t \in [0,T]$, then the integrability condition $|W^*| \ast \mu^{S^*} \in \mathcal{A}^{+}_{\text{loc}}$ is satisfied.

The proof is given in the appendix.

**Remark 1.1.2.** If for the support of the law of $S^*_t$ it holds true that $\text{supp}(P^{*S^*_t}) = \mathbb{R}_+^d$, for all $t \in [0,T]$, then $TH(t,s) = 0$ follows for all $(t,s) \in [0,T] \times \mathbb{R}_+^d$. This condition is satisfied if $c^{*ii}(t,s) > 0$, for all $i \leq d$. For one-dimensional Lévy exponential models $S^* = e^{L^*}$, where $L^*$ is a Lévy process, various conditions that imply $\text{supp}(P^{*S^*_t}) = \mathbb{R}_+$ are given in Sato (1999, Theorem 24.10).

Throughout this chapter we postulate enough smoothness of the transition probability $P^{*S^*_t}_{s,T}(s,dy)$ corresponding to the Markov process $S^*$ to imply
that the backward function \( G \) is in \( C^{1,2}([0, T] \times \mathbb{R}^d) \).

**Assumption SC\((g)\)** Let \( g : \mathbb{R}^d \to \mathbb{R} \) be given. \( S^* \) satisfies the smoothness condition \( SC(g) \), if the pricing function \( \mathcal{G}(t, s) = \int g(y) \mathcal{P}_{t,T}^*(s,dy) \) is in \( C^{1,2}([0, T] \times \mathbb{R}^d) \). Similarly, \( S^* \) satisfies the smoothness condition \( SC(F_0) \) for some \( F_0 \subset F \), if \( SC(g) \) holds for all \( g \in F_0 \).

A crucial assumption in the general comparison results for special semimartingales of section 1.1 is the propagation of order property: For some ordering function \( g = G(T, \cdot) \in F \) of interest we assume that the ordering at time \( T \) is propagated to earlier times \( t \in [0, T] \), i.e. it holds true that the backward function \( G(t, \cdot) \) also is in the function class \( F \), for all \( t \in [0, T] \).

This order propagating property is known in exponential diffusion models in financial context for \( F = F_{cx} \) as propagation of convexity: The arbitrage price process \( \mathcal{G}(t, s) \) of a European contingent claim with convex payoff function \( g(s) = G(T, s) \) is convex in \( s \), \( \forall t \in [0, T] \) (cp. El Karoui, Jeanblanc-Picqué, and Shreve (1998), Hobson (1998b), Bellamy and Jeanblanc (2000), Gushchin and Mordecki (2002)).

**Assumption PO\((g)\) (Propagation of order)** \( S^* \) satisfies the propagation of order property \( PO(g) \) w.r.t. \( g \in F \), if \( \mathcal{G}(t, \cdot) \in F \), for all \( t \in [0, T] \). Similarly, \( S^* \) satisfies the propagation of order property \( PO(F_0) \) for some \( F_0 \subset F \) if \( PO(g) \) holds for all \( g \in F_0 \).

The basic idea of the stochastic calculus approach is to study the backward linking process \( \mathcal{G}(t, S_t) \) which relates the processes \( S, S^* \) in a suitable way. We need additional conditions concerning the integrability and boundedness of the backward linking process \( \mathcal{G}(t, S_t) \).

**Assumption BIC\((g)\)** Let \( g : \mathbb{R}^d \to \mathbb{R} \). \( S, S^* \) satisfy the boundedness and integrability condition \( BIC(g) \), if the backward linking process \( \mathcal{G}(t, S_t) \) is bounded from below and \( \mathbb{E}\mathcal{G}(t, S_t) < \infty \), for all \( t \in [0, T] \). Similarly, \( S, S^* \) satisfy the boundedness and integrability condition \( BIC(F_0) \) for some \( F_0 \subset F \), if \( BIC(g) \) holds for all \( g \in F_0 \).

**Assumption CD\((g)\)** Let \( g : \mathbb{R}^d \to \mathbb{R} \). \( S, S^* \) satisfy the integrability condition \( CD(g) \), if the backward linking process \( \mathcal{G}(t, S_t) \) is a process of class \( (D) \). Similarly, \( S, S^* \) satisfy the integrability condition \( CD(F_0) \) for some \( F_0 \subset F \), if \( CD(g) \) holds for all \( g \in F_0 \).

The comparison result for (directionally) convex functions is as follows.

**Theorem 1.1.3 ((Directionally) convex order, upper bound).** Let \( S, S^*, S_0 = S^*_0 \), satisfy Assumption MG.
1.1. Multivariate comparison results

1. For $g \in \mathcal{F}_{\text{dcx}}$ assume that $S, S^*$ satisfy Assumption BIC$(g)$ (or CD$(g)$). Additionally, assume that $S^*$ satisfies Assumptions SC$(g)$, and PO$(g)$, and that the Kolmogorov backward equation $T_{Eg} G(t, S_{t-} (\omega)) = 0$ is satisfied $\lambda \times P$-a.e.. Let further $|W| \ast \mu_t^S \in A^+_\text{loc}$.

If for the differential characteristics of the stochastic logarithms $X, X^*$ it holds true that

$$c_t^{ij}(\omega) \leq c^{*ij}(t, S_{t-} (\omega)), \quad i, j \leq d,$$

$$\int_{(-1, \infty)^d} f(t, S_{t-} (\omega), x) K_{\omega, t}(dx) \leq \int_{(-1, \infty)^d} f(t, S_{t-} (\omega), x) K^*_t(S_{t-} (\omega), dx),$$

(1.3)

$\lambda \times P$-a.e., for all $f : [0, T] \times \mathbb{R}^d_+ \times (-1, \infty)^d \rightarrow \mathbb{R}$ with $f(t, s, \cdot) \in \mathcal{F}_{\text{dcx}}$ such that the integrals exist, then

$$Eg(S_T) \leq Eg^*(S^*_T).$$

2. For $g \in \mathcal{F}_c$ assume that $S, S^*$ satisfy Assumption BIC$(g)$ (or CD$(g)$). Additionally, assume that $S^*$ satisfies Assumptions SC$(g)$ and PO$(g)$, and that the Kolmogorov backward equation $T_{Eg} G(t, S_{t-} (\omega)) = 0$ is satisfied $\lambda \times P$-a.e..

If for the differential characteristics of the stochastic logarithms $X, X^*$ it holds true that

$$c_t(\omega) \leq psd c^*(t, S_{t-} (\omega)),$$

$$\int_{(-1, \infty)^d} f(t, S_{t-} (\omega), x) K_{\omega, t}(dx) \leq \int_{(-1, \infty)^d} f(t, S_{t-} (\omega), x) K^*_t(S_{t-} (\omega), dx),$$

(1.4)

$\lambda \times P$-a.e., for all $f : [0, T] \times \mathbb{R}^d \times (-1, \infty)^d \rightarrow \mathbb{R}$ with $f(t, s, \cdot) \in \mathcal{F}_c$ such that the integrals exist, then

$$Eg(S_T) \leq E^*g(S^*_T).$$

Proof. We establish that the backward linking process $G(t, S_t)$ is an $(\mathcal{A}_t)$-supermartingale under $P$. Then, using $S_0 = S_0^0$, we obtain the stated comparison result

$$Eg(S_T) = EG(T, S_T) \leq G(0, S_0) = E^*g(S^*_T).$$

1. Let $g \in \mathcal{F}_{\text{dcx}}$. Itô’s formula implies that $G(t, S_t)$ is a semimartingale with evolution

$$G(t, S_t) = G(0, 1) + M_t + \int_{[0,t]} D_t G(u, S_{u-}) du$$

$$+ \frac{1}{2} \sum_{i,j \leq d} \int_{[0,t]} D^2_{ij} G(u, S_{u-}) dC^S_{u,i} + W \ast \mu^S_t,$$
where $M_t := \sum_{i\leq d} \int_{[0,t]} \mathbf{D}_t \mathcal{G}(u, S_u -) dS_u^i$ is a one-dimensional local $(\mathcal{A}_t)$-martingale under $P$. As $|W| \ast \mu^S \in \mathcal{A}_t^{loc}$, there is a local $(\mathcal{A}_t)$-martingale $\tilde{M}$ such that

$$\mathcal{G}(t, S_t) = \mathcal{G}(0, 1) + M_t + \tilde{M}_t + \int_{[0,t]} \mathbf{D}_t \mathcal{G}(u, S_u -) du$$

$$+ \frac{1}{2} \sum_{i,j\leq d} \int_{[0,t]} \mathbf{D}_t^2 \mathcal{G}(u, S_u -) dS_u^{ij} + W * \nu^S_t,$$

where $W * \nu^S_t$ is defined as in (1.2) with $\mu^{S*}$ replaced by $\nu^S$. Using Lemma A.1.2 we obtain in terms of differential characteristics of the stochastic logarithm $X$ that $\mathcal{G}(t, S_t) = \mathcal{G}(0, 1) + M_t + \tilde{M}_t + V_t$, where

$$V_t := \int_{[0,t]} \left\{ \mathbf{D}_t \mathcal{G}(u, S_u -) + \frac{1}{2} \sum_{i,j\leq d} \mathbf{D}_t^2 \mathcal{G}(u, S_u -) S_u^{ij} - S_u^{ij} c_u^{ij}ight.$$ 

$$+ \int_{(-1,\infty)^d} (\Lambda \mathcal{G})(u, S_u -, x) K_u(dx) \right\} du$$

is predictable and of finite variation. As $\mathcal{G}(t, S_t -)$ satisfies the Kolmogorov backward equation we obtain

$$V_t = \int_{[0,t]} \left\{ \frac{1}{2} \sum_{i,j\leq d} \mathbf{D}_t^2 \mathcal{G}(u, S_u -) S_u^{ij} - S_u^{ij} c_u^{ij} - c^{*ij}(u, S_u -)ight.$$ 

$$+ \int_{(-1,\infty)^d} (\Lambda \mathcal{G})(u, S_u -, x) (K_u(dx) - K_u^*(S_u -, dx)) \right\} du.$$

(1.5)

Now we establish that the orderings (1.3) of the differential characteristics of $X$ imply that the backward linking process $\mathcal{G}(t, S_t)$ is a local $(\mathcal{A}_t)$-supermartingale under $P$. Due to the comparison assumption on the quadratic characteristics of $X, X^*$ in (1.3) and the directional convexity of $\mathcal{G}$ in the space variable, the first term of the integrand of $V_t$ is non-positive. For fixed $(\omega, u) \in (\Omega, [0, T])$ define the function

$$\Upsilon(x) := \Lambda \mathcal{G}(u, S_u - (\omega), x).$$

(1.6)

As $D^2_{ij} \Upsilon(x) = \nu_S^{ij} - \nu_S^{ij} D^2_{ij} \mathcal{G}(u, S_u - (1 + x)) \geq 0$ for all $x$, $\Upsilon(\cdot)$ is directionally convex. Hence the ordering on the jump kernels in (1.3) implies that also the second term of the integrand of $V_t$ is non-positive. This yields $-V_t \in \mathcal{A}_t^{loc}$ and it follows that $\mathcal{G}(t, S_t)$ is a local $(\mathcal{A}_t)$-supermartingale under $P$.

It remains to prove that $\mathcal{G}(t, S_t)$ is a supermartingale. In the case that $\mathcal{G}(t, S_t)$ is bounded below with $E \mathcal{G}(t, S_t) < \infty, \forall t$, $M_t$ is bounded below and, therefore, is an $(\mathcal{A}_t)$-supermartingale under $P$. It follows that
Remark 1.1.4. 1. As seen in the proof, it suffices to assume

\[ \mathcal{G}(t, S_t) \] is a supermartingale, as it is integrable. In the case that \( \mathcal{G}(t, S_t) \) is a process of class \((D)\) we consider a localizing sequence \( \tau_n \) for local supermartingale \( \mathcal{G}(t, S_t) \). As for all \( t \in [0, T] \) we have \( P\)-a.s. that \( (\mathcal{G}(t, S_t))^{\tau_n} \to \mathcal{G}(t, S_t), n \to \infty, \) and \( \mathcal{G}(t, S_t) \) is of class \((D)\), the convergence takes place in \( L^1 \) and therefore \( \mathcal{G}(t, S_t) \) is an \((\mathcal{A}_t)\)-supermartingale under \( P \).

2. Let \( g \in \mathcal{F}_{cx} \). Similar to the proof of the first part we show that the orderings in (1.4) imply that the backward linking process is a local \((\mathcal{A}_t)\)-supermartingale under \( P \). The evolution of \( \mathcal{G}(t, S_t) \) is given by \( \mathcal{G}(t, S_t) = \mathcal{G}(0, S_0) + M_t + \tilde{M}_t + V_t \), with \( V_t \) as in (1.5) and we have to show that \( -V_t \in \mathcal{A}_{loc}^+ \). Due to the positive semidefiniteness of the symmetric matrix \( \left( c^{ij} - c^j \right)_{i,j \leq d} := \left( c^{ij}(t, S_{\tau_n}(\omega)) - c^j(\omega) \right)_{i,j \leq d} \) for fixed \( (\omega, t) \), its spectral decomposition is given by \( \left( c^{ij} - c^j \right)_{i,j \leq d} = (\sum_{k \leq d} \lambda_k a_k a_k^\ast)_{i,j \leq d} \) where \( \lambda_k \geq 0 \) are the eigenvalues and \( a_k \) are the eigenvectors of the matrix \( (c^* - c) \). Thus the first term of the integrand of \( V_t \) takes the form \(-\frac{1}{2} \sum_{k \leq d} \lambda_k \sum_{i,j \leq d} D^2_{ij} \mathcal{G}(u, S_{\tau_n}) S^i_{u-} S^j_{u-} a_k^\ast a_k \).

\[ \mathcal{G}(t, S_t) \] is of class \((D)\), \( \lambda \) is non-positive due to the propagation of convexity property in Assumption \( PO(g) \) and the characterization of the convex order. Also by the convexity assumption on \( \mathcal{G}(t, \cdot) \), \( (\Lambda \mathcal{G})(u, S_{\tau_n}(\omega), x) \) is non-negative and \( \Upsilon(x) := \Lambda \mathcal{G}(u, S_{\tau_n}, x) \in \mathcal{F}_{cx} \). Furthermore, by the ordering of the jump-compensators, the second term in the integrand of \( V_t \) is non-positive. Therefore, \( -V_t \in \mathcal{A}_{loc}^+ \) and \( \mathcal{G}(t, S_t) \) is a local \((\mathcal{A}_t)\)-supermartingale under \( P \). The \((\mathcal{A}_t)\)-supermartingale property under \( P \) follows similar to the proof the first part of the theorem.

\[ \square \]

Remark 1.1.4. 1. As seen in the proof, it suffices to assume

\[ \int_{\mathbb{R}^d} (\Lambda \mathcal{G})(t, S_{\tau_n}(\omega), x) K_{\omega,t}(dx) \leq \int_{\mathbb{R}^d} (\Lambda \mathcal{G})(t, S_{\tau_n}(\omega), x) K^\ast_{\tau_n}(S_{\tau_n}(\omega), dx) \]

for \( \lambda \times P\)-a.e. \((t, \omega)\), instead of the orderings on the jump kernels in (1.3) and (1.4).

2. In (1.3) and (1.4) it suffices to consider functions \( f : [0, T] \times \mathbb{R}_+^d \times (-1, \infty)^d \to \mathbb{R} \), with \( f(t, s, 0) = 0 \). For the directionally convex case it further suffices to consider such \( f(t, s, \cdot) \in \mathcal{F}_{d cx} \) that are increasing on \( \mathbb{R}_+^d \) and decreasing on \((-1, 0)^d \). This follows from the properties of \( \Upsilon \) in (1.6). By definition, \( \Upsilon(0) = 0 \) and the first partial derivatives are given by

\[ D_i \Upsilon(x) = D_i \mathcal{G}(u, S_{u-}(1 + x)) S^i_{u-} - D_i \mathcal{G}(u, S_{u-}) S^i_{u-}, \quad i \leq d. \]
Directional convexity of $\mathcal{G}(t, \cdot)$ implies that $D_i \mathcal{G}(t, \cdot)$ is increasing, hence from $S \geq 0$ it follows that $D_i \Upsilon(x) \geq 0$ for $x \geq 0$ and $D_i \Upsilon(x) \leq 0$ for $x \leq 0$. For $d = 1$ this also holds true for $S$ with values in $\mathbb{R}$.

The corresponding result for the lower bound is similar with reversed inequalities. In this case one has to establish that the backward linking process $\mathcal{G}(t, S_t)$ is a $(\mathcal{A}_t)$-submartingale under $P$. Observe that in this case the result only holds true under Assumption CD($g$), Assumption BIC($g$) leads to trivial cases. To establish that $\mathcal{G}(t, S_t)$ is a true submartingale under $P$ one would have to modify the bounding condition in Assumption BIC($g$) to boundedness from above. Then, necessarily, $\mathcal{G}(t, \cdot)$ is constant, as it is (directionally) convex.

**Theorem 1.1.5 ((Directionally) convex order, lower bound).** Let $S, S^*, S_0 = S^*_0$, satisfy Assumption MG.

1. For $g \in \mathcal{F}_{dcx}$ assume that $S, S^*$ satisfy Assumption CD($g$). Additionally, assume that $S^*$ satisfies Assumptions SC($g$) and PO($g$), and that the Kolmogorov backward equation $T_{\log} \mathcal{G}(t, S_t-(\omega)) = 0$ is satisfied $\lambda \times P$-a.e.. Let further $|W| * \mu_t^S \in \mathcal{A}_{loc}^+$.

If for the differential characteristics of the stochastic logarithms $X, X^*$ it holds true that

$$ c^{*ij}(t, S_t-(\omega)) \leq c_t^{ij}(\omega), \qquad i, j \leq d, $$

$$ \int_{(-1,\infty)^d} f(t, S_t-(\omega), x) K^*_t(S_t-(\omega), dx) \leq \int_{(-1,\infty)^d} f(t, S_t-(\omega), x) K_{\omega,t}(dx), $$

$\lambda \times P$-a.e., for all $f : [0,T] \times \mathbb{R}_+^d \times (-1, \infty)^d \to \mathbb{R}$ with $f(t, s, \cdot) \in \mathcal{F}_{dcx}$ such that the integrals exist, then

$$ Eg(S^*_T) \leq Eg(S_T). $$

2. For $g \in \mathcal{F}_{cx}$ assume that $S, S^*$ satisfy Assumption CD($g$). Additionally, assume that $S^*$ satisfies Assumptions SC($g$) and PO($g$), and that the Kolmogorov backward equation $T_{\log} \mathcal{G}(t, S_t-(\omega)) = 0$ is satisfied $\lambda \times P$-a.e..

If for the differential characteristics of the stochastic logarithms $X, X^*$ it holds true that

$$ c^*(t, S_t-(\omega)) \leq_{psd} c_t(\omega), $$

$$ \int_{(-1,\infty)^d} f(t, S_t-, x) K^*(t, S_t-(\omega), dx) \leq \int_{(-1,\infty)^d} f(t, S_t-, x) K_{\omega,t}(dx), $$

$\lambda \times P$-a.e., for all $f : [0,T] \times \mathbb{R}_+^d \times (-1, \infty)^d \to \mathbb{R}$ with $f(t, s, \cdot) \in \mathcal{F}_{cx}$ such that the integrals exist, then

$$ Eg(S^*_T) \leq Eg(S_T). $$
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\[ \lambda \times P\text{-a.e., for all } f : [0, T] \times \mathbb{R}_+^d \times (-1, \infty)^d \to \mathbb{R}_+ \text{ with } f(t, s, \cdot) \in \mathcal{F}_t \text{ such that the integrals exist, then} \]

\[ E^* g(S_t^*) \leq E g(S_T). \]

**Proof.** The proof is similar to the proof of Theorem 1.1.3. The crucial points to establish are that \( V \) in (1.5) is an increasing process, which follows similarly to the proof of Theorem 1.1.3 from the ordering conditions on the differential characteristics of \( X, X^* \), and that then \( G(t, S_t) \) is a true \((\mathcal{A}_t)\)-submartingale under \( P \). As in Theorem 1.1.3, this follows from Assumption CD\((g)\). \( \square \)

1.1.2 Comparison in terms of local characteristics

In this subsection we extend the comparison results of the previous subsection to the case where \( S, S^* \) are not necessarily martingales. The comparisons are investigated with respect to the order generating function classes \( \mathcal{F} \) in (1) or w.r.t. single elements \( g \) in these classes. We consider \( d \)-dimensional special semimartingales \( S \sim (b, c, K) \) and \( S^* \sim (b^*(t, S^*_t), c^*(t, S^*_t), K^*(t, S^*_t)) \) on stochastic bases \((\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \in [0, T]}, P)\) and \((\Omega^*, \mathcal{A}^*, (\mathcal{A}^*_t)_{t \in [0, T]}, P^*)\), respectively. As in the previous subsection the approach relies on the Kolmogorov backward equation for the backward function \( G(t, s) = E^*(g(S_t^*)|S_t^* = s) \), now in terms of the differential characteristics of the Markovian comparison process \( S^* \). For \( \mathcal{H} \in C^{1,2}([0, T] \times \mathbb{R}^d) \), we define the functional \( T \) in terms of the differential characteristics of \( S^* \) by

\[ T\mathcal{H}(t, s) := D_t \mathcal{H}(t, s) + \sum_{i \leq d} D_i \mathcal{H}(t, s) b^{*i}(t, s) + \frac{1}{2} \sum_{i, j \leq d} D^2_{ij} \mathcal{H}(t, s) c^{*ij}(t, s) \]

\[ + \int (\Lambda \mathcal{H})(t, s, x) K^*(t, s; dx), \]

where \( (\Lambda \mathcal{H})(t, s, x) := \mathcal{H}(t, s + x) - \mathcal{H}(t, s) - \sum_{i \leq d} D_i \mathcal{H}(t, s) x^i \), and \( b^*(t, s), c^*(t, s) \) and \( K^*(t, s) \) are the differential characteristics of \( S^* \). The next lemma is parallel to Lemma 1.1.1, the proof is omitted.

**Lemma 1.1.6 (Kolmogorov backward equation).** Assume that \( \mathcal{H} \in C^{1,2}([0, T] \times \mathbb{R}^d) \) and that \( |W^*| \ast \mu^{S^*} \in \mathcal{A}^+_{\text{loc}} \). If \( \mathcal{H}(t, S^*_t) \) is a local \((\mathcal{A}^*_t)\)-martingale under \( P^* \), then there is a \( P^*\)-null set \( N^* \), s.th.

\[ T\mathcal{H}(t, S^*_t(\omega^*)) = 0, \quad \forall t \in [0, T], \forall \omega^* \in N^{*c}. \]

If \( \mathcal{H}(t, \cdot) \in \mathcal{F}_{\text{ex}}, \) for all \( t \in [0, T] \), then the integrability condition \( |W^*| \ast \mu^{S^*} \in \mathcal{A}^+_{\text{loc}} \) is satisfied.
The increasing (directionally) convex comparison result in terms of the differential characteristics of \( S, S^* \) is as follows.

**Theorem 1.1.7 (Increasing (directionally) convex comparison of semimartingales, upper bound).** Let \( S, S^* \in S^d \) have differential local characteristics \( S \sim (b, c, K)_{id} \) and \( S^* \sim (b^*(t, S^*_t), c^*(t, S^*_t), K^*(t, S^*_t))_{id} \).

1. For \( g \in F_{idcx} \) assume that \( S, S^* \) satisfy Assumption BIC\((g)\) (or CD\((g)\)). Additionally, assume that \( S^* \) satisfies Assumptions SC\((g)\) and PO\((g)\), and that the Kolmogorov backward equation \( T G(t, S_{t-}(\omega)) = 0 \) is satisfied \( \lambda \times \mathbb{P} \)-a.e.. Let further \( |W| \ast \mu_t \in A_{loc}^+ \).

If for the differential characteristics it holds true that
\[
\begin{align*}
& b^j_i(\omega) \leq b^*_{i}(t, S_{t-}(\omega)), \quad c^j_t(\omega) \leq c^*_{i}(t, S_{t-}(\omega)), \quad i, j \leq d, \\
& \int_{\mathbb{R}^d} f(t, S_{t-}(\omega), x) K_{\omega,t}(dx) \leq \int_{\mathbb{R}^d} f(t, S_{t-}(\omega), x) K^*_{t}(S_{t-}(\omega), dx),
\end{align*}
\] (1.7)
\( \lambda \times \mathbb{P} \)-a.e., for all \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) with \( f(t, s, \cdot) \in F_{dcx} \), such that the integrals exist, then
\[
E_g(S_T) \leq E^*_{g}(S^*_T).
\]

2. For \( g \in F_{icx} \) assume that \( S, S^* \) satisfy Assumption BIC\((g)\) (or CD\((g)\)). Additionally, assume that \( S^* \) satisfies Assumptions SC\((g)\) and PO\((g)\), and that the Kolmogorov backward equation \( T G(t, S_{t-}(\omega)) = 0 \) is satisfied \( \lambda \times \mathbb{P} \)-a.e..

If for the differential characteristics it holds true that
\[
\begin{align*}
& b^j_i(\omega) \leq b^*_{i}(t, S_{t-}(\omega)), \quad i \leq d, \quad c_t(\omega) \leq_{psd} c^*_{i}(t, S_{t-}(\omega)), \\
& \int_{\mathbb{R}^d} f(t, S_{t-}(\omega), x) K_{\omega,t}(dx) \leq \int_{\mathbb{R}^d} f(t, S_{t-}(\omega), x) K^*_t(S_{t-}(\omega), dx),
\end{align*}
\] (1.8)
\( \lambda \times \mathbb{P} \)-a.e., for all \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \) with \( f(t, s, \cdot) \in F_{cx} \), such that the integrals exist, then
\[
E_g(S_T) \leq E^*_{g}(S^*_T).
\]

**Proof.** The proof is similar to the proof of Theorem 1.1.3. Let the canonical decomposition of the special semimartingale \( S \) be given by \( S = S_0 + N + B \). In terms differential characteristics of \( S \), the backward linking process \( \mathcal{G}(t, S_t) \) is of the form \( \mathcal{G}(t, S_t) = \mathcal{G}(0, S_0) + M_t + \hat{M}_t + V_t \), where \( M_t = \)
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\[\sum_{i \leq d} \int D_t G(u, S_u) dN^i_u\] and \(\hat{M}_t = W^* (\mu - \nu)_t\) are local \((\mathcal{A}_t)\)-martingales under \(P\), and

\[V_t = \int_{[0,t]} \left\{ \sum_{i \leq d} D_t G(u, S_{u-}) b^i_u + \frac{1}{2} \sum_{i,j \leq d} D^2_{ij} G(u, S_{u-}) c^{ij}_u + \int_{\mathbb{R}^d} (\Lambda G)(u, S_{u-}, y) K_u(dy) \right\} du\]

is a predictable process of finite variation. As \(G(t, S_{t-})\) satisfies the Kolmogorov backward equation \(TG(t, S_{t-}) = 0\), it follows that

\[V_t = \int_{[0,t]} \left\{ \sum_{i \leq d} D_t G(u, S_{u-}) (b^i_u - b^{*_i}(u, S_{u-})) + \frac{1}{2} \sum_{i,j \leq d} D^2_{ij} G(u, S_{u-}) (c^{ij}_u - c^{*_ij}(u, S_{u-})) + \int_{\mathbb{R}^d} (\Lambda G)(u, S_{u-}, x) (K_u(dx) - K^*_u(S_{u-}, dx)) \right\} du.\] (1.9)

Similar to the proof of Theorem 1.1.3 the ordering conditions on the differential characteristics imply that \(V_t\) is decreasing, as the propagation of order property \(PO(g)\) is satisfied.

\[\square\]

**Remark 1.1.8.** 1. Similar comparison results are obtained for \(g \in \mathcal{F}\), where \(\mathcal{F}\) is one of the order generating function classes in (1). This follows from the crucial equation (1.9) and the propagation of order property \(PO(g)\). The following table lists sufficient conditions that are similar to the ordering conditions on the differential characteristics in (1.7) and (1.8) that imply \(Eg(S_T) \leq E^* g(S^*_T)\) for \(g \in \mathcal{F}\).

<table>
<thead>
<tr>
<th>Ordering</th>
<th>Drift</th>
<th>Diffusion</th>
<th>Jump</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{F}_{st})</td>
<td>(b \leq b^{*})</td>
<td>(c = c^{*})</td>
<td>(K = K^{*})</td>
</tr>
<tr>
<td>(\mathcal{F}_{idcx})</td>
<td>(b \leq b^{*})</td>
<td>(c \leq c^{*})</td>
<td>(K \leq_{d\text{ex}} K^{*})</td>
</tr>
<tr>
<td>(\mathcal{F}_{icx})</td>
<td>(b \leq b^{*})</td>
<td>(c \leq_{\text{psd}} c^{*})</td>
<td>(K \leq_{\text{ex}} K^{*})</td>
</tr>
<tr>
<td>(\mathcal{F}_{ism})</td>
<td>(b \leq b^{*})</td>
<td>(c \leq c^{*}, c^{ii} = c^{*ii}, i \leq d)</td>
<td>(K \leq_{\text{sm}} K^{*})</td>
</tr>
<tr>
<td>(\mathcal{F}_{d\text{ex}})</td>
<td>(b = b^{*})</td>
<td>(c \leq c^{*})</td>
<td>(K \leq_{d\text{ex}} K^{*})</td>
</tr>
<tr>
<td>(\mathcal{F}_{\text{cx}})</td>
<td>(b = b^{*})</td>
<td>(c \leq_{\text{psd}} c^{*})</td>
<td>(K \leq_{\text{ex}} K^{*})</td>
</tr>
<tr>
<td>(\mathcal{F}_{\text{sm}})</td>
<td>(b = b^{*})</td>
<td>(c \leq c^{*}, c^{ii} = c^{*ii}, i \leq d)</td>
<td>(K \leq_{\text{sm}} K^{*})</td>
</tr>
</tbody>
</table>
Similar to the first point of Remark 1.1.4 it suffices to consider
\[
\int_{\mathbb{R}^d} (\Lambda G)(t, S_{t-}(\omega), x)K_{\omega,t}(dx) \leq \int_{\mathbb{R}^d} (\Lambda G)(t, S_{t-}(\omega), x)K^*(t, S_{t-}(\omega), dx),
\]
for $\lambda \times P$-a.e. $(t, \omega)$, instead of the orderings on the jump kernels in (1.7) and (1.8), and in the last row of the table, respectively.

2. Similar to the second point of Remark 1.1.4, it suffices to consider functions $f : [0, T] \times \mathbb{R}_+^d \times \mathbb{R}_d \rightarrow \mathbb{R}$ with $f(t, s, 0) = 0$. Additionally, if $S \geq 0$ then in (1.7) it suffices to consider such functions $f(t, s, \cdot) \in \mathcal{F}_{dcx}$ that are increasing on $\mathbb{R}_+^d$ and decreasing on $\mathbb{R}_d^d$. This also holds true for the increasing directionally convex ordering result, and for $d = 1$ the assumption $S \geq 0$ is not necessary.

The corresponding result for the lower bound holds true under Assumption CD$(g)$. The boundedness Assumption BIC$(g)$ no longer is useful as now one has to establish that $G(t, S_t)$ is an $(\mathcal{A}_t)$-submartingale, cp. the discussion before Theorem 1.1.5. We omit the proof which is similar to the proof of Theorem 1.1.3.

**Theorem 1.1.9 (Increasing (directionally) convex comparison of semimartingales, lower bound).** Let $S, S^* \in S_d^d$ have differential local characteristics $S \sim (b, c, K)_{id}$ and $S^* \sim (b^*(t, S_t^*), c^*(t, S_t^*), K^*(t, S_t^*))_{id}$.

1. For $g \in \mathcal{F}_{dcx}$ assume that $S, S^*$ satisfy Assumption CD$(g)$. Additionally, assume that $S^*$ satisfies Assumptions SC$(g)$ and PO$(g)$, and that the Kolmogorov backward equation $T\hat{G}(t, S_{t-}(\omega)) = 0$ is satisfied $\lambda \times P$-a.e. Let further $|W| * \mu_i \in \mathcal{A}_{loc}^+$. If for the differential characteristics it holds true that

\[
b^i(t, S_{t-}(\omega)) \leq b^j_i(\omega), \quad c^{ij}(t, S_{t-}(\omega)) \leq c^{ij}_i(\omega), \quad i, j \leq d,
\]

\[
\int_{\mathbb{R}^d} f(t, S_{t-}(\omega), x)K^*(t, S_{t-}(\omega), dx) \leq \int_{\mathbb{R}^d} f(t, S_{t-}(\omega), x)K_{\omega,t}(dx),
\]

for $\lambda \times P$-a.e., for all $f \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_d^d \rightarrow \mathbb{R}$ with $f(t, s, \cdot) \in \mathcal{F}_{dcx}$, such that the integrals exist, then

\[
E^*g(S^*_T) \leq Eg(S_T).
\]
2. For \( g \in \mathcal{F}_{\text{icx}} \) assume that \( S, S^* \) satisfy Assumption CD\((g)\). Additionally, assume that \( S^* \) satisfies Assumptions SC\((g)\) and PO\((g)\), and that the Kolmogorov backward equation \( \int G(t, S_t(\omega)) = 0 \) is satisfied \( \lambda \times P \)-a.e.

If for the differential characteristics it holds true that

\[
\begin{align*}
& b^i(t, S_{t-}(\omega)) \leq b^i_t(\omega), \quad i \leq d, \\
& c^*(t, S_{t-}(\omega)) \leq \text{psd} \ c_t(\omega), \\
& \int_{\mathbb{R}^d} f(t, S_{t-}(\omega), x) K_t^*(S_{t-}(\omega), dx) \leq \int_{\mathbb{R}^d} f(t, S_{t-}(\omega), x) K_{t}(dx),
\end{align*}
\]

\( \lambda \times P \)-a.e., for all \( f \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \) with \( f(t, s, \cdot) \in \mathcal{F}_{\text{icx}} \), such that the integrals exist, then

\[
E^* g(S^*_T) \leq E g(S_T).
\]

**Remark 1.1.10.** A table similar to the table in Remark 1.1.8, as well as the claim about the ordering of the jump terms in that remark, also holds true for the lower bound.

Comparison of \( d \)-dimensional random variables implies comparison of their exponentials for most of the orderings that are considered in this thesis, as the following lemma shows.

**Lemma 1.1.11.** Let \( \mathcal{F} \in \{ \mathcal{F}_{\text{st}}, \mathcal{F}_{\text{icx}}, \mathcal{F}_{\text{idc}}, \mathcal{F}_{\text{ism}}, \mathcal{F}_{\text{sm}} \} \) and \( \bar{X}, \bar{X}^* \) be \( d \)-dimensional semimartingales. If \( \bar{X}_t \leq_{\mathcal{F}} \bar{X}_t^* \), then \( S_t = e^{\bar{X}_t} \leq_{\mathcal{F}} e^{\bar{X}_t^*} = S^*_t \).

**Proof.** The result follows from the fact that \( f \in \mathcal{F} \) implies \( f \circ \exp \in \mathcal{F} \), where \( \exp : \mathbb{R}^d \rightarrow \mathbb{R}^d, (x^1, \ldots, x^d) \mapsto (e^{x^1}, \ldots, e^{x^d}) \), with \( e \) the one-dimensional exponential function. For monotone \( f \), monotonicity of \( f \circ \exp \) is obvious. As \( e : \mathbb{R} \rightarrow \mathbb{R} \) is convex, \( f \in \mathcal{F}_{\text{icx}} \) implies for \( x, y \in \mathbb{R}^d \) and \( \alpha \in (0, 1) \) that \( f \circ \exp(\alpha x + (1 - \alpha)y) \leq f(\alpha \exp(x) + (1 - \alpha) \exp(y)) \leq \alpha f \circ \exp(x) + (1 - \alpha) f \circ \exp(y) \). For increasing directionally convex functions the result follows from Müller and Stoyan (2002, Theorem 3.12.3) and for \( \mathcal{F} \in \{ \mathcal{F}_{\text{ism}}, \mathcal{F}_{\text{sm}} \} \) the result follows from Müller and Stoyan (2002, Theorem 3.9.3).

For some of the orderings also the reverse direction can be established.

**Lemma 1.1.12.** For \( \mathcal{F} \in \{ \mathcal{F}_{\text{st}}, \mathcal{F}_{\text{ism}}, \mathcal{F}_{\text{sm}} \} \) and positive \( d \)-dimensional semimartingales \( S, S^* \) let \( S_t \leq_{\mathcal{F}} S^*_t \). Then for \( \bar{X} = \log(S) \), \( \bar{X}^* = \log(S^*) \) it holds true that \( \bar{X}_t \leq_{\mathcal{F}} \bar{X}_t^* \).

**Proof.** This follows from the fact that for \( f \in \mathcal{F} \) it holds true that \( f \circ \log \in \mathcal{F} \), where \( \log \) is defined componentwise and is increasing in each component. The result for \( \mathcal{F}_{\text{st}} \) is obvious and for \( \mathcal{F} \in \{ \mathcal{F}_{\text{sm}}, \mathcal{F}_{\text{ism}} \} \) the claim follows from Müller and Stoyan (2002, Theorem 3.9.3).
1.2 Propagation of order property

In financial mathematics, the propagation of order property is well-known mainly in one-dimensional diffusion models for $\mathcal{F} = \mathcal{F}_{c_t}$. Bergman, Grundy, and Wiener (1996) establish the propagation of convexity property by PDE techniques, in El Karoui, Jeanblanc-Picqué, and Shreve (1998) propagation of convexity is proved via the theory of stochastic flows and in Hobson (1998b) via coupling. A propagation of convexity result for univariate Markov processes is given in Martini (1999) and a typical Markov argument is also given in Gushchin and Mordecki (2002) for the one-dimensional diffusion case. The proofs of these papers do not seem to be applicable to jump processes or in the multivariate case. A partial extension to multivariate diffusions is given in Janson and Tysk (2004).

The role of convexity can be understood most easily in the following simple example.

Example 2 (Convexity). Let $f$ be a convex function and let $S^*_t = S^*_0 \mathcal{E}(\sigma \cdot W)_t$ be a univariate diffusion model with diffusion coefficient $\sigma$, $\mathcal{E}$ the stochastic exponential. Let $E^*$ denote the corresponding expectation. Then the Black–Scholes price at time $t = 0$,

$$G(s) = E^*(f(S^*_T) \mid S^*_0 = s),$$

is a convex function in $s$. Let $S_t = s \mathcal{E}(\sigma \cdot W)_t \mathcal{E}(\phi \cdot M)_t$ be a diffusion with jumps model, with compensated Poisson martingale $M$ and jump size $\phi$. If the two stochastic exponentials are stochastically independent (as in the case of deterministic $\sigma, \phi$ and intensity) then one obtains by Jensen’s inequality

$$G(s) = G(E(s \mathcal{E}(\phi \cdot M)_T)) \leq E G(s \mathcal{E}(\phi \cdot M)_T)$$

$$= E f(s \mathcal{E}(\phi \cdot M)_T \mathcal{E}(\sigma \cdot W)_T) = E f(S_T)$$

i.e. the price of the European option for the jump diffusion model dominates the price for the diffusion model.

The argument for the comparison in (1.10) is valid also in the multivariate case as soon as one has defined an analog of the stochastic exponential for this case, in subsection 1.1.1 we give a componentwise definition. For this and some related models similar comparison results were given by Henderson and Hobson (2003) using related coupling arguments.

In this section we derive the propagation of order property for several types of univariate and multivariate models in this section. We establish propagation of (monotone) convexity for some classes of univariate and multivariate
diffusions with zero and non-zero drift, respectively, in Subsection 1.2.1. In Subsection 1.2.2 we derive propagation of (monotone) convexity for some classes of diffusion with jumps models that have zero and non-zero drift, respectively. This result is also new in the univariate case. Finally, in Subsection 1.2.3, we establish the propagation of order property \( \text{PO}(g) \), \( g \in \mathcal{F} \), for all \( \mathcal{F} \) in (1), for comparison processes that have independent increments.

### 1.2.1 Monotone convexity for diffusions

We firstly derive propagation of convexity in the case where the comparison process \( S^* \) is a multivariate diffusion with zero drift component and diffusion coefficient \( \sigma^* \) that is convex in the space variable. In a next step we establish propagation of monotone convexity for multivariate diffusions \( S^* \) with non-zero drift in the case where the coefficients are monotone and convex in the space variable. To obtain the results we use an Euler approximation scheme \( \tilde{S}^*_K \) of the respective comparison process \( S^* \). We establish (increasing) convex monotonicity of the corresponding Markov transition operator and then derive propagation of convexity for the Euler scheme \( \tilde{S}^*_K \), which yields the propagation of convexity property for the comparison process \( S^* \) by an approximation argument.

Let \( g \in \mathcal{F}_{\text{cx}} \) and assume that \( W^* \) is a \( d \)-dimensional Brownian motion on a stochastic basis \((\Omega^*, \mathcal{A}^*, (\mathcal{A}^*_t)_{t \in [0,T]}, P^*)\). For \( \sigma^* : [0,T] \times \mathbb{R}^d \to M_+(d, \mathbb{R}) \), \( M_+(d, \mathbb{R}) \) the set of positive semidefinite matrices with values in \( \mathbb{R}^{d \times d} \), and let \( S^* \) be the unique strong solution of the SDE

\[
dS^*_t = \sigma^*(t, S^*_t) dW^*_t, \quad S^*_0 = 1. \tag{1.11}
\]

For the proof of the propagation of convexity, we use the following Euler scheme \( \tilde{S}^*_K \) for \( S^* \). Let \( t_0 \in [0,T] \) and discretize the time interval \([t_0, T]\) into \( K + 1 \) equidistant points \( t_i := \frac{iT-t_0}{K} + t_0, i \in \{0, \ldots, K\} \). We denote the Euler scheme of \( (S^*_t)_{t \in [t_0,T]} \), \( S^*_{t_0} = s \), by

\[
\tilde{S}^*_{K,t_{i+1}} = \tilde{S}^*_{K,t_i} + \sigma^*(t_i, \tilde{S}^*_{K,t_i})(W^*_{t_{i+1}} - W^*_{t_i}), \quad i \in \{0, \ldots, K-1\}, \tag{1.12}
\]

The corresponding backward function is given by

\[
\tilde{g}_K(t, s) := E^*(g(\tilde{S}^*_{K,T}) \mid \tilde{S}^*_{K,t} = s).
\]

We assume that the Euler approximation scheme satisfies the following approximation property, which is satisfied under smoothness and linear growth
Assumption AP(\(g\)) (Approximation property) \(\text{Let } g : \mathbb{R}^d \to \mathbb{R}. \text{ The Euler scheme } \tilde{S}^*_K \text{ of } S^* \text{ satisfies the approximation property AP}(g), \text{ if for } K \to \infty \text{ it holds true that}
\[
\tilde{G}_K(t,s) := \mathbb{E}^* \left( g(\tilde{S}^*_{K,T}) \mid \tilde{S}^*_{K,t} = s \right) \to G(t,s), \quad \forall t \in [0,T], s \in \mathbb{R}^d.
\]
Similarly, the Euler scheme \(\tilde{S}^*_K\) of \(S^*\) satisfies the approximation property \(\text{AP}(\mathcal{F}_0)\) for some \(\mathcal{F}_0 \subset \mathcal{F}\) if \(\text{AP}(g)\) holds true for all \(g \in \mathcal{F}_0\).

A sufficient condition for the propagation of convexity of the Euler scheme is the \(\leq_{\text{cx}}\)-monotonicity of the corresponding transition operator \(T\), i.e. \(S_1 \leq_{\text{cx}} S_2\) implies \(TS_1 \leq_{\text{cx}} TS_2\). To establish \(\leq_{\text{cx}}\)-monotonicity of \(T\) we assume that \(\sigma^* : [t_0, T] \times \mathbb{R}^d \to M_+(d, \mathbb{R})\) is convex in the second component with the positive semidefinite partial ordering \(\leq_{\text{psd}}\) on \(M_+(d, \mathbb{R})\). By \(W \sim \mathcal{N}(\mu, \Sigma)\) we denote that \(W\) is normally distributed with expectation vector \(\mu\) and covariance matrix \(\Sigma\). For better readability we suppress the superscript asterix in the next lemma.

Lemma 1.2.1 (A convex ordering result for Markov operators). Let \(S\) be a \(d\)-dimensional random vector that is independent of \(W \sim \mathcal{N}(0, \Delta I)\), where \(I \in M_+(d, \mathbb{R})\) is the identity, \(\Delta \in \mathbb{R}_+\), and let \(\sigma : \mathbb{R}^d \to M_+(d, \mathbb{R})\). Assume that \(\sigma\) is convex, where \(M_+(d, \mathbb{R})\) is supplied with the positive semidefinite ordering \(\leq_{\text{psd}}\). Then the Markov operator \(T\) on the set of probability measures with state space \((\mathbb{R}^d, \mathcal{B}^d)\) defined by
\[
TS \overset{d}{=} S + \sigma(S)W
\]
is \(\leq_{\text{cx}}\)-monotone.

Proof. Let \(S_1, S_2\) be \(d\)-dimensional random vectors that are independent of \(W\) and satisfy \(S_1 \leq_{\text{cx}} S_2\). Due to Strassen’s Theorem there are random vectors \(\hat{S}_i \overset{d}{=} S_i, \ i = 1, 2,\) on a probability space \((\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})\) such that \(\hat{E}(\hat{S}_2|\hat{S}_1) = \hat{S}_1\), where \(\hat{E}\) denotes the expectation with respect to \(\hat{P}\). We assume without loss of generality that \(S_1, S_2\) are these versions. Then, for \(f \in \mathcal{F}_{\text{cx}}\), Jensen’s inequality implies
\[
Ef(TS_2) = Ef(S_2 + \sigma(S_2)W) = EE(f(S_2 + \sigma(S_2)W)|S_1,W) \\
\geq Ef(S_1 + E(\sigma(S_2)|S_1)W) = E^{S_1}Ef(s_1 + E(\sigma(S_2)|s_1)W),
\]
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where the last equality follows from conditioning on $S_1 = s_1$ and $E^{S_1}$ denotes the expectation with respect to the distribution of $S_1$. As the convex order is stable under mixtures, it remains to establish

$$E(\sigma(S_2) | s_1) W \geq_{cx} \sigma(s_1) W. \quad (1.13)$$

As $\sigma \in \mathcal{F}_{cx}$, Jensen’s inequality implies

$$g(s_1) := E(\sigma(S_2) | s_1) \geq_{psd} \sigma(s_1),$$

and as $\sigma(s_1)$ and $g(s_1)$ are positive semidefinite it follows that $g(s_1)^{T}g(s_1) \geq_{psd} \sigma(s_1)^{T}\sigma(s_1)$, where the superscript $T$ denotes the transpose. As $\sigma(s_1) W \sim N(0, \Delta \sigma(s_1)^{T}\sigma(s_1))$ and $g(s_1) W \sim N(0, \Delta g(s_1)^{T}g(s_1))$, this implies $E(\sigma(S_2) | s_1) W = g(s_1) W \geq_{cx} \sigma(s_1) W$ (cp. Müller and Stoyan (2002, Theorem 3.4.7)).

Propagation of convexity of $\mathcal{G}$ follows from the convexity of the transition operator of $\tilde{S}^*_K$ and the approximation property.

**Theorem 1.2.2 (Propagation of convexity, diffusion case).** Let $g \in \mathcal{F}_{cx}$, $S^*$ be a $d$-dimensional diffusion and assume that the corresponding Euler scheme $\tilde{S}^*_K$ of $S^*$ satisfies Assumption AP$(g)$. If $\sigma^*(t, \cdot)$ is convex for all $t \in [0, T]$, then propagation of convexity property PO$(g)$ for the backward function $\mathcal{G}$ holds true, i.e.

$$\mathcal{G}(t, \cdot) \in \mathcal{F}_{cx}, \forall t \in [0, T].$$

**Proof.** We consider the Euler approximation scheme $\tilde{S}^*_K$ defined in (1.12) with interpolation points $t_i$ and define the corresponding transition operator by $\mathcal{T}_t S \overset{\text{d}}{=} S + \sigma(t_i, S)W$, where $W \overset{\text{d}}{=} W_{t_i+1}^* - W_{t_i}^* \sim N(0, (t_{i+1} - t_i) I)$. Then for $t_0 \in [0, T]$ we have by the Markov property of this scheme

$$\tilde{\mathcal{G}}_K(t_0, y) = E^*(g(\tilde{S}^*_K, t) | \tilde{S}^*_K_{t_0} = y) = E^* g(T_{t_{K-1}} \ldots T_{t_0} y).$$

For $y_1, y_2 \in \mathbb{R}^d$ and $\alpha \in (0, 1)$ let $Y$ be a Bernoulli random vector with distribution $P^Y = \alpha \varepsilon_{(y_1)} + (1 - \alpha) \varepsilon_{(y_2)}$. Then

$$\alpha y_1 + (1 - \alpha) y_2 = EY \leq_{cx} Y.$$

Using the $\leq_{cx}$-monotonicity of the operator $\mathcal{T}_t$ for all $t \in [0, T]$ from Lemma 1.2.1 we obtain

$$\tilde{\mathcal{G}}_K(t_0, \alpha y_1 + (1 - \alpha) y_2) \leq \tilde{\mathcal{G}}_K(t_0, EY) \leq E^* g(T_{t_{K-1}} \ldots T_{t_0} EY) \leq E^* g(T_{t_{K-1}} \ldots T_{t_0} Y) = \tilde{\mathcal{G}}_K(t_0, Y). \quad (1.14)$$

Taking expectations on both sides of (1.14) implies convexity of $\tilde{\mathcal{G}}_K(t_0, \cdot)$. The approximation property AP then implies $\mathcal{G}(t_0, \cdot) \in \mathcal{F}_{cx}$. \qed
The results of Lemma 1.2.1 and Theorem 1.2.2 extend to multivariate diffusions with drift

\[ dS_t^* = b^*(t, S_t^*)dt + \sigma^*(t, S_t^*)dW_t^*, \]

if the coefficients \( b^* : \mathbb{R}^d \to \mathbb{R}^d, \sigma^* : \mathbb{R}^d \to M_+(d, \mathbb{R}), \) are increasing and convex in the space variable, where \( M_+(d, \mathbb{R}) \) is supplied with the positive semidefinite ordering \( \leq_{\text{psd}}. \) In this case the corresponding Euler scheme is of the form

\[ \tilde{S}_{K,t_{i+1}}^* = \tilde{S}_{K,t_i}^* + b^*(t_i, \tilde{S}_{K,t_i}^*)\Delta t_i + \sigma^*(t_i, \tilde{S}_{K,t_i}^*)(W_{t_{i+1}}^* - W_{t_i}^*), \]

and the extension of Lemma 1.2.1 reads as follows. Again, we suppress the superscript asterix.

**Lemma 1.2.3 (A monotone convex ordering result for Markov operators).** Let \( S \) be a \( d \)-dimensional random vector that is independent of, \( W \sim N(0, \Delta I), \Delta \in \mathbb{R}_+. \) Assume that \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to M_+(d, \mathbb{R}) \) are increasing and convex. Then the transition operator \( T \) given by

\[ TS \overset{d}{=} S + \Delta b(S) + \sigma(S)W \]

is \( \leq_{\text{icx}} \)-monotone, i.e. \( S_1 \leq_{\text{icx}} S_2 \) implies \( TS_1 \leq_{\text{icx}} TS_2. \)

**Proof.** Assume that \( S_1, S_2, \) are \( d \)-dimensional random vectors that are independent of \( W \) and satisfy \( S_1 \leq_{\text{icx}} S_2. \) Without loss of generality, we choose \( S_1, S_2 \) such that \( E(S_2|S_1) \geq S_1 \) by Strassen’s Theorem. For \( f \in \mathcal{F}_{\text{icx}} \) it follows from Jensen’s inequality that

\[ Ef(TS_2) \geq EE^{S_1}f(E(S_2|s_1) + \Delta E(b(S_2)|s_1) + E(\sigma(S_2)|s_1)W), \]
\[ \geq EE^{S_1}f(s_1 + \Delta b(s_1) + E(\sigma(S_2)|s_1)W), \]

where the last inequality follows from monotonicity of \( f \) and as monotonicity and convexity of \( b \) imply \( E(b(S_2)|s_1) \geq b(E(S_2|s_1)) \geq b(s_1). \) Convexity and monotonicity of \( \sigma \) similarly imply \( E(\sigma(S_2)|s_1) \geq_{\text{psd}} \sigma(s_1) \), hence it follows as in the proof of Lemma 1.2.1 that \( \sigma(s_1)W \leq_{\text{cx}} E(\sigma(S_2)|s_1)W. \) The claim follows from the stability under mixtures property (MI) of the increasing convex order.

Lemma 1.2.3 and the approximation property AP\( (g) \) imply propagation of monotone convexity for multivariate diffusions with drift.
1.2. Propagation of order property

**Theorem 1.2.4 (Propagation of monotone convexity, diffusion with drift case).** Let \( g \in \mathcal{F}_{\text{inc}} \), \( S^* \) be a \( d \)-dimensional diffusion with drift and assume that the corresponding Euler scheme \( \overline{S}^*_k \) of \( S^* \) satisfies Assumption AP\((g)\). If \( b^*(t, \cdot) \) and \( \sigma^*(t, \cdot) \) are increasing and convex for all \( t \in [0, T] \), then propagation of monotone convexity property PO\((g)\) for the backward function \( \mathcal{G} \) holds true, i.e.,

\[
\mathcal{G}(t, \cdot) \in \mathcal{F}_{\text{inc}}, \quad \forall t \in [0, T].
\]

**1.2.2 Monotone convexity for diffusions with jumps**

We extend the propagation of (monotone) convexity results of the previous subsection to some classes of multivariate diffusion with jumps models. First, we establish propagation of convexity for the case with zero drift, and then derive propagation of monotone convexity for multivariate diffusion with jumps models that have a non-zero drift component. The results are also new in the one-dimensional case. The procedure is similar to the methodology of the previous subsection.

Let \( g \in \mathcal{F}_{\text{inc}} \) and assume that \( W^* \) is a \( d \)-dimensional Brownian motion on a stochastic basis \((\Omega^*, \mathcal{A}^*, (\mathcal{A}^t)^*_{t \in [0, T]}, P^*)\). Assume that \( N^* \) is a Poisson random measure on \([0, T] \times \mathbb{R}\), where the mark space \( \mathbb{E} \) is \( \mathbb{R} \) or \( \mathbb{R}^d \). Let \( N^* \) have deterministic intensity \( \lambda^*(dy)dt \) and \( \lambda^*(\mathbb{E}) < \infty \). For \( \sigma^* : [0, T] \times \mathbb{R}^d \to M_+(d, \mathbb{R}) \) and \( \phi^* : [0, T] \times \mathbb{R}^d \times \mathbb{E} \to \mathbb{R}^d \) let \( S^* \) be a unique solution of

\[
dS^*_t = \sigma^*(t, S^*_t)dW^*_t + \phi^*(t, S^*_t, y)(N^*(dt, dy) - \lambda^*(dy)dt), \quad (1.15)\]

\[
S^*_0 = 1.
\]

Again, we assume that \( \sigma^*(t, \cdot) \) is convex with \( M_+(d, \mathbb{R}) \) supplied with the positive semidefinite ordering and, additionally, we assume that the jump coefficient \( \phi^* \) of the comparison process \( S^* \) satisfies one of the following conditions.

**(J1)** \( \phi^* : [t_0, T] \times \mathbb{R}_+^d \times \mathbb{E} \to \mathbb{R}^d \) factorizes into \( \phi^*(t, s, y) := \varphi^*(t, s)\psi^*(t, y) \), where \( \varphi^* : [t_0, T] \times \mathbb{R}_+^d \to \mathbb{R}_+, \varphi^*(t, \cdot) \) is convex, \( \forall t \in [t_0, T] \), and \( \psi^* : [t_0, T] \times \mathbb{E} \to \mathbb{R}^d \).

**(J2)** \( \phi^* : [t_0, T] \times \mathbb{R}_+^d \times \mathbb{E} \to \mathbb{R}^d \) factorizes into \( \phi^*(t, s, y) := \varphi^*(t, s)\psi^*(t, y) \), where \( \varphi^* : [t_0, T] \times \mathbb{R}_+^d \to \mathbb{R}_+, \varphi^*(t, \cdot) \) is affine-linear, \( \forall t \in [t_0, T] \), and \( \psi^* : [t_0, T] \times \mathbb{E} \to \mathbb{R} \).

**(J3)** \( d = 1, \mathbb{E} = \mathbb{R} \) and \( \phi^* : [t_0, T] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \) factorizes into \( \phi^*(t, s, y) := \sum_{i \leq m} \varphi^*_i(t, s)\psi^*_i(t, y) \), where \( \varphi^*_i : [t_0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \) is convex, \( \forall t \in [t_0, T], i \leq m \) and \( \psi^*_i : [t_0, T] \times \mathbb{R} \to \mathbb{R}_+ \) is non-decreasing, \( \forall i \leq m \).
Observe that in case (J3) the model $S^*$ only has positive jumps.

**Remark 1.2.5.** Under the jump conditions (J1) and (J2), the SDE in (1.16) can be seen as a diffusion driven by a process with independent increments (PII). For time-independent $\psi^*$ it is a Lévy driven diffusion. In the case $d = 1$ the class of functions in (J3) allows to approximate general linking functions $\phi^*$, thus in this case the assumption allows to deal with more general diffusions with jumps.

To establish the propagation of convexity property $\text{PO}(g)$ we proceed as in the diffusion case. We establish the propagation of convexity for the Euler scheme corresponding to the comparison process $S^*$ and then make use of the approximation property. For $t_0 \in [0, T]$ we discretize $[t_0, T]$ into $K + 1$ equidistant points $t_i := t_0 + \frac{i(T-t_0)}{K}$, $i \in \{0, \ldots, K\}$, s.th. $\lambda^*(\mathbb{E}) \Delta t_i < 1$, $\Delta t_i := t_{i+1} - t_i$, and denote the Euler scheme of $S^*$ by

$$
\widetilde{S}_{t_0}^* = S_{t_0}^* + \sigma^*(t_i, S_{t_i}^*) \left( W_{t_{i+1}}^* - W_{t_i}^* \right) + \phi^*(t_i, S_{t_i}^*, Y^*) \tilde{N}^* - E^{Y^*} \phi^*(t_i, S_{t_i}^*, Y^*) \lambda^*(\mathbb{E}) \Delta t_i, \quad (1.16)
$$

where $\tilde{N}^*$ is binomial with $P^{\tilde{N}^*} = \left( 1 - \lambda^*(\mathbb{E}) \Delta t_i \right) \delta_{(0)} + \lambda^*(\mathbb{E}) \Delta t_i \delta_{(1)}$, $Y^*$ has distribution $\frac{\lambda^*(dy)}{\lambda^*(\mathbb{E})}$ on $(\mathbb{E}, \mathcal{E})$ and $E^{Y^*} f(\cdot, \cdot, Y^*) := \frac{1}{\lambda^*(\mathbb{E})} \int f(\cdot, \cdot, y) \lambda^*(dy)$. Also in this case we assume that the approximation property $\text{AP}(g)$ is satisfied. Sufficient condition like smoothness and linear growth of the coefficients, and smoothness and growth conditions on the test function $g$ are given e.g. in Liu and Li (2000). For suitable convergence results for Lévy driven diffusions we refer to Protter and Talay (1997), and Jacod, Kurtz, Méléard, and Protter (2005).

**Theorem 1.2.6 (Propagation of convexity, diffusion with jumps case).** Let $g \in \mathcal{F}_c$, $S^*$ be a $d$-dimensional diffusion with jumps and assume that the Euler scheme $\tilde{S}_K^*$ of $S^*$ satisfies Assumption $\text{AP}(g)$. If $\sigma^*(t, \cdot)$ is convex for all $t \in [0, T]$, where $M_+(d, \mathbb{R})$ is supplied with the positive semidefinite ordering $\leq_{\text{psd}}$, and the jump part satisfies one of the conditions (J1)–(J3) then the propagation of convexity property $\text{PO}(g)$ holds, i.e.

$$
\mathcal{G}(t, \cdot) \in \mathcal{F}_c, \quad \forall t \in [0, T].
$$

**Proof.** The proof uses similar to that of Theorem 1.2.2 the Euler approximation scheme $\tilde{S}_K^*$ in (1.16). The main part that needs to be established is the $\leq_{\text{ex}}$-monotonicity of the corresponding Markov operator $T_t S^* \overset{d}{=} S^* + \sigma^*(t, S^*) W^* + \phi^*(t, S^*, Y^*) \tilde{N}^* - E^{Y^*} \phi^*(t, S^*, Y^*) E^{Y^*} \tilde{N}^*$. This is the content of the following lemma, in which we suppress the superscript asterix for better readability. \qed
Lemma 1.2.7 (A convex ordering result for Markov operators).

Let \( S, W, N, Y \) be independent integrable random variables, where \( S, W \) are \( \mathbb{R}^d \)-valued, \( W \sim \mathcal{N}(0, \Delta I) \), \( \Delta \in \mathbb{R}_{++} \), \( Y \) is \( \mathbb{E} \)-valued and \( N \) has values in \( \mathbb{R} \). Assume that \( \sigma : \mathbb{R}^d \to M_+(d, \mathbb{R}) \) is convex and consider the Markov operator \( T \) of \( S \) with \( \sigma(S)W + \phi(S, Y)N - E\psi(S, Y)EN \).

Then \( T \) is \( \leq_{\text{cx}} \)-monotone, if one of the following conditions holds true:

1. \( \phi : \mathbb{R}^d_+ \times \mathbb{E} \to \mathbb{R}^d \) factorizes into \( \phi(s, y) = \varphi(s)\psi(y) \), where \( \varphi : \mathbb{R}^d_+ \to \mathbb{R}_+ \) is convex and \( \psi : \mathbb{E} \to \mathbb{R}^d \).
2. \( \phi : \mathbb{R}^d_+ \times \mathbb{E} \to \mathbb{R}^d \) factorizes into \( \phi(s, y) = \varphi(s)\psi(y) \), where \( \varphi : \mathbb{R}^d_+ \to \mathbb{R}^d \) is affine-linear and \( \psi : \mathbb{E} \to \mathbb{R} \).
3. \( d = 1 \), \( \mathbb{E} = \mathbb{R} \), \( N \geq 0 \) and \( \phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \) factorizes into \( \phi(s, y) = \sum_{i \leq m} \varphi_i(s)\psi_i(y) \), where \( \varphi_i : \mathbb{R}_+ \to \mathbb{R}_+ \) are convex, and \( \psi_i : \mathbb{R} \to \mathbb{R}_+ \) are non-decreasing, \( i \leq m \).

Proof. We first consider the cases 1 and 2 simultaneously. Assume that \( S_1, S_2 \) are \( d \)-dimensional random vectors that are independent of \( W, N \) and \( Y \) and satisfy \( S_1 \leq_{\text{cx}} S_2 \). Due to Strassen’s Theorem there are \( \hat{S}_i \overset{d}{=} S_i \) s.t. \( E(\hat{S}_2|S_1) = \hat{S}_1 \), without loss of generality we assume that \( \hat{S}_i = S_i \). For \( f \in \mathcal{F}_{\text{cx}} \) Jensen’s inequality implies

\[
Ef(TS_2) = EE\left(f(S_2 + \sigma(S_2)W + \varphi(S_2)(\psi(Y)N - E(\psi(Y)N)) | S_1, W, Y, N\right) \\
\geq Ef\left(E(S_2|S_1) + E(\sigma(S_2)|S_1)W + E(\varphi(S_2)|S_1)(\psi(Y)N - E(\psi(Y)N))\right) \\
= E^{S_1}Ef\left(s_1 + E(\sigma(S_2)|s_1)W + E(\varphi(S_2)|s_1)(\psi(Y)N - E(\psi(Y)N))\right).
\]

Due to Lemma 1.2.1 and as the convex order satisfies the stability under convolutions property (C) and the stability under mixtures property (M) it suffices to establish

\[
C := E(\varphi(S_2)|s_1)(\psi(Y)N - E(\psi(Y)N)) \\
\geq_{\text{cx}} \varphi(s_1)(\psi(Y)N - E(\psi(Y)N)) =: B.
\]

From now on we consider the cases 1 and 2 separately.

1. For convex \( \varphi : \mathbb{R}^d_+ \to \mathbb{R}_+ \) Jensen’s inequality implies that \( \vartheta(s_1) := E(\varphi(S_2)|s_1) - \varphi(s_1) \geq 0 \). For \( j \leq d \) we define \( R_j = \vartheta(s_1)(\psi_j(Y)N - E(\psi_j(Y)N)) \). Then it follows from convexity of \( f \) that

\[
Ef(C) \geq Ef(B) + E(\nabla f(B), R),
\]
where \( R = (R_1, \ldots, R_d) \), \( \nabla \) is the gradient and \( \langle \cdot, \cdot \rangle \) stands for the scalar product in \( \mathbb{R}^d \). From \( R = \frac{\partial (s_1)}{\varphi(s_1)}B \) it follows that \( E\langle \nabla f(B), R \rangle = \frac{\varphi(s_1)}{\varphi(s_1)}E\langle \nabla f(B), B \rangle \). Due to a characterization result of optimal couplings in Rüschendorf and Rachev (1990, Theorem 1) it follows that \( (B, \nabla f(B)) \) is an optimal \( \ell_2 \)-coupling. This implies \( E\langle \nabla f(B), B \rangle \geq \langle E\nabla f(B), EB \rangle = 0 \), as \( EB = 0 \).

2. From affine-linearity of \( \varphi \) it follows that \( E(\varphi(S_2)|s_1) = \varphi(s_1) \), hence \( C \stackrel{d}{=} B \).

3. For the third case let \( d = 1 \) and \( \phi(s, y) = \sum_{i \leq m} \varphi_i(s)\psi_i(y) \). As for \( f \in \mathcal{F}_{cx} \) Jensen’s inequality implies

\[
Ef(T S_2) \geq E^{S_1}Ef\bigg(s_1 + E(\sigma(S_2)|s_1)W
\bigg) + \sum_{i \leq m} E(\varphi_i(S_2)|s_1)(\psi_i(Y)N - E(\psi_i(Y)N)) \bigg),
\]

it suffices to establish

\[
C := \sum_{i \leq m} E(\varphi_i(S_2)|s_1)(\psi_i(Y)N - E(\psi_i(Y)N))
\geq_{cx} \sum_{i \leq m} \varphi_i(s_1)(\psi_i(Y)N - E(\psi_i(Y)N)) =: B.
\]

Due to Jensen’s inequality \( \vartheta_i(s_1) := E(\varphi_i(S_2)|s_1) - \varphi_i(s_1) \) is non-negative. For \( f \in \mathcal{F}_{cx} \cap C^2 \) and \( R := \sum_{i \leq m} \vartheta_i(s_1)(\psi_i(Y)N - E(\psi_i(Y)N)) \), convexity of \( f \) implies

\[
Ef(C) \geq Ef(B) + Ef'(B)R.
\]

To prove that \( Ef'(B)R \) is non-negative, we make use of some results on association of random vectors (cp. Müller and Stoyan (2002, Theorems 3.10.5, 3.10.7.)). As \( Y, N \) are independent random variables, \( (Y, N) \) is associated. From monotonicity of \( \psi_i \geq 0 \) it follows that \( \Psi_i(y, n) := \psi_i(y)n \), \( n \geq 0 \), is non-decreasing in \( (y, n) \), \( \forall i \leq m \), and, therefore \( \Psi_i \geq 0 \) implies that

\[
Z := (Z_1, \ldots, Z_m) = (\Psi_1(Y, N), \ldots, \Psi_m(Y, N))
\]

is associated, thus \( Z = Z - EZ \) is associated (see Müller and Stoyan (2002, Theorem 3.10.7.)). From non-negativity of \( \varphi_i(s_1) \) and \( \vartheta_i(s_1) \) it follows that \( (B, R) = (\sum_{i \leq m} \varphi_i(s_1)\tilde{Z}_i, \sum_{i \leq m} \vartheta_i(s_1)\tilde{Z}_i) \) is non-decreasing in \( Z \) and, therefore, is associated. Thus, \( EF_1(B, R)F_2(B, R) \geq EF_1(B, R) EF_2(B, R) \) for all non-decreasing \( F_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, k = 1, 2 \). As \( f \in \mathcal{F}_{cx} \), \( f' \) is non-decreasing and with \( F_1(B, R) := f'(B) \), \( F_2(B, R) := R \) it follows that \( Ef'(B)R \geq Ef'(B)ER = 0 \).
Similarly to the previous subsection, there is also a monotone convex ordering result for diffusions with jumps that have an additional drift component. The SDE of the comparison process \( S^* \) in this case is given by

\[
dS^*_t = b^*(t, S^*_t)dt + \sigma^*(t, S^*_t)dW^*_t + \phi^*(t, S^*_t, y)(N^*(dt, dy) - \lambda^*(dy)dt),
\]

\( S^*_0 = s_0, \) (1.17)

where \( b^*: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma^*: [0, T] \times \mathbb{R}^d \to M_+(d, \mathbb{R}) \) are increasing and convex in the space variable. Additionally, we assume that the jump coefficient \( \phi^* \) factorizes similar to (J1) and (J3), with an additional monotonicity assumption in the space variable.

(J1') \( \phi^*: [t_0, T] \times \mathbb{R}^d \times \mathbb{E} \to \mathbb{R}^d \) factorizes into \( \phi^*(t, s, y) := \varphi^*(t, s)\psi^*(t, y) \), where \( \varphi^*: [t_0, T] \times \mathbb{R} \to \mathbb{R}^+ \), \( \varphi^*(t, \cdot) \) is increasing and convex, \( \forall t \in [t_0, T] \), and \( \psi^*: [t_0, T] \times \mathbb{E} \to \mathbb{R}^d \).

(J3') \( d = 1, \mathbb{E} = \mathbb{R}, N \geq 0 \) and \( \phi^*: [t_0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \) factorizes into \( \phi^*(t, s, y) := \sum_{i \leq m} \varphi_i^*(t, s)\psi_i^*(t, y) \), where \( \varphi_i^*: [t_0, T] \times \mathbb{R} \to \mathbb{R}^+ \) is increasing and convex in the space variable, \( \forall t \in [t_0, T], i \leq m \) and \( \psi_i^*: [t_0, T] \times \mathbb{R} \to \mathbb{R}^+ \) is non-decreasing, \( \forall i \leq m \).

The Euler approximation scheme \( \bar{S}^*_{K,t} \) of \( S^* \) is of the form

\[
\bar{S}^*_{K,t+1} = \bar{S}^*_{K,t} + b^*(t_i, \bar{S}^*_{K,t})\Delta t_i + \sigma^*(t_i, \bar{S}^*_{K,t})(W^*_{t_i+1} - W^*_{t_i}) + \phi^*(t_i, \bar{S}^*_{K,t}, Y^*)\tilde{N}^* - EY^*\phi^*(t_i, \bar{S}^*_{K,t}, Y^*)\lambda^*(\mathbb{R})\Delta t_i,
\]

\( \bar{S}^*_{K,t_0} = s. \)

The main point is to establish \( \leq_{\text{icx}} \) monotonicity of the Markov transition operator \( T_t \) corresponding to \( \bar{S}^*_K \), which is given in the next lemma. As before we suppress the superscript asterix.

**Lemma 1.2.8 (A monotone convex ordering result for Markov operators).** Let \( S, W, N, Y \) be independent integrable random variables, where \( S, W \) are \( \mathbb{R}^d \)-valued, \( W \sim N(0, \Delta I) \), \( \Delta \in \mathbb{R}^+ \), \( Y \) is \( \mathbb{E} \)-valued, and \( N \) has values in \( \mathbb{R} \). Assume that \( b: \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma: \mathbb{R}^d \to M_+(d, \mathbb{R}) \) are increasing and convex, where \( M_+(d, \mathbb{R}) \) is supplied with the positive semidefinite ordering \( \preceq_{\text{psd}} \). Then the Markov operator \( T \) that is given by

\( T S = S + \Delta b(S) + \sigma(S)W + \phi(S,Y)N - EY\phi(S,Y)EN \) is \( \leq_{\text{icx}} \) monotone, if one of the following conditions holds true
1. $\phi : \mathbb{R}^d \times E \to \mathbb{R}^d$ factorizes into $\phi(s, y) = \varphi(s)\psi(y)$, where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is increasing and convex and $\psi : E \to \mathbb{R}$.

2. $d = 1$, $E = \mathbb{R}$, $N \geq 0$ and $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ factorizes into $\phi(s, y) = \sum_{i \leq m} \varphi_i(s)\psi_i(y)$, where $\varphi_i : \mathbb{R} \to \mathbb{R}_+$ are increasing and convex, and $\psi_i : \mathbb{R} \to \mathbb{R}_+$ are non-decreasing, $i \leq m$.

Proof. 1. Due to stability of the increasing convex order under convolutions and mixtures, and due to Lemma 1.2.3 it remains to establish

$$C := E(\varphi(S_2)|s_1)(\psi(Y)N - E\psi(Y)N) \geq_{\text{icx}} \varphi(s_1)(\psi(Y)N - E\psi(Y)N) =: B.$$  

As $\varphi$ is convex and increasing it holds true that $E(\varphi(S_2)|s_1) \geq \varphi(E(S_2|s_1)) \geq \varphi(s_1)$, and similarly to the proof of Lemma 1.2.7 it follows that $B \leq_{\text{cx}} C$.

2. It remains to verify $C := \sum_{i \leq m} E(\varphi_i(S_2)|s_1)(\psi_i(Y)N - E\psi_i(Y)N) \geq_{\text{icx}} \sum_{i \leq m} \varphi_i(s_1)(\psi_i(Y)N - E\psi_i(Y)N) =: B$. As $\varphi_i$ is convex and increasing it holds true that $\varphi_i(s_1) := E(\varphi_i(S_2)|s_1) - \varphi_i(s_1) \geq 0$ and similarly to the proof of Lemma 1.2.7 it follows that $B \leq_{\text{cx}} C$. \qed

Remark 1.2.9. The analogue of (J2) for the monotone convex case is to assume that $\varphi : \mathbb{R}_+^d \to \mathbb{R}_+^d$ is affine-linear and increasing. In this case it holds true that $E(\varphi(S_2)|s_1) = \varphi(E(S_2|s_1)) \geq \varphi(s_1)$, which implies directionally convex ordering $B \leq_{\text{dcx}} C$: For $a, b \in \mathbb{R}^d$, $a \leq b$ and a random variable $X$ with values in $\mathbb{R}$ and $EX = 0$ it holds true that $a_iX \leq_{\text{cx}} b_iX$, for all $i \leq d$. As $aX$ and $bX$ are comonotone, it follows from Müller and Stoyan (2002, Lemma 3.12.13) that $aX \leq_{\text{dcx}} bX$.

The propagation of monotone convexity of the backward function $G$ in a multivariate diffusion model with drift reads as follows.

Theorem 1.2.10 (Propagation of monotone convexity, diffusion with jumps and drift case). Let $g \in \mathcal{F}_{\text{icx}}$, $S^*$ be a $d$-dimensional diffusion with jumps and additional drift component with SDE (1.17) and assume that the corresponding Euler scheme $\tilde{S}_K^*$ of $S^*$ satisfies Assumption $\text{AP}(g)$. If $\sigma^*(t, \cdot)$ is convex for all $t \in [0, T]$ and the jump part satisfies the condition (J1') or (J3') then the propagation of convexity property $\text{PO}(g)$ holds true, i.e.

$$G(t, \cdot) \in \mathcal{F}_{\text{cx}}, \quad \forall t \in [0, T].$$

1.2.3 $\text{PO}(g)$ for processes with independent increments

Now we establish the propagation of order property $\text{PO}(g)$, $g \in \mathcal{F}$, for all orders generated by $\mathcal{F}$ in (1) in the case where the paths of $S^*$, $X^* = \text{Log}(S^*)$
and $\bar{X}^* = \log(S^*)$, respectively, have independent increments (PII). From our assumption that the corresponding characteristics are absolutely continuous w.r.t. the Lebesgue measure, the PIIs have no fixed times of discontinuity, cp. Jacod and Shiryaev (2003, Theorem II.4.18). In the terminology of Sato (1999, Definition 1.1.6) this type of processes is called additive processes. Additive processes are càdlàg modifications of additive processes in law, which are characterized as Markov processes with spatially homogeneous transition functions, cp. Sato (1999, Theorem 2.10.4). The transition function $P^{S^*}_{s,t}(x,B) := P^{S^*}(S^*_t - S^*_s \in B - x)$ of a Markov process $S^*$ is called spatially homogeneous, if it is of the form

$$P^{S^*}_{s,t}(x,B) = P^{S^*}_{s,t}(0,B - x) =: P^{S^*}(B - x), \quad B \in \mathcal{B}^d.$$ 

The propagation of order property $\text{PO}(g)$ for PII mainly follows from a representation of the backward function $G(t,s)$ in terms of the corresponding spatially homogeneous transition function.

**Lemma 1.2.11 (PO$(g)$ for PII).** Let $g \in \mathcal{F}$, where $\mathcal{F}$ is one of the order generating function classes in (1).

1. If $S^*$ is a PII, then $S^*$ satisfies $\text{PO}(g)$.

2. If $\bar{X}^*$ is a PII, then $S^* = e^{\bar{X}^*}$ satisfies $\text{PO}(g)$.

3. If $X^*$ is a PII, then $S^* = \mathcal{E}(X^*)$ satisfies $\text{PO}(g)$.

**Proof.** 1. As $S^*$ is an additive process, it has a spatially homogeneous transition function $P^{S^*}_{v,w}$, hence the backward function $G$ has representation

$$G(t,s) = \int g(y+s)P^{S^*}_{t,T}(dy).$$

As the property $g \in \mathcal{F}$ is characterized by first and/or second order differences, the result follows from this representation.

2. The PII $\bar{X}^*$ has a spatially homogeneous transition function $\bar{P}^{\bar{X}^*}_{v,w}$, hence the backward function $G$ is of the form

$$G(t,s) = \int g(se^y)\bar{P}^{\bar{X}^*}_{t,T}(dy).$$

3. Due to Lemma A.1.1, $\bar{X}^* = \log(\mathcal{E}(X^*))$ is a PII, thus the result follows from the representation $G(t,s) = E^*(g(\mathcal{E}(X^*_T) \mid \mathcal{E}(X^*)_t = s)) = E^*(g(e^{X^*_t}) \mid e^{X^*_t} = s)$ and the previous result. \qed
As corollary of Lemma 1.2.11, Theorem 1.1.7 and Remark 1.1.8 we obtain a well-known ordering result for multivariate normal random variables, cp. Müller and Stoyan (2002, Section 3.13).

**Corollary 1.2.12 (Ordering of normal random vectors).** Let $N \sim N(\mu, \Sigma), \ N^* \sim (\mu^*, \Sigma^*)$ be normally distributed random variables with expectation vectors $\mu, \mu^*$ and covariance matrices $\Sigma, \Sigma^*$.

1. If $\mu \leq \mu^*$ and $\Sigma = \Sigma^*$ then $N \leq_{st} N^*$.
2. If $\mu = \mu^*$ and $\Sigma \leq_{psd} \Sigma^*$ then $N \leq_{cx} N^*$.
3. If $\mu = \mu^*$ and $\Sigma_{ij} \leq \Sigma^*_{ij}, \forall i, j \leq d$, then $N \leq_{dcx} N^*$.
4. If $\mu \leq \mu^*$ and $\Sigma \leq_{psd} \Sigma^*$ then $N \leq_{icx} N^*$.
5. If $\mu \leq \mu^*$ and $\Sigma_{ij} \leq \Sigma^*_{ij}, \forall i, j \leq d$, then $N \leq_{idcx} N^*$.
Chapter 2

Comparison of Lévy processes

In this chapter we obtain several comparison results for finite-dimensional distributions of univariate and multivariate Lévy processes $S^{(i)}$, $i = 1, 2$, w.r.t. the order generating function classes $\mathcal{F}$ in (1). Thereto, we introduce the class of functions $\mathcal{F}^{(m)} := \{ f := (\mathbb{R}^d)^m \to \mathbb{R} : f(s_1, \ldots, s_{i-1}, \cdot, s_{i+1}, \ldots, s_m) \in \mathcal{F}, s_i \in \mathbb{R}^d, i \leq m \}, m, d \in \mathbb{N}$, that are componentwise in $\mathcal{F}$. A $d$-dimensional process $S^{(1)}$ is said to have smaller finite-dimensional distributions with respect to the product ordering induced by $\mathcal{F}$ than a $d$-dimensional process $S^{(2)}$, if for every $m \in \mathbb{N}$ and all $0 \leq t_1 < \cdots < t_m \leq T$ it holds true that $E g(S^{(1)}_{t_1}, \ldots, S^{(1)}_{t_m}) \leq E g(S^{(2)}_{t_1}, \ldots, S^{(2)}_{t_m})$, for all $g \in \mathcal{F}^{(m)}$. We denote this ordering by

$$(S^{(1)}_t) \leq_{\mathcal{F}} (S^{(2)}_t).$$

For time-homogeneous Markov processes the existence of a $\leq_{\mathcal{F}}$-monotone transition kernel that separates the transition kernels of $S^{(i)}$ is sufficient to establish ordering of the finite-dimensional distributions. A useful tool is the following separation result.

**Lemma (Ordering of finite-dimensional distributions)** Two time-homogeneous Markov processes $(S^{(1)}_t)_{t \in [t_1, T]}$ and $(S^{(2)}_t)_{t \in [t_1, T]}$ with transition kernels $Q^{(1)}_t$ and $Q^{(2)}_t$ satisfy

$$(S^{(1)}_t) \leq_{\mathcal{F}} (S^{(2)}_t),$$

if $S^{(1)}_{t_1} \leq_{\mathcal{F}} S^{(2)}_{t_1}$ and if a family $(Q_t)$ of $\leq_{\mathcal{F}}$-monotone transition kernels exists such that

$$Q^{(1)}_t(x, \cdot) \leq_{\mathcal{F}} Q_t(x, \cdot) \leq_{\mathcal{F}} Q^{(2)}_t(x, \cdot),$$

for all $x$ and all $t > 0$. 
Proof. The proof uses similar arguments as the proof of Theorem 5.2.15 in Müller and Stoyan (2002). We consider the case of dimension \( m = 2 \).

For \( f \in \mathcal{F}(2) \), \( \preceq \)-monotonicity of \( Q_{t_2-t_1}(s_1, \cdot) \) is equivalent to \( g(s_1) := \int f(s_1, s_2)Q_{t_2-t_1}(s_1, ds_2) \in \mathcal{F} \). As \( f(s_1, \cdot) \in \mathcal{F} \), the ordering of the transition kernels implies for \( f^{(i)}(s_1) := \int f(s_1, s_2)Q^{(i)}_{t_2-t_1}(s_1, ds_2) \) that \( f^{(1)}(s_1) \preceq g(s_1) \preceq f^{(2)}(s_1) \). Hence \( S^{(1)}_{t_1} \preceq F S^{(2)}_{t_1} \) implies

\[
Ef(S^{(1)}_{t_1}, S^{(1)}_{t_2}) = \int f^{(1)}(s_1)P^{S^{(1)}_{t_1}}(ds_1) \leq \int g(s_1)P^{S^{(1)}_{t_1}}(ds_1) \\
\leq \int g(s_1)P^{S^{(2)}_{t_1}}(ds_1) \leq \int f^{(2)}(s_1)P^{S^{(2)}_{t_1}}(ds_1) = Ef(S^{(2)}_{t_1}, S^{(2)}_{t_2})
\]

The result for \( m > 2 \) follows by induction.

In section 2.1 we obtain stochastic and convex type comparison results for the particular case of compound Poisson processes with drift. These are mainly due to a representation of compound Poisson processes as random sum processes with a Poisson number of summands, which allows comparison results by a natural coupling argument. We make intensive use of the fact that the integral stochastic orders \( \preceq \) generated by \( \mathcal{F} \) in (1) satisfy the stability under convolution property (C) and the stability under mixtures property (MI). In the case where the Lévy measures \( F^{(i)} \) of the compound Poisson processes \( S^{(i)} \), \( i = 1, 2 \), have the same finite total mass, the ordering results are of the type that ordering the Lévy measures \( F^{(1)} \preceq F^{(2)} \) implies the corresponding ordering of the compound Poisson processes \( (S^{(1)}_{t_1}) \preceq (S^{(2)}_{t_1}) \), if the drift parts are suitably chosen. If the Lévy measures \( F^{(i)} \) have different total mass we modify the Lévy measure with smaller total mass by adding the difference mass \(|F^{(1)}(\mathbb{R}^d) - F^{(2)}(\mathbb{R}^d)|\) as point mass into the origin. This technique is also used in Möller (2004).

As corollaries we establish several cut and domination criteria for one-dimensional compound Poisson processes in terms of the corresponding Lévy measures.

In section 2.2 we extend the comparison results for compound Poisson processes to pure jump Lévy processes that have infinite Lévy measures, what is due to the following procedure. We truncate the Lévy measures around the origin and give ordering conditions on the drift and the truncated jump parts that imply ordering of the induced compound Poisson processes. Establishing weak convergence yields to the corresponding ordering of the limit processes. In the one-dimensional case we obtain several cut and domination criteria in terms of Lévy measures that imply ordering of the corresponding Lévy processes. For most of the assertions we state several versions that depend on the regularity of the paths of the processes. This is due to the
2.1 Compound Poisson processes

As we want to compare Lévy processes whose Lévy measures do not necessarily have the same finite total mass, we make use of the modified Lévy measures, which for $k \in \{1, 2\}$ s.t. $\|F^{(k)}\| \leq \|F^{(3-k)}\|$ are given by $\tilde{F}^{(k)}(dx) = F^{(k)}(dx) + (\|F^{(3-k)}\| - \|F^{(k)}\|)^+ \delta_{\{0\}}(dx), \tag{2.1}$ where $a^+ = \max\{0, a\}$ is the positive part of $a \in \mathbb{R}$ and $\delta_{\{0\}}$ denotes the Dirac measure in the origin. Observe that by the Lévy–Khintchine formula it holds true that $S^{(i)} \sim (b^{(i)}(0), 0, \tilde{F}^{(i)})_0, \ i = 1, 2$, cp. Lemma A.2.3. Even though we leave aside the standard convention that the jump kernel has no point mass in the origin, the modified Lévy measures $\tilde{F}^{(i)}$ uniquely characterize the distributions of $S^{(i)}$ modulo this point mass. The general comparison result for (multivariate) compound Poisson processes is as follows.

Lemma 2.1.1 (Ordering of compound Poisson processes). Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0, \ i = 1, 2$, be $d$-dimensional compound Poisson processes and assume that the Lévy measures $F^{(i)}$ satisfy $\int_{|x| > 1} |x| F^{(i)}(dx) < \infty$. Let the modified Lévy measures $\tilde{F}^{(i)}$ be given by (2.1). If $\tilde{F}^{(1)} \lesssim \tilde{F}^{(2)}$ holds true for
1. $\mathcal{F} \in \{\mathcal{F}_{st}, \mathcal{F}_{icx}, \mathcal{F}_{idcx}, \mathcal{F}_{ism}\}$ and, additionally, $b^{(1)}(0) \leq b^{(2)}(0)$,

or for

2. $\mathcal{F} \in \{\mathcal{F}_{cx}, \mathcal{F}_{dcx}, \mathcal{F}_{sm}\}$ and, additionally, $b^{(1)}(0) = b^{(2)}(0)$,

then $(S^{(1)}_t) \leq_F (S^{(2)}_t)$.

Proof. The compound Poisson processes $S \sim (b^{(i)}(0), 0, \tilde{F}^{(i)})_0$, $i = 1, 2$, with modified Lévy measures $\tilde{F}^{(i)}$ given in (2.1) are representable as random sum processes

$$S^{(i)}_t = b^{(i)}(0)t + \sum_{j=1}^{N_t} X^{(i)}_j, \quad i = 1, 2,$$

where $(X^{(i)}_j)$ are iid random vectors with distributions $\tilde{R}^{(i)}(dx) = \frac{1}{\lambda} \tilde{F}^{(i)}(dx)$, $\lambda := \|\tilde{F}^{(i)}\|$, that are independent of the Poisson process $N_t$ with intensity $\lambda$. Thus $S^{(1)}$ and $S^{(2)}$ are naturally coupled by the same Poisson process $N$. As $\|\tilde{F}^{(i)}\| = \lambda$, $i = 1, 2$, the condition $\tilde{F}^{(1)} \leq_F \tilde{F}^{(2)}$ implies $X^{(1)}_j \leq_F X^{(2)}_j$, for all $j$. Due to property (C) and property (MI), the coupling of $S^{(1)}$ and $S^{(2)}$ implies for $f \in \mathcal{F}$

$$Ef\left(\sum_{j=1}^{N_t} X^{(1)}_j\right) = E^{N_t}Ef\left(\sum_{j=1}^{n} X_j^{(1)}\right) \leq E^{N_t}Ef\left(\sum_{j=1}^{n} X_j^{(2)}\right)$$

$$= Ef\left(\sum_{j=1}^{N_t} X_j^{(2)}\right),$$

(2.2)

where $E^{N_t}$ denotes the expectation with respect to the distribution of $N_t$. Now we consider the two different cases.

1. For $\mathcal{F} \in \{\mathcal{F}_{st}, \mathcal{F}_{icx}, \mathcal{F}_{idcx}, \mathcal{F}_{ism}\}$ the ordering $\leq_F$ satisfies property (T), hence it follows from $b^{(1)}(0) \leq b^{(2)}(0)$ that

$$b^{(1)}(0)t + \sum_{j=1}^{N_t} X^{(2)}_j \leq_F b^{(1)}(0)t + \sum_{j=1}^{N_t} X^{(2)}_j + (b^{(2)}(0) - b^{(1)}(0))t,$$

The ordering (2.2) and property (T) imply

$$S^{(1)}_t = b^{(1)}(0)t + \sum_{j=1}^{N_t} X^{(1)}_j \leq_F b^{(1)}(0)t + \sum_{j=1}^{N_t} X^{(2)}_j \leq_F b^{(2)}(0)t + \sum_{j=1}^{N_t} X^{(2)}_j = S^{(2)}_t.$$
2. For $\mathcal{F} \in \{\mathcal{F}_{\text{cx}}, \mathcal{F}_{\text{dex}}, \mathcal{F}_{\text{sm}}\}$ and $b^{(1)}(0) = b^{(2)}(0)$ the ordering (2.2) implies

$$S^{(1)}_t = b^{(1)}(0)t + \sum_{j=1}^{N_t} X^{(1)}_j \leq_{\mathcal{F}} b^{(2)}(0)t + \sum_{j=1}^{N_t} X^{(2)}_j = S^{(2)}_t.$$ 

3. It remains to prove that also the finite-dimensional distributions are ordered. Due to the separation lemma in the introduction of this chapter we have to establish that $Q^{(1)}_t(x, \cdot) = P(S^{(1)}_t \in \cdot | S^{(1)}_0 = x)$ is $\leq_{\mathcal{F}}$-monotone. To that aim, it suffices to prove that for $f \in \mathcal{F}$ the function

$$f_{Q^{(1)}_t}(x) := \int f(y)Q^{(1)}_t(x, dy)$$

belongs to $\mathcal{F}$. From spatial homogeneity it follows that $f_{Q^{(1)}_t}$ is of the form $f_{Q^{(1)}_t}(x) = \int f(x+y)Q^{(1)}_t(dy)$, where $Q^{(1)}_t(dy) := Q^{(1)}_t(0, dy)$. As for $f \in \mathcal{F}$ it holds true that $f_x(\cdot) := f(x + \cdot)$ is $f_{Q^{(1)}_t}$ the result follows. \hfill $\square$

Lemma 2.1.1 implies several cut and domination criteria for one-dimensional compound Poisson processes $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$ in terms of their Lévy measures $F^{(i)}$ which imply $(S^{(1)}_t) \leq_{\mathcal{F}} (S^{(2)}_t)$ for $\mathcal{F} \in \{\mathcal{F}_{\text{st}}, \mathcal{F}_{\text{cx}}, \mathcal{F}_{\text{sm}}\}$ under appropriate drift conditions. First we establish three versions of a cut criterion that is parallel to the classical cut criterion for probability distribution functions due to Karlin and Novikoff (1963). If a Lévy measure $F^{(2)}$ has more mass in the tails than a Lévy measure $F^{(1)}$, and less mass near the center, then it is bigger with respect to the (increasing) convex order. Observe that in the case of Lévy measures with different finite total mass the different versions of the result depend on the locus where the ordering of the Lévy measures changes. This is due to the fact that the modified Lévy measure $\bar{F}^{(k)}$ has some point mass in the origin. In the case where the two order changes of the Lévy measures $F^{(i)}$ take place on the negative half axis, we obtain the following result.

**Theorem 2.1.2** (Cut criterion for compound Poisson processes, $k_\ell < k_r \leq 0$). Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, be one-dimensional compound Poisson processes and assume that $\int_{\{|x| > 1\}} |x| F^{(i)}(dx) < \infty$. For $k_\ell < k_r \leq 0$ assume that

$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in B((-\infty, k_\ell)), \quad \text{(2.3)}$$

$$F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in B((k_\ell, k_r)), \quad \text{(2.4)}$$

$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in B((k_r, \infty)). \quad \text{(2.5)}$$

In the case $\|F^{(1)}\| < \|F^{(2)}\|$ additionally assume that

$$F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-), \quad \text{if } F^{(1)}((-\infty, k_r]) > F^{(2)}((-\infty, k_r]).$$
1. If $b^{(1)}(0) \leq b^{(2)}(0)$ and $\int xF^{(1)}(dx) \leq \int xF^{(2)}(dx)$, then $(S^{(1)}_t) \leq_{\text{lex}} (S^{(2)}_t)$.

2. If $b^{(1)}(0) = b^{(2)}(0)$ and $\int xF^{(1)}(dx) = \int xF^{(2)}(dx)$, then $(S^{(1)}_t) \leq_{\text{cx}} (S^{(2)}_t)$.

Proof. We consider the increasing convex comparison result, the convex case is treated similarly. Let $\tilde{F}^{(i)}$, $i = 1, 2$, be the modified Lévy measures introduced in (2.1). Then $S^{(1)} \sim (b^{(1)}(0), 0, \tilde{F}^{(1)})_0$ by Lemma A.2.3. As $\|F^{(i)}\| = \lambda$, $i = 1, 2$, we are in the position to apply the classical cut criterion to the corresponding distribution functions $G^{(i)}(x) := \tilde{F}^{(i)}((-\infty, x])$.

As on $\mathbb{R}_-$ it holds true that $\tilde{F}^{(i)} = F^{(i)}$, it follows from (2.3) that

$$G^{(2)}(x) \geq G^{(1)}(x) \text{ on } (-\infty, k_\ell).$$

For $x \in (0, \infty)$ the condition (2.5) yields $\lambda - G^{(1)}(x) = F^{(1)}((x, \infty)) \leq F^{(2)}((x, \infty)) = \lambda - G^{(2)}(x)$, hence $G^{(1)} \geq G^{(2)}$ on $(0, \infty)$ and from right-continuity of $G^{(i)}$ it follows that

$$G^{(1)} \geq G^{(2)}, \text{ on } [0, \infty).$$

For the comparison of $G^{(1)}$ and $G^{(2)}$ on $[k_\ell, 0)$ we consider two different cases.

i. Assume that $\|F^{(2)}\| \leq \|F^{(1)}\|$. Then $\tilde{F}^{(2)}(dx) = F^{(2)}(dx) + (\|F^{(1)}\| - \|F^{(2)}\|)\delta_{\{0\}}(dx)$ and $\tilde{F}^{(1)} = F^{(1)}$. Then we have $G^{(1)}(0_-) = G^{(1)}(0) \geq G^{(2)}(0) = G^{(2)}(0_-) + \|F^{(1)}\| - \|F^{(2)}\| \geq G^{(2)}(0_-)$ and from (2.5) it follows for $x \in (k_\ell, 0)$ that $G^{(1)}(0_-) - G^{(1)}(x) = F^{(1)}((x, 0)) \leq F^{(2)}((x, 0)) = G^{(2)}(0_-) - G^{(2)}(x)$, thus $G^{(1)}(x) \geq G^{(2)}(x)$. Right-continuity of $G^{(i)}$ implies that also $G^{(1)}(k_\ell) \geq G^{(2)}(k_\ell)$, hence

$$G^{(1)} \geq G^{(2)} \text{ on } [k_\ell, 0).$$

As $G^{(1)}(k_\ell) \geq G^{(2)}(k_\ell)$ and $G^{(1)}(k_\ell-) \leq G^{(2)}(k_\ell-)$ it follows from (2.4) that there is a $\kappa \in [k_\ell, k_r)$ s.th. $G^{(1)}(k) \leq G^{(2)}(k)$, for all $k < \kappa$ and $G^{(1)}(k) \geq G^{(2)}(k)$, for all $k \in [\kappa, k_r]$. Consequently, $G^{(1)}$ and $G^{(2)}$ cross once and the sign sequence of the difference $G^{(1)} - G^{(2)}$ is $-, +$, hence the cut criterion implies $\tilde{F}^{(i)} \leq_{\text{lex}} \tilde{F}^{(2)}$, as $\int x\tilde{F}^{(1)}(dx) = \int xF^{(1)}(dx) \leq \int xF^{(2)}(dx) = \int x\tilde{F}^{(2)}(dx)$ by assumption.

ii. Assume that $\|F^{(1)}\| < \|F^{(2)}\|$. Then $\tilde{F}^{(1)}(dx) = F^{(1)}(dx) + (\|F^{(2)}\| - \|F^{(1)}\|)\delta_{\{0\}}(dx)$ and $\tilde{F}^{(2)} = F^{(2)}$. Again, we denote the corresponding distribution functions by $G^{(i)}$. 

First consider the case where $G^{(1)}(k_r) \leq G^{(2)}(k_r)$. In this case it follows from (2.4) and right-continuity of $G^{(i)}$ that

$$G^{(1)} \leq G^{(2)} \text{ on } [k_\ell, k_r].$$

As (2.5) then implies $G^{(1)} \leq G^{(2)}$ on $(k_r, 0)$ it follows that $G^{(1)}$ and $G^{(2)}$ cross once, namely in the origin. The cut criterion implies $\overline{F}^{(1)} \leq_{\text{lex}} \overline{F}^{(2)}$, as additionally it holds true that $\int x\overline{F}^{(1)}(dx) \leq \int x\overline{F}^{(2)}(dx)$.

In the case when $G^{(1)}(k_r) \geq G^{(2)}(k_r)$ we make the additional assumption that $F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-)$, which is $G^{(1)}(0-) \geq G^{(2)}(0-)$ in terms of $G^{(i)}$. From (2.5) and right-continuity of $G^{(i)}$ it then follows that

$$G^{(1)} \geq G^{(2)} \text{ on } [k_r, 0),$$

as for $x \in (k_r, 0)$ it holds true that $G^{(1)}(0-) - G^{(1)}(x) = F^{(1)}((x, 0)) \leq F^{(2)}((x, 0)) = G^{(2)}(0-) - G^{(2)}(x)$. As $G^{(1)}(k_r) > G^{(2)}(k_r)$ and $G^{(1)}(k_\ell-) \leq G^{(2)}(k_\ell-)$ it follows from (2.4) that $G^{(1)}$ and $G^{(2)}$ cross once in $[k_\ell, k_r]$ with sign sequence $-,$ for $G^{(1)} - G^{(2)}$. The cut criterion implies $\overline{F}^{(1)} \leq_{\text{lex}} \overline{F}^{(2)}$, as $\int x\overline{F}^{(1)}(dx) \leq \int x\overline{F}^{(2)}(dx)$ by assumption.

Then the drift condition $b^{(1)}(0) \leq b^{(2)}(0)$ and $\int xF^{(1)}(dx) \leq \int xF^{(2)}(dx)$ imply $(S^{(1)}_t) \leq_{\text{lex}} (S^{(2)}_t)$, due to Lemma 2.1.1. □

**Remark 2.1.3.** We did not make any assumptions on the ordering of $F^{(i)}$ in the points of order changes $k_\ell, k_r$.

If the pointwise ordering of the Lévy measures changes once on the negative and once on the positive half axis, we obtain the following result.

**Theorem 2.1.4 (Cut criterion for compound Poisson processes, $k_\ell \leq 0 \leq k_r$).** Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, be one-dimensional compound Poisson processes and assume that $\int_{\{x>0\}} |x|F^{(i)}(dx) < \infty$. For $k_\ell \leq 0 \leq k_r$ assume that

$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell]), \quad (2.6)$$

$$F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, k_r]), \quad (2.7)$$

$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)). \quad (2.8)$$

In the case $\|F^{(1)}\| > \|F^{(2)}\|$ additionally assume that

$$F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-) + \|F^{(1)}\| - \|F^{(2)}\|,$$

if there is a $\kappa \in [k_\ell, 0)$ s.th. $F^{(1)}((\infty, k]) \leq F^{(2)}((\infty, k])$, for all $k < \kappa$ and $F^{(1)}((\infty, k]) \geq F^{(2)}((\infty, k])$, for all $k \in [\kappa, 0)$.
1. If \( b^{(1)}(0) \leq b^{(2)}(0) \) and \( \int xF^{(1)}(dx) \leq \int xF^{(2)}(dx) \), then \( (S^{(1)}_t) \leq_{\text{lex}} (S^{(2)}_t) \).

2. If \( b^{(1)}(0) = b^{(2)}(0) \) and \( \int xF^{(1)}(dx) = \int xF^{(2)}(dx) \), then \( (S^{(1)}_t) \leq_{\text{cx}} (S^{(2)}_t) \).

**Proof.** Similarly to the previous proof we establish \( \tilde{F}^{(1)} \leq_{\text{lex}} \tilde{F}^{(2)} \) by using the cut criterion and then make use of Lemma 2.1.1. We establish the increasing convex ordering result for the case \( k_t < 0 < k_r \), the cases where one of the (or both) \( k_t = k_r = 0 \), as well as the convex result easily follow. Similar to the proof of Theorem 2.1.2 for the distributions \( G^{(i)} \) corresponding to the modified Lévy measures \( F^{(i)} \) it holds true that

\[
G^{(1)}(0) \leq G^{(2)}(0) \quad \text{on} \ (-\infty, k_t) \quad \text{and} \quad G^{(1)}(0) \geq G^{(2)}(0) \quad \text{on} \ [k_r, \infty).
\]

To prove the ordering of the distribution functions \( G^{(i)} \) on \([k_t, k_r]\) we consider two different cases.

i. If \( \| F^{(1)} \| \leq \| F^{(2)} \| \) the modified Lévy measures are given by \( \tilde{F}^{(1)}(dx) = F^{(1)}(dx) + (\| F^{(2)} \| - \| F^{(1)} \|) \delta_{(0)}(dx) \) and \( \tilde{F}^{(2)} = F^{(2)} \). From (2.7) it follows that \( G^{(i)} \) cross once or never on \([k_t, 0]\).

Assume that \( G^{(1)} \) crosses \( G^{(2)} \) once from below on \([k_t, 0]\). Then \( G^{(1)}(0_+) \geq G^{(2)}(0_+) \), hence \( G^{(1)}(0) \geq G^{(2)}(0) \), as \( \tilde{F}^{(1)} \) has some point mass in the origin. From (2.7) it follows that \( G^{(1)} \geq G^{(2)} \) on \([0, k_r]\), hence \( G^{(i)} \) cross once on \( \mathbb{R} \).

If \( G^{(i)} \) do not cross on \((-\infty, 0)\), there occur two cases, depending on the ordering of \( G^{(i)} \) in 0. If \( G^{(1)}(0) \geq G^{(2)}(0) \), it follows similar to the previous case that \( G^{(1)} \geq G^{(2)} \) on \([0, k_r]\), hence \( G^{(i)} \) cross once, namely in the origin. If \( G^{(1)}(0) < G^{(2)}(0) \) it follows from (2.7) and \( G^{(1)}(k_r) \geq G^{(2)}(k_r) \) that \( G^{(i)} \) cross once in \((0, k_r]\).

ii. If \( \| F^{(1)} \| > \| F^{(2)} \| \) the modified Lévy measures are given by \( \tilde{F}^{(2)}(dx) = F^{(2)}(dx) + (\| F^{(1)} \| - \| F^{(2)} \|) \delta_{(0)}(dx) \) and \( \tilde{F}^{(1)} = F^{(1)} \). We consider two different cases depending on the relative location of \( G^{(i)}(0) \).

If \( G^{(1)}(0_-) \leq G^{(2)}(0_-) \) it holds true that also \( G^{(1)}(0) \leq G^{(2)}(0) \) and from (2.7) and \( G^{(1)}(k_r) \geq G^{(2)}(k_r) \) it follows that \( G^{(i)} \) cross once on \((0, k_r]\).

If \( G^{(1)}(0_-) > G^{(2)}(0_-) \) then by (2.7) and \( G^{(1)}(k_t_-) \leq G^{(2)}(k_t_-) \), the distributions \( G^{(i)} \) cross once in \([k_t, 0]\). Hence we additionally assume that \( F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-) + \| F^{(1)} \| - \| F^{(2)} \| \), what is equivalent to \( G^{(1)}(0) = F^{(1)}(\mathbb{R}_-) + \| F^{(2)} \| - \| F^{(1)} \| \geq F^{(2)}(\mathbb{R}_-) = G^{(2)}(0) \), as \( G^{(2)} \) has no jump in 0. Now the same argument as in the case where \( \| F^{(1)} \| \leq \| F^{(2)} \| \) and \( G^{(1)}(0) \geq G^{(2)}(0) \) holds true, applies. \( \square \)
In the case where the dominance changes of the Lévy measures takes place on the positive half axis, we obtain the following result. We omit the proof that is along the same lines as the proofs of Theorems 2.1.2 and 2.1.4.

**Theorem 2.1.5 (Cut criterion for compound Poisson processes, \(0 \leq k_\ell < k_r\)).** Let \(S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_t, i = 1, 2,\) be one-dimensional compound Poisson processes and assume that \(\int_{\{|x| > 1\}} |x| F^{(i)}(dx) < \infty.\) For \(0 \leq k_\ell < k_r\) assume that

\[
F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)),
\]
\[
F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, k_r)),
\]
\[
F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)).
\]

In the case \(\|F^{(1)}\| < \|F^{(2)}\|\) additionally assume that

\[
F^{(2)}(\mathbb{R}_-) \geq F^{(1)}(\mathbb{R}_-) + \|F^{(2)}\| - \|F^{(1)}\|,
\]

if \(F^{(1)}((-\infty, k_\ell]) + \|F^{(2)}\| - \|F^{(1)}\| < F^{(2)}((-\infty, k_\ell]).\)

1. If \(b^{(1)}(0) \leq b^{(2)}(0)\) and \(\int x F^{(1)}(dx) \leq \int x F^{(2)}(dx),\) then \((S^{(1)}_t) \leq_{\text{icx}} (S^{(2)}_t).\)

2. If \(b^{(1)}(0) = b^{(2)}(0)\) and \(\int x F^{(1)}(dx) = \int x F^{(2)}(dx),\) then \((S^{(1)}_t) \leq_{\text{cx}} (S^{(2)}_t).\)

**Remark 2.1.6.** The second part of the previous theorem is a generalization of Theorem 6.1 in Møller (2004), which is the main comparison tool for compound Poisson processes of that paper. The condition \(b^{(1)}(0) = b^{(2)}(0)\) is implicitly assumed as he considers \(S^{(i)}\) under martingale measures and assumes \(\int x F^{(1)}(dx) = \int x F^{(2)}(dx) = d,\) hence \(b^{(i)}(0) = -\int x F^{(i)}(dx) = d,\) \(i = 1, 2.\) In addition to condition \(\int x F^{(1)}(dx) = \int x F^{(2)}(dx),\) he assumes that

\[
\frac{1}{\|F^{(1)}\|} \int x F^{(1)}(dx) \leq \frac{1}{\|F^{(2)}\|} \int x F^{(2)}(dx),\]

hence \(\|F^{(1)}\| \geq \|F^{(2)}\|,\) to obtain ordering of the terminal values \(S^{(1)}_T \leq_{\text{cx}} S^{(2)}_T.\)

For finite Lévy measures \(F^{(i)}, i = 1, 2,\) the following domination criterion implies (increasing) convex ordering of the corresponding compound Poisson processes. \(F^{(2)}\) dominates \(F^{(1)}, F^{(1)} \leq F^{(2)},\) if for all \(A \in \mathcal{B}\) it holds true that \(F^{(1)}(A) \leq F^{(2)}(A).\) Observe that this domination criterion is a generalization of Theorem 2.1.4 with \(k_\ell = k_r = 0.\)

**Theorem 2.1.7 (Domination criterion for compound Poisson processes).** Let \(S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_t, i = 1, 2,\) be one-dimensional compound Poisson processes and assume that the Lévy measures \(F^{(i)}\) satisfy \(\int_{\{|x| > 1\}} |x| F^{(i)}(dx) < \infty\) and \(F^{(1)} \leq F^{(2)}\).

1. If \(b^{(1)}(0) \leq b^{(2)}(0)\) and \(\int x F^{(1)}(dx) \leq \int x F^{(2)}(dx),\) then \((S^{(1)}_t) \leq_{\text{icx}} (S^{(2)}_t).\)
2. If \( b^{(1)}(0) = b^{(2)}(0) \) and \( \int xF^{(1)}(dx) = \int xF^{(2)}(dx) \), then \( (S_t^{(1)}) \leq_{cx} (S_t^{(2)}) \).

**Proof.** Again, we establish the classical cut criterion for the distribution functions \( G^{(i)} \) corresponding to the modified Lévy measures \( \tilde{F}^{(i)} \), and then make use of Lemma 2.1.1 to obtain (increasing) convex ordering. From \( F^{(1)} \leq F^{(2)} \) it follows that \( \|F^{(1)}\| \leq \|F^{(2)}\| \), thus \( F^{(1)} \) is modified to \( \tilde{F}^{(1)}(dx) = F^{(1)}(dx) + (\|F^{(2)}\| - \|F^{(1)}\|)\delta_{\{0\}}(dx) \). The ordering \( F^{(1)} \leq F^{(2)} \) implies

\[
G^{(1)} \leq G^{(2)} \text{ on } \mathbb{R}_- \quad \text{and} \quad G^{(1)} \geq G^{(2)} \text{ on } \mathbb{R}_+,
\]

hence the distribution functions \( G^{(i)} \) cross in the origin and the cut criterion is satisfied. \( \square \)

Now we establish three versions of a domination criterion for finite Lévy measures \( F^{(i)}, i = 1, 2 \), that implies stochastic ordering of the corresponding compound Poisson processes \( S^{(i)} \). If \( F^{(1)} \) has more mass on small values than \( F^{(2)} \) and less mass on big values, then stochastic ordering \( (S_t^{(1)}) \leq_{st} (S_t^{(2)}) \) follows under a suitable drift condition. Again, the versions depend on the locus of the dominance change. If the dominance change takes place on the negative half axis, the result is as follows.

**Theorem 2.1.8 (Criterion for stochastic ordering of compound Poisson processes, \( k \leq 0 \)).** Let \( S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0, i = 1, 2, \) be one-dimensional compound Poisson processes. Assume that \( \int_{\{|x| > 1\}} |x|F^{(i)}(dx) < \infty \) and that for \( k \leq 0 \) it holds true that

\[
F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k)),
\]

\[
F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k, \infty)),
\]

and \( b^{(1)}(0) \leq b^{(2)}(0) \). In the case \( \|F^{(1)}\| < \|F^{(2)}\| \) additionally assume that \( F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-) \). Then

\[
(S_t^{(1)}) \leq_{st} (S_t^{(2)}).
\]

**Proof.** We establish that the distribution functions \( G^{(i)} \) corresponding to the modified Lévy measures \( \tilde{F}^{(i)}, i = 1, 2 \), are ordered as \( G^{(1)} \geq G^{(2)} \). Then \( \tilde{F}^{(1)} \leq_{st} \tilde{F}^{(2)} \) holds true and the result follows from Lemma 2.1.1. From \( F^{(1)}(A) \geq F^{(2)}(A) \) for all \( A \in \mathcal{B}((-\infty, k)) \) it follows that

\[
G^{(1)} \geq G^{(2)} \text{ on } (-\infty, k).
\]

For the comparison of \( G^{(i)} \) on \([k, \infty)\) we consider two cases.
1. Assume that \(\|F^{(1)}\| < \|F^{(2)}\|\) and let \(\tilde{F}^{(1)}(dx) = F^{(1)}(dx) + (\|F^{(2)}\| - \|F^{(1)}\|)\delta_{\{0\}}(dx)\), \(\tilde{F}^{(2)} = F^{(2)}\) and \(\lambda := \|F^{(1)}\|\). For \(A := (x, \infty), x \in (0, \infty)\), it holds true that \(\lambda - G^{(2)}(x) = F^{(2)}(A) \geq F^{(1)}(A) = \lambda - G^{(1)}(x)\), hence \(G^{(2)}(x) \leq G^{(1)}(x), x \in (0, \infty)\), and from right-continuity of \(G^{(i)}\) it follows that
\[
G^{(2)} \leq G^{(1)} \text{ on } [0, \infty).
\]
Assumption \(F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-)\) is \(G^{(1)}(0-) \geq G^{(2)}(0-)\) in terms of distribution functions and a consideration that is similar to the step that yielded \(G^{(2)} \leq G^{(1)} \text{ on } [0, \infty)\) implies
\[
G^{(2)} \leq F^{(1)} \text{ on } [k, 0),
\]
hence \(G^{(2)} \leq G^{(1)}\), thus \(\tilde{F}^{(1)} \leq_{st} \tilde{F}^{(2)}\).

2. Assume that \(\|F^{(2)}\| \leq \|F^{(1)}\|\) and let \(\tilde{F}^{(2)}(dx) = F^{(2)}(dx) + (\|F^{(1)}\| - \|F^{(2)}\|)\delta_{\{0\}}(dx)\) and \(\tilde{F}^{(1)} = F^{(1)}\). Similar to the previous case it follows from \(F^{(1)}(A) \leq F^{(2)}(A), \forall A \in \mathcal{B}((k, \infty))\), that
\[
G^{(2)} \leq G^{(1)} \text{ on } [0, \infty)
\]
and \(G^{(2)}(0-) = G^{(2)}(0) - (\|F^{(1)}\| - \|F^{(2)}\|) \leq G^{(1)}(0) = G^{(1)}(0-)\). A similar argument as in the previous case yields
\[
G^{(2)} \leq G^{(1)} \text{ on } [k, 0)
\]
and the result follows. \(\square\)

If the dominance change of the Lévy measures takes place on the positive half axis, we obtain the following result.

**Theorem 2.1.9 (Criterion for stochastic ordering of compound Poisson processes, \(k \geq 0\)).** Let \(S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_{0}, i = 1, 2\), be one-dimensional compound Poisson processes. Assume that \(\int_{\{|x| > 1\}} |x| F^{(i)}(dx) < \infty\) and that for \(k \geq 0\) it holds true that
\[
F^{(1)}(A) \geq F^{(2)}(A), \forall A \in \mathcal{B}((-\infty, k)),
\]
\[
F^{(1)}(A) \leq F^{(2)}(A), \forall A \in \mathcal{B}((k, \infty)),
\]
and \(b^{(1)}(0) \leq b^{(2)}(0)\). In the case \(\|F^{(2)}\| < \|F^{(1)}\|\) additionally assume that \(F^{(1)}(\mathbb{R}_+) \leq F^{(2)}(\mathbb{R}_+)\). Then
\[
(S^{(1)}_t) \leq_{st} (S^{(2)}_t).
\]
Proof. Similar to Theorem 2.1.8 we establish $G^{(1)} \geq G^{(2)}$. From $F^{(1)}(A) \geq F^{(2)}(A)$ for all $A \in \mathcal{B}((-\infty, k))$, $k \geq 0$, it follows that

$$G^{(1)} \geq G^{(2)}$$
onumber

on $(-\infty, 0)$.

To establish $G^{(1)} \geq G^{(2)}$ on $[0, \infty)$ we consider two cases.

1. If $\|F^{(1)}\| \leq \|F^{(2)}\|$ let $\tilde{F}^{(1)}(dx) = F^{(1)}(dx) + (\|F^{(1)}\| - \|F^{(2)}\|)\delta_{\{0\}}(dx)$ and $\tilde{F}^{(2)} = F^{(2)}$. Then it holds true that $G^{(1)}(0) = G^{(1)}(0_-) + \|F^{(2)}\| - \|F^{(1)}\| \geq G^{(1)}(0_-) \geq G^{(2)}(0_-) = G^{(2)}(0)$, as $F^{(2)}(\{0\}) = 0$. For $x \in (0, k)$ we have $G^{(1)}(x) - G^{(1)}(0) = F^{(1)}((0, x]) \geq F^{(2)}((0, x]) = G^{(2)}(x) - G^{(2)}(0)$, hence

$$G^{(1)} \geq G^{(2)}$$

on $[0, k]$.

For $x \in (k, \infty)$ we obtain $\lambda - G^{(1)}(x) = F^{(1)}((x, \infty)) \leq F^{(2)}((x, \infty)) = \lambda - G^{(2)}(x)$, thus by right-continuity of $G^{(i)}$

$$G^{(1)} \geq G^{(2)}$$

on $[k, \infty)$.

2. If $\|F^{(2)}\| < \|F^{(1)}\|$ let $\tilde{F}^{(2)}(dx) = F^{(2)}(dx) + (\|F^{(1)}\| - \|F^{(2)}\|)\delta_{\{0\}}(dx)$ and $\tilde{F}^{(1)} = F^{(1)}$. From the additional assumption $F^{(1)}(\mathbb{R}_+) \leq F^{(2)}(\mathbb{R}_+)$ and $\|\tilde{F}^{(i)}\| = \lambda$ it follows that $G^{(1)}(0) \geq G^{(2)}(0)$. The ordering $G^{(1)} \geq G^{(2)}$ on $(0, \infty)$ follows similar to the previous case.

Theorems 2.1.8 and 2.1.9 imply stochastic ordering without additional conditions, if the dominance change takes place in $k = 0$.

Corollary 2.1.10 (Criterion for stochastic ordering of compound Poisson processes, $k = 0$). Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, be one-dimensional compound Poisson processes and let $\int_{|x|>1} |x| F^{(i)}(dx) < \infty$. If $b^{(1)}(0) \leq b^{(2)}(0)$ and

$$F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, 0)),
$$

$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((0, \infty)),$$

then

$$(S^{(1)}_t) \leq_{st} (S^{(2)}_t).$$
2.2 Lévy processes with infinite Lévy measures

In this section we extend the ordering results for compound Poisson processes from section 2.1 to ordering of Lévy processes with infinite Lévy measures. First we give two versions of a general comparison assertion that are parallel to Lemma 2.1.1 and depend on the regularity of the paths of the processes. Then we establish several cut and domination criteria for one-dimensional Lévy measures with infinite total mass that imply orderings of the corresponding Lévy processes. The proofs of the general ordering results are of the following type. First we truncate the Lévy measures $F^{(i)}$ around the origin to obtain truncated Lévy measures

$$F_n^{(i)}(dx) := \mathbb{I}_{(\varepsilon_n^{(i)}, \varepsilon_n^{(i)})} F^{(i)}(dx), \quad \varepsilon_n^{(i)} \uparrow 0, \varepsilon_n^{(i)} \downarrow 0,$$

that have finite total mass. Then we establish ordering of the corresponding compound Poisson processes $S_n^{(i)} \sim (ES^{(i)}_1, 0, F_n^{(i)})_{id}$ by using the appropriate ordering result stated in the previous section. Along the lines of the notion of modified Lévy measures given in (2.1) we make use of the modified truncated Lévy measures that we define by

$$\tilde{F}_n^{(k)}(dx) := F_n^{(k)}(dx) + (\|F_n^{(3-k)}\| - \|F_n^{(k)}\|)^+ \delta_0(dx).$$

Establishing functional weak convergence $S_n^{(i)} \overset{F}{\Rightarrow} S^{(i)}$ yields finite-dimensional ordering of the limit processes, if the ordering $\leq_F$ satisfies the stability under weak convergence property (W). Due to Lemma A.2.2, property (W) is satisfied by the orders $\leq_{st}, \leq_{ism}, \leq_{sm}$, whereas the orders generated by $F_{cx}, F_{icx}, F_{dcx}, F_{idcx}$ are not stable with respect to weak convergence. If for these orders additionally convergence of the expectations $ES_{n,t}^{(i)} \rightarrow ES_t^{(i)}$ holds true, then the ordering $(S_{n,t}^{(1)}) \leq_F (S_{n,t}^{(2)})$ is propagated to the limit processes, cp. Müller and Stoyan (2002, Theorems 3.4.6 and 3.12.8). A suitable functional weak convergence result is the following corollary of Jacod and Shiryaev (2003, Corollary VII.3.6), which is proved in the Appendix.

**Lemma 2.2.1 (Functional weak convergence).** Let $S \sim (b(h), 0, F)_h$ be a $d$-dimensional Lévy process whose Lévy measure $F$ has infinite total mass and for $\varepsilon_n \uparrow 0, \varepsilon_n \downarrow 0$ let $F_n$ be the corresponding truncated Lévy measure given in (2.9). If $b_n(h) \rightarrow b(h)$ then for the compound Poisson processes $S_n \sim (b_n(h), 0, F_n)_h$ functional weak convergence

$$S_n \overset{F}{\Rightarrow} S$$

holds true.
For a Lévy process $S$ with Lévy measure $F$ the existence of the first moments $ES_t$ is equivalent to the statement $\int_{\{|x|>1\}} |x| F(dx) < \infty$, which we will assume for all of the next assertions. In this case $S$ has representation $S \sim (ES_1, 0, F)_{\text{id}}$. The fact that a Lévy process $S$ has paths of finite variation, is characterized in terms of the Lévy measure by $\int_{\{|x|<1\}} |x| F(dx) < \infty$. In this case, $S$ has representation $S \sim (b(0), 0, F)_0$, and we implicitly assume that $S$ has paths of finite variation, if we give this representation. In the sequel we make use of the following corollary of Lemma 2.2.1, which is also proved in the Appendix.

**Corollary 2.2.2 (Functional weak convergence).** Let $F$ be a Lévy measure with infinite total mass and for sequences $\varepsilon_n \uparrow 0$, $\varepsilon_n \downarrow 0$ let $F_n$ be the corresponding truncated Lévy measure given in (2.9).

1. If $\int_{\{|x|>1\}} |x| F(dx) < \infty$, then for $S \sim (ES_1, 0, F)_{\text{id}}$ and $S_n \sim (ES_1, 0, F_n)_{\text{id}}$ it holds true that $S_n \xrightarrow{L} S$.

2. If for a continuous truncation function $h$ it holds true that $\int |h(x)| F(dx) < \infty$, then for $S \sim (b(0), 0, F)_0$ and $S_n \sim (b(0), 0, F_n)_0$ it follows that $S_n \xrightarrow{L} S$.

**Remark 2.2.3.** Observe that the integrability conditions in the previous corollary are necessary s.th. in the first case the modified second characteristics $\tilde{C}_n(\text{id}), \tilde{C}(\text{id})$ and in the second case the drift parts $b_n(h) := b_n(0) + \int h(x) F_n(dx), b(h) := b(0) + \int h(x) F(dx)$ are well-defined. Observe that the integrability condition in the first case characterizes the semimartingale $S$ to be locally square-integrable, cp. Jacod and Shiryaev (2003, Theorem II.2.29).

Now we establish the extension of Lemma 2.1.1 to the comparison of multivariate Lévy processes $S^{(i)}, i = 1, 2$ with infinite Lévy measures. If $S^{(i)}$ do not necessarily have paths of finite variation, we obtain the following result.

**Theorem 2.2.4 (Ordering of Lévy processes with infinite Lévy measures).** Let $S^{(i)} \sim (ES_1^{(i)}, 0, F^{(i)})_{\text{id}}, i = 1, 2$, be $d$-dimensional Lévy processes with Lévy measures $F^{(i)}$ that have infinite total mass and assume that $\int_{\{|x|>1\}} |x|^2 F^{(i)}(dx) < \infty$. Let $\varepsilon_n^{(i)} \uparrow 0$, $\varepsilon_n^{(i)} \downarrow 0$ be sequences s.th. for the modified truncated Lévy measures $\tilde{F}_n^{(i)}$ it holds true that

$$\tilde{F}_n^{(1)} \leq \tilde{F}_n^{(2)}$$

for
1. \( \mathcal{F} \in \{ \mathcal{F}_{st}, \mathcal{F}_{ix}, \mathcal{F}_{dix}, \mathcal{F}_{ism} \} \) and, additionally, it holds true that
\[ \int x F_n^{(2)}(dx) - \int x F_n^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)}, \]
or for
\[ \mathcal{F} \in \{ \mathcal{F}_{ix}, \mathcal{F}_{dix}, \mathcal{F}_{ism} \} \] and, additionally, it holds true that
\[ ES_1^{(1)} = ES_1^{(2)} \] and \( \int x F_n^{(1)}(dx) = \int x F_n^{(2)}(dx) \).

Then it follows that
\[ (S_t^{(1)}) \leq \mathcal{F} (S_t^{(2)}). \]

**Proof.** We establish the increasing convex type assertion, the proof of the convex type claim is similar. For \( \varepsilon \in \{ \varepsilon_n(1) \uparrow 0, \varepsilon_n(2) \downarrow 0, \ i = 1, 2, \) let \( F_n^{(i)} \) be the truncated Lévy measures given in (2.9). Then \( S_n^{(i)} \sim (ES_1^{(i)}, 0, F_n^{(i)})_{\text{id}} \) are compound Poisson processes and from Jacod and Shiryaev (2003, Proposition II.2.24) it follows that \( S_n^{(i)} \sim (ES_1^{(i)} - \int x F_n^{(i)}(dx), 0, F_n^{(i)})_0 \). As the modified truncated Lévy measures \( \tilde{F}_n^{(i)} \) are ordered as \( \tilde{F}_n^{(1)} \leq \tilde{F}_n^{(2)} \) and have the same finite total mass, and as the condition \( \int x F_n^{(2)}(dx) - \int x F_n^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)} \) implies \( b_n^{(1)}(0) \leq b_n^{(2)}(0) \), it follows from Lemma 2.1.1 that
\[ (S_n^{(1), t}) \leq \mathcal{F} (S_n^{(2), t}). \]

By functional weak convergence from Corollary 2.2.2 and as \( ES_n^{(i), 1} = ES_1^{(i)} \) we obtain \( (S_t^{(1)}) \leq \mathcal{F} (S_t^{(2)}) \). \( \square \)

The assumption \( \int x F_n^{(2)}(dx) - \int x F_n^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)} \) is a bounding condition for the distance of the truncated Lévy measures \( F_n^{(i)} \) that ensures \( b_n^{(1)}(0) \leq b_n^{(2)}(0) \). The second version of the general ordering result for multivariate Lévy processes with infinite Lévy measures that have paths of finite variation that does not require such a condition.

**Theorem 2.2.5 (Ordering of Lévy processes with infinite Lévy measures and paths of finite variation).** Let \( S_t^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0, \ i = 1, 2, \) be Lévy processes with infinite Lévy measures \( F^{(i)} \) that satisfy \( \int |x| F^{(i)}(dx) < \infty \). Let \( \varepsilon_n^{(1)} \uparrow 0, \varepsilon_n^{(2)} \downarrow 0 \) be sequences s.th. for the modified truncated Lévy measures \( \tilde{F}_n^{(i)} \) it holds true that
\[ \tilde{F}_n^{(1)} \leq \mathcal{F} \tilde{F}_n^{(2)} \]
and in the case where

1. \( \mathcal{F} \in \{ \mathcal{F}_{st}, \mathcal{F}_{ix}, \mathcal{F}_{dix}, \mathcal{F}_{ism} \} \), additionally it holds true that \( b^{(1)}(0) \leq b^{(2)}(0) \) and \( \int x F_n^{(1)}(dx) \leq \int x F_n^{(2)}(dx) \).
2. \[ \mathcal{F} \in \{ \mathcal{F}_{ex}, \mathcal{F}_{dx}, \mathcal{F}_{sm} \}, \] additionally it holds true that \( b^{(1)}(0) = b^{(2)}(0) \) and \( \int xF_n^{(1)}(dx) = \int xF_n^{(2)}(dx) \).

Then it follows that

\[ (S_t^{(1)}) \leq_{\mathcal{F}} (S_t^{(2)}). \]

**Proof.** Again we establish the first case, the proof of the second case is similar. For the compound Poisson processes \( S_n^{(i)} \sim (b^{(i)}(0), 0, F_n^{(i)})_0 \) it follows from Lemma 2.1.1 that \( (S_n^{(1)}) \leq_{\mathcal{F}} (S_n^{(2)}) \), due to the fact that \( S_n^{(i)} \sim (b^{(i)}(0), 0, \tilde{F}_n^{(i)})_0 \). By Corollary 2.2.2, functional weak convergence \( S_n^{(i)} \xrightarrow{w} S^{(i)} \) holds true. As, additionally, we have \( ES_n^{(i)} = b^{(i)}(0) + \int xF_n^{(i)}(dx) \rightarrow b^{(i)}(0) + \int xF^{(i)}(dx) = ES_1^{(i)} \), it follows that \( (S_1^{(1)}) \leq_{\mathcal{F}} (S_1^{(2)}). \)

Theorems 2.2.4 and 2.2.5 imply several cut and domination criteria for Lévy processes with (possibly) infinite Lévy measures that are parallel to the criteria for compound Poisson processes stated in section 2.1. We make use of the following notation. For a Lévy measure \( F \) let the negative jump part \( F_- \) be given by \( F_- (dx) := 1_{\mathbb{R}_-} (x) F(dx) \) and let the positive jump part \( F_+ \) be given by \( F_+ (dx) := 1_{\mathbb{R}_+} (x) F(dx) \). If \( S_- \sim (ES_{-1}, 0, F_-)_{id} \) and \( S_+ \sim (ES_{+1}, 0, F_+)_{id} \) are independent and if \( ES_1 = ES_{-1} + ES_{+1} \), then it holds true that \( S \overset{d}{=} S_- + S_+ \sim (ES_1, 0, F)_{id} \). First give two versions of a cut criterion that depend on the regularity of the Lévy processes. If the processes do not necessarily have paths of finite variation, the result is as follows.

**Theorem 2.2.6 (Cut criterion for Lévy processes).** Let \( S^{(i)} \sim (ES_1^{(i)}, 0, F^{(i)})_{id}, i = 1, 2, \) be one-dimensional Lévy processes and assume that \( \| F^{(1)} \| = \infty \) and \( \int_{\{|x|>1\}} |x|^2 F^{(i)}(dx) < \infty \). For \( k_\ell < 0 < k_r \) assume that

\[
\begin{align*}
F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in B((-\infty, k_\ell)), \\
F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in B(k_\ell, 0)), \\
F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in B((0, k_r)), \\
F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in B((k_r, \infty)),
\end{align*}
\]

(2.11)

1. If \( ES_1^{(1)} \leq ES_1^{(2)} \) and there are sequences \( \xi_n^{(i)} \uparrow 0, \bar{\xi}_n^{(i)} \downarrow 0, \) s.th.

(a) \( \xi_n^{(2)} \leq \xi_n^{(1)} < 0 < \bar{\xi}_n^{(1)} < \bar{\xi}_n^{(2)} \),

(b) \( 0 \leq \int xF_n^{(2)}(dx) - \int xF_n^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)} \), and

(c) if \( \| F_n^{(1)} \| > \| F_n^{(2)} \| \) and if there is a \( \kappa \in [k_\ell, 0) \) s.th. \( F_n^{(1)}(-\infty, k_\ell) \leq F_n^{(2)}(-\infty, k_\ell) \), for all \( k < \kappa \) and \( F_n^{(1)}(-\infty, k) \geq F_n^{(2)}(-\infty, k) \), for all \( k \in [\kappa, 0) \), then additionally assume that \( F_n^{(1)}(\mathbb{R}_-) \geq F_n^{(2)}(\mathbb{R}_-) + \| F_n^{(1)} \| - \| F_n^{(2)} \| \).
then $(S_t^{(1)}) \leq_{icx} (S_t^{(2)})$.

2. If for numbers $ES_{+1}^{(i)}, ES_{-1}^{(i)}, i = 1, 2,$ with $ES_{+1}^{(i)} + ES_{-1}^{(i)} = ES_1^{(i)}$, that satisfy $ES_{+1}^{(i)} \leq ES_{+1}^{(2)}$, there are sequences $\xi_n^{(i)} \uparrow 0$, $\xi_n^{(2)} \downarrow 0$, s.th.

(a) $0 \leq \int xF_n^{(2)}(dx) - \int xF_n^{(1)}(dx) \leq ES_{-1}^{(2)} - ES_{-1}^{(1)}$, and

(b) $0 \leq \int xF_n^{(2)}(dx) - \int xF_n^{(1)}(dx) \leq ES_{+1}^{(2)} - ES_{+1}^{(1)}$.

and if additionally it holds true that

(c) if $\|F_n^{(1)}\| < \|F_n^{(2)}\|$ and $\xi_n^{(1)} < \xi_n^{(2)}$,
then $F_n^{(1)}(\langle -\infty, \xi_n^{(1)} \rangle) \leq F_n^{(2)}(\langle -\infty, \xi_n^{(1)} \rangle)$, and

(d) if $\|F_n^{(1)}\| < \|F_n^{(2)}\|$ and $\xi_n^{(2)} < \xi_n^{(1)}$,
then $F_n^{(1)}(\langle -\infty, \xi_n^{(2)} \rangle) + \|F_n^{(2)}\| - \|F_n^{(1)}\| \geq F_n^{(2)}(\langle -\infty, \xi_n^{(1)} \rangle)$,

then $(S_t^{(1)}) \leq_{icx} (S_t^{(2)})$.

Proof. 1. For the first part we make use of Theorem 2.1.4. Let $F_n^{(i)}$ denote the truncated Lévy measures and let $S_n^{(i)} \sim (ES_n^{(i)}, 0, F_n^{(i)})_{id}$ be the corresponding compound Poisson processes. From the pointwise ordering in (2.11) and $\xi_n^{(2)} \leq \xi_n^{(1)} < 0 < \xi_n^{(1)} < \xi_n^{(2)}$ there is a $N \in \mathbb{N}$ s.th.

\[
F_n^{(1)}(A) \leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)),
\]

\[
F_n^{(1)}(A) \geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}([k_\ell, k_r)),
\]

\[
F_n^{(1)}(A) \leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}([k_r, \infty)),
\]

for all $n \geq N$. Due to conditions 1b and c, Theorem 2.1.4 implies $(S_n^{(1)}) \leq_{icx} (S_n^{(2)})$. The result follows from Corollary 2.2.2 and the fact that $ES_n^{(1)} = ES_1^{(i)}$.

2. Let $\xi_n^{(i)} \uparrow 0$, $\xi_n^{(2)} \downarrow 0$ be sequences s.th. the conditions 2a–d are satisfied. Let $S_{n-}^{(i)} \sim (ES_{n-1}^{(i)}, 0, F_{n-}^{(i)})_{id}$ and $S_{n-}^{(i)} \sim (ES_{n+1}^{(i)}, 0, F_{n+}^{(i)})_{id}$ be independent Lévy processes and let $S_{n-}^{(i)} \sim (ES_{n-1}^{(i)}, 0, F_{n-}^{(i)})_{id}$ and $S_{n+}^{(i)} \sim (ES_{n+1}^{(i)}, 0, F_{n+}^{(i)})_{id}$ be the corresponding compound Poisson processes that are due to truncation of the Lévy measures $F^{(i)}$. We establish $(S_{n-}^{(1)}) \leq_{icx} (S_{n-}^{(2)})$ and $(S_{n+}^{(1)}) \leq_{icx} (S_{n+}^{(2)})$, then it follows from weak convergence that $(S_{n-}^{(1)}) \leq_{icx} (S_{n-}^{(2)})$ and $(S_{n+}^{(1)}) \leq_{icx} (S_{n+}^{(2)})$, hence $(S_t^{(1)}) \leq_{icx} (S_t^{(2)})$ by independence of $S_{n-}^{(i)}$ and $S_{n+}^{(i)}$ and the stability under convolutions property (C).
We first establish the ordering of \( S_{n-}^{(i)} \). If \( \xi_n^{(2)} \leq \xi_n^{(1)} \), the ordering (2.11) implies that there is a \( N \in \mathbb{N} \) s.th.

\[
\begin{align*}
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)), \\
F_n^{(1)}(A) &\geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}(k_\ell, \xi_n^{(1)})), \\
0 &= F_n^{(1)}(A) = F_n^{(2)}(A), \quad \forall A \in \mathcal{B}(\xi_n^{(1)}, \infty)),
\end{align*}
\]

for all \( n \geq N \). If \( \| F_n^{(1)} \| \leq \| F_n^{(2)} \| \), Theorem 2.1.4 then yields \( (S_{n-}^{(1)})_{\leq} (S_{n-}^{(2)})_{\leq} \) due to condition 2a. Similarly, if \( \| F_n^{(1)} \| \geq \| F_n^{(2)} \| \) the result follows from Theorem 2.1.2.

In the case \( \xi_n^{(1)} < \xi_n^{(2)} \), the ordering (2.11) implies that there is a \( N \in \mathbb{N} \) s.th.

\[
\begin{align*}
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)), \\
F_n^{(1)}(A) &\geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}(k_\ell, \xi_n^{(1)})), \\
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}(\xi_n^{(1)}, \infty)),
\end{align*}
\]

for all \( n \geq N \), and the result follows from Theorem 2.1.2, due to conditions 2a and c.

The ordering \( (S_{n+}^{(1)})_{\leq} (S_{n+}^{(2)})_{\leq} \) similarly follows from Theorems 2.1.4 and 2.1.5. In the case \( \pi_n^{(1)} \leq \pi_n^{(2)} \) it holds true that there is a \( N \in \mathbb{N} \) s.th.

\[
\begin{align*}
0 &= F_n^{(1)}(A) = F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, \pi_n^{(1)})), \\
F_n^{(1)}(A) &\geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}(\pi_n^{(1)}, k_r)), \\
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}(k_r, \infty)),
\end{align*}
\]

for all \( n \geq N \), hence the result follows from Theorem 2.1.4 if \( \| F_n^{(1)} \| \leq \| F_n^{(2)} \| \) and from Theorem 2.1.5 if \( \| F_n^{(1)} \| \geq \| F_n^{(2)} \| \). In the case \( \pi_n^{(2)} \leq \pi_n^{(1)} \), the result is implied by Theorem 2.1.5 as the conditions 2b and d are satisfied.

\[ \square \]

**Remark 2.2.7.** 1. Observe that the second case of the previous ordering result does not hold true for Lévy processes \( S^{(i)} \), where one of the processes has paths of finite variation, whereas the other has paths of infinite variation. This is obvious in the case \( \int_{(-1,1)} |x| F^{(1)}(dx) < \infty \) and \( \int_{(-1,1)} |x| F^{(2)}(dx) = \infty \), where the ordering in (2.11) does not hold true. In this case also the first part of the theorem fails. If \( \int_{(-1,1)} |x| F^{(1)}(dx) = \infty \) and \( \int_{(-1,1)} |x| F^{(2)}(dx) < \infty \), it follows that there is a \( N_{-} \in \mathbb{N} \) s.th. \( \int x F^{(1)}_{n_{-}}(dx) < \int x F^{(2)}_{n_{-}} + ES_{n_{-}}^{(1)} - ES_{n_{-}}^{(2)} \), for all \( n \geq N_{-} \), or there is a \( N_{+} \in \mathbb{N} \) s.th. \( \int x F^{(4)}_{n_{+}}(dx) > \int x F^{(2)}_{n_{+}}(dx) \), \( \forall n \geq N_{+} \), hence conditions 2a and 2b are violated.
2. Observe that in Theorem 2.2.6 we do not require \( F^{(2)} \) to have infinite total mass, hence the theorem also treats the case where \( S^{(1)} \) is a Lévy process with infinite Lévy measure (that has paths of finite variation in the second case) and \( S^{(2)} \) is a Poisson process.

In the case where both Lévy processes \( S^{(i)} \) have paths of finite variation, we obtain the following version of the previous result. We omit the proof, which is similar to the proof of Theorem 2.2.6.

**Theorem 2.2.8 (Cut criterion for Lévy processes with paths of finite variation).** Let \( S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)}_0), i = 1, 2, \) be one-dimensional Lévy processes and assume that \( \| F^{(1)} \| = \infty \). For \( k_\ell < 0 < k_r \) assume that

\[
F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)),
F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, 0)),
F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((0, k_r)),
F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)),
\tag{2.12}
\]

1. If \( b^{(1)}(0) \leq b^{(2)}(0) \) and there are sequences \( \xi_n^{(i)} \uparrow 0, \varpi_n^{(i)} \downarrow 0, \) s.th.

(a) \( \xi_n^{(2)} \leq \xi_n^{(1)} < 0 < \varpi_n^{(1)} < \varpi_n^{(2)} \),
(b) \( \int x F^{(1)}_{n^-}(dx) \leq \int x F^{(2)}_{n^-}(dx) \), and
(c) if \( \| F^{(1)}_{n^-} \| > \| F^{(2)}_{n^-} \| \) and if there is a \( \kappa \in [k_\ell, 0) \) s.th. \( F^{(1)}_{n^-}((-\infty, \kappa]) \leq F^{(2)}_{n^-}((-\infty, \kappa]) \) for all \( k < \kappa \) and \( F^{(1)}_{n^-}((-\infty, k]) \geq F^{(2)}_{n^-}((-\infty, k]) \) for all \( k \in [\kappa, 0) \), then additionally assume that \( F^{(1)}_{n^-}(\mathbb{R}_-) \geq F^{(2)}_{n^-}(\mathbb{R}_-) \) + \( \| F^{(1)}_{n^-} \| - \| F^{(2)}_{n^-} \| \)

then \( (S^{(1)}_t) \leq_{icx} (S^{(2)}_t) \).

2. If for numbers \( b^{(i)}_+(0), b^{(i)}_-(0), i = 1, 2, \) with \( b^{(i)}_+(0) + b^{(i)}_-(0) = b^{(i)}(0) \) that satisfy \( b^{(i)}_+(0) \leq b^{(2)}_-(0) \), there are sequences \( \xi_n^{(i)} \uparrow 0, \varpi_n^{(i)} \downarrow 0, \) s.th.

(a) \( \int x F^{(1)}_{n^-}(dx) \leq \int x F^{(2)}_{n^-}(dx) \), and
(b) \( \int x F^{(1)}_{n^+}(dx) \leq \int x F^{(2)}_{n^+}(dx) \),

and if additionally it holds true that

(c) if \( \| F^{(1)}_{n^-} \| < \| F^{(2)}_{n^-} \| \) and \( \xi_n^{(1)} < \xi_n^{(2)} \),
then \( F^{(1)}_{n^-}((-\infty, \xi_n^{(1)})] \leq F^{(2)}_{n^-}((-\infty, \xi_n^{(1)})] \), and
(d) if \( \| F^{(1)}_{n^+} \| < \| F^{(2)}_{n^+} \| \) and \( \varpi_n^{(2)} < \varpi_n^{(1)} \),
then \( F^{(1)}_{n^+}((-\infty, \varpi_n^{(1)})] + \| F^{(2)}_{n^+} \| - \| F^{(1)}_{n^+} \| \geq F^{(2)}_{n^+}((-\infty, \varpi_n^{(1)})] \),
1. Let

\( S^{(1)}_t \overset{\text{i.e.}}{\leq} S^{(2)}_t \).

For Lévy processes with infinite Lévy measure the following domination criterion, which is an extension of Theorem 2.1.7, holds true. For Lévy measures \( F^{(i)} \) with infinite total mass domination \( F^{(1)} \leq F^{(2)} \) is defined by \( F^{(1)}(A) \leq F^{(2)}(A) \) for all \( A \in \mathcal{B}((-\infty, 0)) \) and for all \( A \in \mathcal{B}((0, \infty)) \).

**Theorem 2.2.9 (Domination criterion for Lévy processes).** Let \( S^{(i)}_t \sim (ES^{(i)}_t, 0, F^{(i)})_{id}, i = 1, 2, \) be one-dimensional Lévy processes. Assume that \( \int_{\{|x|>1\}} |x|^2 F^{(i)}(dx) < \infty \) and that the Lévy measure \( F^{(2)} \) has infinite total mass and that \( F^{(i)} \) are ordered as \( F^{(1)} \leq F^{(2)} \).

1. If \( ES^{(1)}_1 \leq ES^{(2)}_1 \) and there are sequences \( \varepsilon^{(i)}_n \uparrow 0, \varepsilon^{(i)}_n \downarrow 0, i = 1, 2, \) s.th.

a) \( 0 \leq \int x F^{(2)}_n(dx) - \int x F^{(1)}_n(dx) \leq ES^{(2)}_1 - ES^{(1)}_1 \), and

b) in the case \( \varepsilon^{(2)}_n < \varepsilon^{(1)}_n < \varepsilon^{(2)}_n \) and \( \|F_n^{(1)}\| > \|F_n^{(2)}\| \) it additionally holds true that \( F_n^{(1)}(\mathbb{R}+) \geq F_n^{(2)}(\mathbb{R}+) + \|F_n^{(1)}\| - \|F_n^{(2)}\| \) if there is a \( \kappa_n \in [\varepsilon^{(2)}_n, 0) \) s.th. \( F_n^{(1)}((\infty, k]) \leq F_n^{(2)}((\infty, k]), \forall k < \kappa_n \) and \( F_n^{(1)}((\infty, k]) \geq F_n^{(2)}((\infty, k]), \forall k \in [\kappa_n, 0) \),

then \( (S^{(1)}_t) \overset{\text{i.e.}}{\leq} (S^{(2)}_t) \).

2. If for numbers \( ES^{(i)}_{-1}, ES^{(i)}_{+1}, i = 1, 2, \) with \( ES^{(i)}_1 = ES^{(i)}_{-1} + ES^{(i)}_{+1} \) that satisfy \( ES^{(1)}_{-1} \leq ES^{(2)}_{+1} \), there are sequences \( \varepsilon^{(i)}_n \uparrow 0, \varepsilon^{(i)}_n \downarrow 0, i = 1, 2, \) s.th.

a) \( \varepsilon^{(2)}_n \leq \varepsilon^{(1)}_n \),

b) \( 0 \leq \int x F^{(2)}_{n-}(dx) - \int x F^{(1)}_{n-}(dx) \leq ES^{(2)}_{-1} - ES^{(1)}_{-1} \),

c) \( 0 \leq \int x F^{(2)}_{n+}(dx) - \int x F^{(1)}_{n+}(dx) \leq ES^{(2)}_{+1} - ES^{(1)}_{+1} \),

then \( (S^{(1)}_t) \overset{\text{i.e.}}{\leq} (S^{(2)}_t) \).

**Proof.** 1. Let \( \varepsilon^{(i)}_n \uparrow 0, \varepsilon^{(i)}_n \downarrow 0 \) be sequences s.th. the conditions 1a and b hold true. Let \( S^{(i)}_n \sim (ES^{(i)}_t, 0, F^{(i)})_{id}, i = 1, 2, \) be compound Poisson processes corresponding to the truncation of \( F^{(i)} \). We establish \( (S^{(1)}_{n,t}) \overset{\text{i.e.}}{\leq} (S^{(2)}_{n,t}) \), the result then follows by weak convergence due to Corollary 2.2.2. We consider different cases depending on the relative location of \( \varepsilon^{(i)}_n \downarrow 0 \).

If \( \varepsilon^{(1)}_n \leq \varepsilon^{(2)}_n \) and \( \varepsilon^{(2)}_n \leq \varepsilon^{(1)}_n \) it holds true that \( F^{(1)}_n \leq F^{(2)}_n \) and Theorem 2.1.7 implies \( (S^{(1)}_{n,t}) \overset{\text{i.e.}}{\leq} (S^{(2)}_{n,t}) \).
2.2. Lévy processes with infinite Lévy measures

In the case $\varepsilon_n^{(1)} \leq \varepsilon_n^{(2)} < 0 < \varepsilon_n^{(1)} \leq \varepsilon_n^{(2)}$ the relative domination of the truncated Lévy measures is

$$F_n^{(1)}(A) \leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, \varepsilon_n^{(2)})),$$

$$F_n^{(1)}(A) = F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\varepsilon_n^{(2)}, \varepsilon_n^{(1)})),$$

$$F_n^{(1)}(A) \geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\varepsilon_n^{(1)}, \varepsilon_n^{(2)})),$$

$$F_n^{(1)}(A) \leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\varepsilon_n^{(2)}, \infty)).$$

If $\|F_n^{(1)}\| \leq \|F_n^{(2)}\|$, Theorem 2.1.4 yields $(S_{n,t}^{(1)}) \leq_{\text{icx}} (S_{n,t}^{(2)})$, and if $\|F_n^{(1)}\| \geq \|F_n^{(2)}\|$ the result follows from Theorem 2.1.5.

The case $\varepsilon_n^{(2)} \leq \varepsilon_n^{(1)} < 0 < \varepsilon_n^{(2)} \leq \varepsilon_n^{(1)}$ similarly follows with help of Theorems 2.1.2 and 2.1.4, depending on the ordering of the total masses of $F_n^{(i)}$.

If $\varepsilon_n^{(2)} \leq \varepsilon_n^{(1)} < 0 < \varepsilon_n^{(1)} \leq \varepsilon_n^{(2)}$ it holds true that

$$F_n^{(1)}(A) \leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, \varepsilon_n^{(2)})),$$

$$F_n^{(1)}(A) \geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\varepsilon_n^{(2)}, \varepsilon_n^{(1)})),$$

$$F_n^{(1)}(A) \leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\varepsilon_n^{(1)}, \infty)).$$

Theorem 2.1.4 implies $(S_{n,t}^{(1)}) \leq_{\text{icx}} (S_{n,t}^{(2)})$, due to condition 1b.

2. The proof is similar to the proof of Theorem 2.2.6. Let $\varepsilon_n^{(i)} \uparrow 0$, $\varepsilon_n^{(i)} \downarrow 0$ be sequences s.t.h. the conditions 2a–c are satisfied. Let $S^{(i)}(0), 0, F_n^{(i)} \text{id}$ and $S^{(i)} \sim (ES^{(i)}(0), 0, F_n^{(i)}) \text{id}$ be independent Lévy processes and let $S_n^{(i)} \sim (ES^{(i)}(0), 0, F_n^{(i)}) \text{id}$ and $S_n^{(i)} \sim (ES^{(i)}(0), 0, F_n^{(i)}) \text{id}$ be independent compound Poisson processes that correspond to $S^{(i)}$, $S^{(i)}$ and are due to truncation. We establish increasing convex ordering of $(S_{n,-}^{(i)})$ and $(S_{n,+}^{(i)})$, then it follows from weak convergence that $(S_{n,-}^{(i)}) \leq_{\text{icx}} (S_{n,-}^{(i)})$ and $(S_{n,+}^{(i)}) \leq_{\text{icx}} (S_{n,+}^{(i)})$, hence $(S^{(i)}) \leq_{\text{icx}} (S^{(i)})$ by independence of $S^{(i)}$ and $S^{(i)}$ and the stability under convolutions property (C).

As $\varepsilon_n^{(2)} \leq \varepsilon_n^{(1)}$ it holds true for the negative truncated jump parts that

$$F_n^{(1)}(A) \leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, \varepsilon_n^{(2)})),$$

$$F_n^{(1)}(A) \geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\varepsilon_n^{(2)}, \varepsilon_n^{(1)})),$$

$$0 = F_n^{(1)}(A) = F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\varepsilon_n^{(1)}, \infty)),$$

hence it follows from Theorems 2.1.2 and 2.1.4, that $(S_{n,-}^{(i)}) \leq_{\text{icx}} (S_{n,-}^{(i)})$, depending on the ordering of $\|F_n^{(i)}\|$. For the ordering of the positive truncated jump parts we consider two cases. If $\varepsilon_n^{(1)} \leq \varepsilon_n^{(2)}$ it follows
similarly to the first case with help of Theorems 2.1.4 and 2.1.5 that \((S_{n,+,t}^{(1)}) \leq_{icx} (S_{n,+,t}^{(2)})\). In the case \(\varepsilon_n^{(2)} \leq \varepsilon_n^{(1)}\) this result is implied by Theorem 2.1.7.

**Remark 2.2.10.** 1. Assumption 2a in the previous theorem is necessary for the increasing convex ordering \(\tilde{F}_{n,-}^{(1)} \leq_{icx} \tilde{F}_{n,-}^{(2)}\) to hold true as for \(\varepsilon_n^{(1)} < \varepsilon_n^{(2)}\), the condition \(F^{(1)} \leq F^{(2)}\) implies \(\int x F_{n,-}^{(2)}(dx) \leq \int x F_{n,-}^{(1)}(dx) \leq 0\).

2. There is a version of previous theorem for the case where \(S^{(i)}, i=1,2\), have paths of finite variation. This similar to the relationship of Theorems 2.2.6 and 2.2.8.

Theorem 2.2.9 implies a comparison criterion for Lévy processes that have paths are of infinite variation and whose Lévy measures are absolutely continuous w.r.t. the Lebesgue measure.

**Theorem 2.2.11 (Domination criterion for Lévy processes with paths of infinite variation).** Let \(S^{(i)} \sim (ES^{(i)}_1, 0, F^{(i)}_1)_{id}, i=1,2\), be one-dimensional Lévy processes with \(\int_{\{|x|>1\}} |x|^2 F^{(i)}(dx) < \infty\), s.th. \(\int_A |x| F^{(i)}(dx) = \infty\) for \(A = (-1,0)\) and \(A = (0,1)\), and assume that \(F^{(i)}\) have densities \(f^{(i)}\). If \(ES^{(1)}_1 \leq ES^{(2)}_1\), and

\[
0 < f^{(1)}(x) \leq f^{(2)}(x), \quad \forall x \in \mathbb{R}, \tag{2.13}
\]

then \((S^{(1)}_t) \leq_{icx} (S^{(2)}_t)\).

**Proof.** We construct sequences \(\varepsilon_n^{(i)} \uparrow 0, \varepsilon_n^{(i)} \downarrow 0, i=1,2\), s.th. the conditions of Theorem 2.2.9 are satisfied. Let \(\varepsilon_n^{(1)} \leq \varepsilon_n^{(2)}\) be given for all \(n \in \mathbb{N}\). Due to the assumption \(\int_A |x| F^{(i)}(dx) = \infty\) for \(A = (-1,0)\) and \(A = (0,1)\), there are sequences \(\varepsilon_n^{(i)} \downarrow 0, i=1,2\), s.th. \(\int x F_{n,-}^{(i)}(dx) = 0\). Hence conditions 1a and b of Theorem 2.2.9 are satisfied.

There is a stochastic ordering result for Lévy measures with infinite total mass that is similar to the result of Corollary 2.1.10.

**Theorem 2.2.12 (Criterion for stochastic ordering of Lévy processes with infinite Lévy measures).** Let \(S^{(i)} \sim (ES^{(i)}_1, 0, F^{(i)}_1)_{id}, i=1,2\), be one-dimensional Lévy processes. Assume that the Lévy measures satisfy \(\int_{\{|x|>1\}} |x|^2 F^{(i)}(dx) < \infty\), \(F^{(i)}\) have infinite total mass and are ordered as

\[
F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty,0)), \text{ and } \tag{2.13}
F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((0,\infty)).
\]
1. If \( E S^{(1)}_1 \leq E S^{(2)}_1 \) and there are sequences \( \xi^{(i)}_n \uparrow 0, \xi^{(i)}_n \downarrow 0, i = 1, 2 \), s.th.
   (a) \( 0 \leq \int x F^{(2)}_n (dx) - \int x F^{(1)}_n (dx) \leq E S^{(2)}_1 - E S^{(1)}_1 \),
   (b) it never holds true that \( \xi^{(1)}_n < \xi^{(2)}_n \) and \( \xi^{(1)}_n < \xi^{(2)}_n \) simultaneously,
   (c) if \( \xi^{(1)}_n < \xi^{(2)}_n < 0 < \xi^{(2)}_n \leq \xi^{(1)}_n \) and \( \|F^{(1)}_n\| < \|F^{(2)}_n\| \) it holds true that
      \( F^{(1)}_n (\mathbb{R}^-) \geq F^{(2)}_n (\mathbb{R}^-) \),
   (d) if \( \xi^{(2)}_n \leq \xi^{(1)}_n < 0 < \xi^{(2)}_n \leq \xi^{(1)}_n \), and \( \|F^{(1)}_n\| > \|F^{(2)}_n\| \) it holds true that
      \( F^{(1)}_n (\mathbb{R}^+) \leq F^{(2)}_n (\mathbb{R}^+) \),
   then \( (S^{(1)}_t) \leq_{st} (S^{(2)}_t) \).

2. If for numbers \( E S^{(i)}_{-1}, E S^{(i)}_{+1}, i = 1, 2 \), with \( E S^{(i)}_1 = E S^{(i)}_{-1} + E S^{(i)}_{+1} \) that satisfy \( E S^{(1)}_{-1} \leq E S^{(2)}_{-1} \), there are sequences \( \xi^{(i)}_n \uparrow 0, \xi^{(i)}_n \downarrow 0, i = 1, 2 \), s.th.
   (a) \( 0 \leq \int x F^{(2)}_{n,-} (dx) - \int x F^{(1)}_{n,-} (dx) \leq E S^{(2)}_{-1} - E S^{(1)}_{-1} \),
   (b) \( 0 \leq \int x F^{(2)}_{n,+} (dx) - \int x F^{(1)}_{n,+} (dx) \leq E S^{(2)}_{+1} - E S^{(1)}_{+1} \),
   and if additionally it holds true that
   (c) if \( \xi^{(1)}_n < \xi^{(2)}_n \), then \( \|F^{(1)}_{n,-}\| \geq \|F^{(2)}_{n,-}\| \),
   (d) if \( \xi^{(1)}_n < \xi^{(2)}_n \), then \( \|F^{(1)}_{n,+}\| \leq \|F^{(2)}_{n,+}\| \),
   then \( (S^{(1)}_t) \leq_{st} (S^{(2)}_t) \).

Proof. 1. Let \( \xi^{(i)}_n \uparrow 0, \xi^{(i)}_n \downarrow 0 \) be sequences s.th. the conditions 1a–d are satisfied. For the compound Poisson processes \( S^{(i)}_n \sim (E S^{(i)}_1, 0, F^{(i)}_0)_{id} \) we establish \( (S^{(1)}_n) \leq_{st} \left( S^{(2)}_n, t \right) \), then the result follows from weak convergence,
   which is established in Corollary 2.2.2.

For \( \xi^{(2)}_n \leq \xi^{(1)}_n < 0 < \xi^{(2)}_n \leq \xi^{(1)}_n \) it follows from the ordering condition on \( F^{(i)}_n \) that
\[
F^{(1)}_n (A) \geq F^{(2)}_n (A), \quad \forall A \in \mathcal{B}((-\infty, \xi^{(1)}_n)), \quad 0 = F^{(1)}_n (A) = F^{(2)}_n (A), \quad \forall A \in \mathcal{B}((\xi^{(1)}_n, \xi^{(2)}_n)), \quad F^{(1)}_n (A) \leq F^{(2)}_n (A), \quad \forall A \in \mathcal{B}((\xi^{(2)}_n, \infty)).
\]

Therefore, Corollary 2.1.10 implies \( (S^{(1)}_{n,t}) \leq_{st} (S^{(2)}_{n,t}) \).

If \( \xi^{(2)}_n \leq \xi^{(1)}_n < 0 < \xi^{(2)}_n \leq \xi^{(1)}_n \) it holds true that
\[
F^{(1)}_n (A) \geq F^{(2)}_n (A), \quad \forall A \in \mathcal{B}((-\infty, \xi^{(1)}_n)), \quad F^{(1)}_n (A) \leq F^{(2)}_n (A), \quad \forall A \in \mathcal{B}((\xi^{(1)}_n, \xi^{(2)}_n)), \quad 0 = F^{(1)}_n (A) = F^{(2)}_n (A), \quad \forall A \in \mathcal{B}((\xi^{(2)}_n, \infty)), \quad F^{(1)}_n (A) \leq F^{(2)}_n (A), \quad \forall A \in \mathcal{B}(\xi^{(2)}_n, \infty)).
\]
hence Theorem 2.1.8 implies \((S_{n,t}^{(1)}) \leq_{st} (S_{n,t}^{(2)})\), due to the additional assumption in the case \(\|F_n^{(1)}\| < \|F_n^{(2)}\|\). Similarly, the case \(\tilde{\varepsilon}_n^{(2)} \leq \tilde{\varepsilon}_n^{(1)} < 0 < \tilde{\sigma}_n^{(1)} \leq \tilde{\sigma}_n^{(2)}\) follows from condition 1d and Theorem 2.1.9.

Observe that in the case \(\tilde{\varepsilon}_n^{(1)} < \tilde{\varepsilon}_n^{(2)} < 0 < \tilde{\sigma}_n^{(1)} < \tilde{\sigma}_n^{(2)}\) three dominance changes for the truncated Lévy measures \(F_n^{(i)}\) take place, hence the results of subsection 2.1 are not directly applicable in this case.

2. The proof is similar to the proof of the second part of Theorem 2.2.6. Let \(\tilde{\varepsilon}_n^{(i)} \uparrow 0, \tilde{\sigma}_n^{(i)} \downarrow 0\) be sequences s.t. the conditions 2a–d are satisfied and assume that \(S^{(i)}_\sim \sim (ES^{(i)}_{n,1}, 0, F^{(i)}_\sim)_{id}\) and \(S^{(i)}_\sim \sim (ES^{(i)}_{n,1}, 0, F^{(i)}_\sim)_{id}\) are independent Lévy processes. For the independent compound Poisson processes \(S^{(i)}_{n,\sim} \sim (ES^{(i)}_{n,1}, 0, F^{(i)}_{n,\sim})_{id}\) and \(S^{(i)}_{n,\sim} \sim (ES^{(i)}_{n,1}, 0, F^{(i)}_{n,\sim})_{id}\) that are due to truncation, we establish \((S^{(1)}_{n,-t}) \leq_{st} (S^{(2)}_{n,-t})\) and \((S^{(1)}_{n,t}) \leq_{st} (S^{(2)}_{n,t})\) and then obtain the result similarly to Theorem 2.2.6.

For \(\tilde{\varepsilon}_n^{(2)} \leq \tilde{\varepsilon}_n^{(1)}\) it holds true that
\[
F^{(1)}_{n}(A) \geq F^{(2)}_{n}(A), \quad \forall A \in \mathcal{B}((-\infty, \tilde{\varepsilon}_n^{(1)})),
\]
\[
0 = F^{(1)}_{n,\sim}(A) = F^{(2)}_{n,\sim}(A), \quad \forall A \in \mathcal{B}(\tilde{\varepsilon}_n^{(1)}, \infty)),
\]
hence it follows from Corollary 2.1.10 that \((S^{(1)}_{n,-t}) \leq_{st} (S^{(2)}_{n,-t})\). In the case \(\tilde{\sigma}_n^{(2)} \leq \tilde{\sigma}_n^{(1)}\) it similarly follows from Corollary 2.1.10 that \((S^{(1)}_{n,t}) \leq_{st} (S^{(2)}_{n,t})\).

For \(\tilde{\varepsilon}_n^{(1)} < \tilde{\varepsilon}_n^{(2)}\) it holds true that
\[
F^{(1)}_{n}(A) \geq F^{(2)}_{n}(A), \quad \forall A \in \mathcal{B}((-\infty, \tilde{\varepsilon}_n^{(1)})),
\]
\[
0 = F^{(1)}_{n,\sim}(A) \leq F^{(2)}_{n,\sim}(A), \quad \forall A \in \mathcal{B}(\tilde{\varepsilon}_n^{(1)}, \tilde{\varepsilon}_n^{(2)})),
\]
\[
0 = F^{(1)}_{n}(A) = F^{(2)}_{n}(A), \quad \forall A \in \mathcal{B}(\tilde{\varepsilon}_n^{(2)}, \infty)),
\]
thus Theorem 2.1.8 yields \((S^{(1)}_{n,-t}) \leq_{st} (S^{(2)}_{n,-t})\), due to condition 2c. Similarly, in the case \(\tilde{\varepsilon}_n^{(1)} < \tilde{\varepsilon}_n^{(2)}\) Theorem 2.1.9 implies \((S^{(1)}_{n,t}) \leq_{st} (S^{(2)}_{n,t})\), due to condition 2d.

**Remark 2.2.13.** If \(\int_{(-1,0)} |x| F^{(i)}(dx) = \infty\) and \(\int_{(0,1)} x F^{(i)}(dx) = \infty\) and \(F^{(i)}\) are absolutely continuous, then one can construct sequences \(\tilde{\varepsilon}_n^{(i)} \uparrow 0, \tilde{\varepsilon}_n^{(i)} \downarrow 0\) s.t. condition 1a of the previous theorem is satisfied. At first, we fix \(\tilde{\varepsilon}_n^{(i)}\) and \(\tilde{\varepsilon}_n^{(i)}\). If
\[
\int_{\mathbb{R}\setminus(\tilde{\varepsilon}_n^{(i)}, \tilde{\varepsilon}_n^{(i)})} x F^{(2)}(dx) - \int_{\mathbb{R}\setminus(\tilde{\varepsilon}_n^{(i)}, \tilde{\varepsilon}_n^{(i)})} x F^{(1)}(dx) \leq ES^{(2)}_{1} - ES^{(1)}_{1}, \quad (2.14)
\]
then condition 1a is satisfied with $\varepsilon_n^{(1)} := \varepsilon_n^{(2)}$ and $\varepsilon_n^{(1)} := \varepsilon_n^{(2)}$. If (2.14) is not satisfied, we choose $\varepsilon_n^{(1)} > \varepsilon_n^{(2)}$ and $\varepsilon_n^{(1)} := \varepsilon_n^{(2)}$ or $\varepsilon_n^{(1)} < \varepsilon_n^{(2)}$ and $\varepsilon_n^{(1)} := \varepsilon_n^{(2)}$ (but not simultaneously, cp. condition 1b), s.th. condition 1a holds. This is possible, due to the Assumptions on $F^{(i)}$. Additionally, we have to check if conditions 1c and 1d, respectively, are satisfied. In the next step we fix $\varepsilon_{n+1}^{(1)} > \varepsilon_n^{(2)}$ and $\varepsilon_{n+1}^{(2)} < \varepsilon_n^{(1)}$ and restart the algorithm.

Hitherto we have considered the comparison of pure jump Lévy processes, now we incorporate the Gaussian part. Firstly, we state a comparison result for continuous Lévy processes which is a corollary of well-known ordering results for normally distributed random variables, cp. Müller and Stoyan (2002, Section 3.13) and Corollary 1.2.12.

**Lemma 2.2.14 (Ordering of Gaussian Lévy processes).** Let $S^{(i)} \sim (ES^{(i)}_1, c^{(i)}_1, 0)_{id}$, $i = 1, 2$, be $d$-dimensional continuous Lévy processes.

1. If $ES^{(1)}_1 \leq ES^{(2)}_1$ and $c^{(1)} = c^{(2)}$, then $(S^{(1)}_t) \leq_{st} (S^{(2)}_t)$.
2. If $ES^{(1)}_1 \leq ES^{(2)}_1$ and $c^{(1)} \leq_{psd} c^{(2)}$, then $(S^{(1)}_t) \leq_{lex} (S^{(2)}_t)$.
3. If $ES^{(1)}_1 \leq ES^{(2)}_1$ and $c^{(1)}_{ij} \leq c^{(2)}_{ij}, \forall i, j \leq d$, then $(S^{(1)}_t) \leq_{ism} (S^{(2)}_t)$.
4. If $ES^{(1)}_1 \leq ES^{(2)}_1$ and $c^{(1)}_{ij} \leq c^{(2)}_{ij}, \forall i, j \leq d, i \neq j$, then $(S^{(1)}_t) \leq_{ism} (S^{(2)}_t)$.

The corresponding convex type orderings hold true if $ES^{(1)}_1 = ES^{(2)}_1$.

**Proof.** For a Gaussian Lévy process $S \sim (ES_1, c, 0)$ it holds true that $S_t \sim N(tES_1, tc)$, where $N$ is the normal distribution. The result follows from Corollary 1.2.12. \qed

As the Gaussian and the jump part of a Lévy process are independent, we obtain ordering of Lévy processes that incorporate these parts from an ordering of the respective parts and the stability under convolutions property (C).

**Lemma 2.2.15 (Ordering of Lévy processes).** Let $F$ be one of the order generating function classes in (1) and assume that $S^{(i)} \sim (ES^{(i)}_1, c^{(i)}_1, F^{(i)}_{id})$, $i = 1, 2$, are $d$-dimensional Lévy processes. If the jump processes $S^{(i)}_J \sim (ES^{(i)}_1, 0, F^{(i)}_{id})$ and the continuous processes $S^{(i)}_C \sim (0, c^{(i)}_1, 0)_{id}$ are independent and if

$$(S^{(1)}_{J,t}) \leq_F (S^{(2)}_{J,t}) \quad \text{and} \quad (S^{(1)}_{C,t}) \leq_F (S^{(2)}_{C,t}),$$

then $(S^{(1)}_t) \leq_F (S^{(2)}_t)$.
Proof. As the orders generated by $\mathcal{F}$ that are considered in this thesis satisfy property (C) and due to the fact that $S^{(i)}_d = S^{(i)}_j + S^{(i)}_c$, the result follows. \qed

Remark 2.2.16. Due to Lemmas 1.1.11 and 1.1.12 the comparison results of this section also imply orderings of the exponentials of Lévy processes for $\mathcal{F} \in \{\mathcal{F}_{st}, \mathcal{F}_{icx}, \mathcal{F}_{idcx}, \mathcal{F}_{ism}, \mathcal{F}_{sm}\}$ and of the logarithms of positive Lévy processes for $\mathcal{F} \in \{\mathcal{F}_{st}, \mathcal{F}_{ism}, \mathcal{F}_{sm}\}$. This also holds true for the finite-dimensional orderings w.r.t. $\mathcal{F}^{(m)}$.

2.3 Extension to PII

In this section we extend the comparison results for Lévy processes of the previous sections to processes with independent increments $S^{(i)}$, whose increments are not necessarily stationary. We make use of the following representation result of Norberg (1993, Theorem 1), for sake of completeness we give a proof in the Appendix.

Theorem 2.3.1 (Norberg (1993)). Let $S \sim (0, 0, K(s; dy))_0$ be a $d$-dimensional PII with $\lambda(s) := K(s; \mathbb{R}^d) < \infty$ and let $Y_t$ be a random sum process that is defined by

$$Y_t = \sum_{j=1}^{N_t} \hat{X}_{t,j}, \quad t \in [0, T],$$

where the extended Poisson process $\hat{N}_t \sim \mathcal{P}(\Lambda(t))$, $\Lambda(t) := \int_0^t \lambda(s)ds$, is independent of the iid sequence $(\hat{X}_{t,j}) \sim \hat{R}_t$, $\hat{R}_t(dy) = \frac{1}{\Lambda(t)} \int_0^t K(s; dy)ds$.

Then for all $t \in [0, T]$ it holds true that $S_t \overset{d}{=} Y_t$.

To obtain a generalization of Lemma 2.1.1 that implies $S^{(1)}_t \leq_S S^{(2)}_t$, for all $t \in [0, T]$, for PII $S^{(i)} \sim (b^{(i)}(t; 0), 0, K^{(i)}(t; \cdot))_0$, we make use of the modified Lévy kernels $\tilde{K}^{(i)}$, which for $t \in [0, T]$ and $k \in \{1, 2\}$ s.th. $K^{(k)}(t; \mathbb{R}^d) \leq K^{(3-k)}(t; \mathbb{R}^d)$ are defined by

$$\tilde{K}^{(k)}(t; dx) = K^{(k)}(t; dx) + (K^{(3-k)}(t; \mathbb{R}^d) - K^{(k)}(t; \mathbb{R}^d)) \overset{d}{=} \delta_{\{0\}}(dx).$$

(2.15)

An extension of Lemma A.2.3 to the PII case then implies $S^{(i)} \sim (b^{(i)}(t; 0), 0, \tilde{K}^{(i)}(t; \cdot))_0$, $i = 1, 2$. Observe that similar to the case of modified Lévy measures, we violate the convention $\tilde{K}(t; \{0\}) = 0$, and the kernels are unique modulo the point mass in the origin.
Lemma 2.3.2 (Ordering of PII with finite Lévy kernels). Let $S^{(i)} \sim (b^{(i)}(t; 0), 0, K^{(i)}(t; \cdot))_0$, $i = 1, 2$, be $d$-dimensional PII with $K^{(i)}(t, \mathbb{R}^d) < \infty$, for all $t \in [0, T]$, and assume that $\int_{|x| > 1} |x| K^{(i)}(t; dx) < \infty$, for all $t \in [0, T]$. Let the modified Lévy kernels $\tilde{K}^{(i)}$ be given by (2.15). If for all $t \in [0, T]$ it holds true that $\tilde{K}^{(1)}(t; \cdot) \leq_{\mathcal{F}} \tilde{K}^{(2)}(t; \cdot)$ for

1. $\mathcal{F} \in \{\mathcal{F}_{st}, \mathcal{F}_{icx}, \mathcal{F}_{dcx}, \mathcal{F}_{ism}\}$ and, additionally, $b^{(1)}(t; 0) \leq b^{(2)}(t; 0)$,

or for

2. $\mathcal{F} \in \{\mathcal{F}_{cx}, \mathcal{F}_{dcx}, \mathcal{F}_{ism}\}$ and, additionally, $b^{(1)}(t; 0) = b^{(2)}(t; 0)$,

then $S^{(1)}_t \leq_{\mathcal{F}} S^{(2)}_t$ for all $t \in [0, T]$.

Proof. Let $t \in [0, T]$ and define $\lambda(s) := \tilde{K}^{(i)}(s, \mathbb{R}^d)$, $s \leq t$ and $\Lambda(t) := \int_0^t \lambda(s) ds$. Theorem 2.3.1 implies that the random vectors $S^{(i)}_t$ are distributed as random sums with additional drift

$$S^{(i)}_t = \int_0^t b^{(i)}(s; 0) ds + \sum_{j=1}^{\tilde{N}_t} \tilde{X}^{(i)}_{t,j},$$

where $\tilde{N}_t \sim \mathcal{P}(\Lambda(t))$ is independent of the iid sequences $(\tilde{X}^{(i)}_{t,j}) \sim \tilde{R}^{(i)}_t$, $\tilde{R}^{(i)}_t(dy) = \frac{1}{\Lambda(t)} \int_0^t \tilde{K}^{(i)}(s; dy) ds$, hence $S^{(1)}$ and $S^{(2)}$ are naturally coupled by the same extended Poisson process $\tilde{N}$. As $\tilde{R}^{(i)}_t$ are mixtures of $\tilde{K}^{(i)}(s; \cdot)$, $s \leq t$ with mixing distribution $\frac{1}{\Lambda(t)} \lambda|_{[0,t]}$, it follows from $\tilde{K}^{(1)}(s; \cdot) \leq_{\mathcal{F}} \tilde{K}^{(2)}(s; \cdot)$, for all $s \in [0, T]$, and the stability under mixtures property (MI) that $\tilde{R}^{(1)}_t \leq_{\mathcal{F}} \tilde{R}^{(2)}_t$. As $b^{(1)}(s; 0) \leq b^{(2)}(s; 0)$ for all $s \in [0, T]$ implies $\int_0^t b^{(1)}(s; 0) ds \leq \int_0^t b^{(2)}(s; 0) ds$, the result follows similarly to the proof of Lemma 2.1.1. \qed

Remark 2.3.3. 1. If one is only interested in the ordering of $S^{(i)}$ at a specific time $t \in [0, T]$, the conditions in the first part of the previous lemma can be reduced to $\int_0^t b^{(1)}(s; 0) ds \leq \int_0^t b^{(2)}(s; 0) ds$ and $\int_0^t \tilde{K}^{(1)}(s; \cdot) ds \leq_{\mathcal{F}} \int_0^t \tilde{K}^{(2)}(s; \cdot) ds$, similar for the conditions of the second part.

2. If the increments of $S^{(i)}$ are not stationary, ordering of the finite-dimensional distributions does not follow.

3. Lemma 2.3.2 implies several cut and domination criteria for one-dimensional PII $S^{(i)} \sim (b^{(i)}(t; 0), 0, K^{(i)}(t; \cdot))_0$, $i = 1, 2$, in terms of the corresponding Lévy kernels $K^{(i)}(t; \cdot)$ under appropriate drift conditions. These results are parallel to the cut and domination criteria in section 2.1.
To obtain ordering results for PII with infinite Lévy kernels that correspond
to the results of subsection 2.2, for $t \in [0, T]$ we introduce the truncated Lévy
kernels $K_n(t; \cdot)$, which are extensions of the truncated Lévy measures given
in (2.9), by

$$K_n(t; dx) := 1_{(\xi_{t,n}, \tau_{t,n})^c}(x) K(t; dx), \quad \xi_{t,n} \uparrow 0, \tau_{t,n} \downarrow 0. \quad (2.16)$$

Similar to section 2.2, ordering of the truncated Lévy kernels $K_n^{(i)}(t; \cdot)$ im-
plies ordering of the limit processes that have Lévy kernels $K^{(i)}(t; \cdot)$, if an
appropriate weak convergence holds true. The proof of the following lemma
is given in the Appendix.

**Lemma 2.3.4 (Functional weak convergence, PII case).** For a con-
tinuous truncation function $h$ let $S \sim (b(t; h), 0, K^{(i)}(t; \cdot))_h$ be a $d$-dimensional
Lévy process whose Lévy kernel $K$ has infinite total mass and for $\xi_{t,n} \uparrow 0, \tau_{t,n} \downarrow 0$ let $K_n$ be the corresponding truncated Lévy kernel introduced
in (2.16). If $\sup_{s \leq t} |B_{n,s} - B_s| \to 0$, for all $t \in [0, T]$, then for $S_n \sim (b_n(t; h), 0, K_n(t; \cdot))_h$ functional weak convergence

$$S_n \xrightarrow{\mathcal{L}} S$$

holds true.

Similar to Corollary 2.2.2 we obtain a weak convergence result that fits to
our situation. Observe that the first part is formulated in terms of the char-
acteristics of $S$, not of the differential characteristics. The proof is displaced
to the Appendix.

**Corollary 2.3.5 (Functional weak convergence, PII case).** Let $K(t; \cdot)$
be a Lévy kernel that has infinite total mass for all $t \in [0, T]$, and for sequences $\xi_{t,n} \uparrow 0, \tau_{t,n} \downarrow 0$ let $K_n(t; \cdot)$ be the corresponding truncated Lévy kernel given in (2.16).

1. If $|x|^2 * \nu_t < \infty$, then for $S \sim (ES_t, 0, \nu_t)_t$ and $S_n \sim (ES_t, 0, \nu_{n,t})_t$, it
   holds true that $S_n \xrightarrow{\mathcal{L}} S$.

2. If for a continuous truncation $h$ it holds true that $\int h(x)K(t; dx) < \infty$,
   for all $t \in [0, T]$, then for $S \sim (b(t; 0), 0, K(t; \cdot))_0$ and $S_n \sim (b(t; 0), 0, K_n(t; \cdot))_0$ it follows that $S_n \xrightarrow{\mathcal{L}} S$.

To obtain an extension of Theorem 2.2.4 to the PII case we introduce the
modified truncated Lévy kernels $K^{(k)}_n$, $k = 1, 2$, by

$$K^{(k)}_n(t; dx) := K^{(k)}_n(t; dx) + (K^{(3-k)}_n(t; \mathbb{R}^d) - K^{(k)}(t, \mathbb{R}^d))^+ \delta_{\{0\}}(dx).$$

The comparison result for $d$-dimensional PII with infinite Lévy kernels is as
follows.
Theorem 2.3.6 (Ordering of PII with infinite Lévy kernels). Let $S^{(i)} \sim (E_{S_t^{(i)}}, 0, \nu_t)_{i \in \mathbb{N}}, i = 1, 2$, be $d$-dimensional PII with Lévy kernels $K^{(i)}$ that satisfy $K^{(i)}(t; \mathbb{R}^d) = \infty$ and $|x|^2 * \nu_t < \infty$, for all $t \in [0, T]$. Let $\xi_{t,n} \uparrow 0, \xi_{t,n} \downarrow 0$ be sequences s.th. for the modified truncated Lévy kernels $\tilde{K}^{(i)}_{n,t}$ it holds true that

$$\tilde{K}^{(1)}_{n,t}(t; \cdot) \leq \xi_{n} \tilde{K}^{(2)}_{n,t}(t; \cdot), \quad \forall t \in [0, T], n \in \mathbb{N},$$

for

1. $\mathcal{F} \in \{\mathcal{F}_{st}, \mathcal{F}_{icx}, \mathcal{F}_{dclx}, \mathcal{F}_{ism}\}$ and, additionally, it holds true that

$$0 \leq \int xK^{(2)}_{n,t}(t; dx) - \int xK^{(1)}_{n,t}(t; dx) \leq ES^{(2)}_{t} - ES^{(1)}_{t}, \quad \forall t \in [0, T], n \in \mathbb{N},$$

or for

2. $\mathcal{F} \in \{\mathcal{F}_{cx}, \mathcal{F}_{dclx}, \mathcal{F}_{ism}\}$ and, additionally, it holds true that $ES^{(1)}_{t} = ES^{(2)}_{t}$ and $\int xK^{(1)}_{n,t}(t; dx) = \int xK^{(2)}_{n,t}(t; dx)$, for all $t \in [0, T], n \in \mathbb{N}$.

Then $S^{(1)}_{t} \leq \xi_{n} S^{(2)}_{t}$ for all $t \in [0, T]$ follows.

Proof. The claim follows from Lemma 2.3.2 and Corollary 2.3.5 in the same manner as Theorem 2.2.4 follows from Lemma 2.1.1 and the corresponding weak convergence result.

Remark 2.3.7. Theorem 2.3.6 implies cut and domination criteria that are parallel to the results obtained in section 2.2.

Similar to Lemma 2.2.14 we also obtain orderings for continuous PII. This is due to the fact that a PII $S \sim (0, C, 0)$ is a Brownian motion with deterministic covariance function $C$, due to Jacod and Shiryaev (2003, Theorems II.4.4 and II.4.15). As in Lemma 2.2.14 the orderings follow from the comparison results for multivariate normal distributions in Müller and Stoyan (2002, Section 3.13) or in Corollary 1.2.12.

Lemma 2.3.8 (Ordering of Gaussian PII). Let $S^{(i)} \sim (E_{S_t^{(i)}}, C^{(i)}(t), 0)_{i \in \mathbb{N}}, i = 1, 2$, be $d$-dimensional PII.

1. If $ES^{(1)}_{t} \leq ES^{(2)}_{t}$ and $c^{(1)}(t) = c^{(2)}(t)$, for all $t \in [0, T]$, then $S^{(1)}_{t} \leq \xi_{n} S^{(2)}_{t}$ for all $t \in [0, T]$.

2. If $ES^{(1)}_{t} \leq ES^{(2)}_{t}$ and $c^{(1)}(t) \leq \mathit{psd} c^{(2)}(t)$, for all $t \in [0, T]$, then $S^{(1)}_{t} \leq \xi_{n} S^{(2)}_{t}$ for all $t \in [0, T]$. 

3. If $ES^{(1)}_t \leq ES^{(2)}_t$ and $c^{(1)ij}(t) \leq c^{(2)ij}(t)$, for all $t \in [0, T]$, $\forall i, j \leq d$, then $S^{(1)}_t \leq_{\text{idex}} S^{(2)}_t$ for all $t \in [0, T]$.

4. If $ES^{(1)}_t \leq ES^{(2)}_t$ and $c^{(1)ij}(t) \leq c^{(2)ij}(t)$, for all $t \in [0, T]$, $\forall i, j \leq d$, $i \neq j$, then $S^{(1)}_t \leq_{\text{ism}} S^{(2)}_t$ for all $t \in [0, T]$.

The corresponding convex type orderings hold true if $ES^{(1)}_t = ES^{(2)}_t$, for all $t \in [0, T]$.

Finally, similar to Corollary 2.2.15 the stability under convolutions property (C) implies an ordering result for PII that incorporate continuous martingale and jump parts.

**Corollary 2.3.9 (Ordering of PII).** Let $F$ be one of the order generating function classes in (1) and assume that $S^{(i)} \sim (ES^{(i)}_t, C^{(i)}(t), \nu^{(i)}_t)_{\text{id}}$, $i = 1, 2$, are $d$-dimensional PII. If for the jump parts $S^{(i)}_J \sim (ES^{(i)}_t, 0, \nu^{(i)}_t)_{\text{id}}$ and the continuous parts $S^{(i)}_C \sim (0, C^{(i)}(t), 0)_{\text{id}}$ it holds true that

$$S^{(1)}_{J,t} \leq_{\mathcal{F}} S^{(2)}_{J,t} \quad \text{and} \quad S^{(1)}_{C,t} \leq_{\mathcal{F}} S^{(2)}_{C,t}, \quad \forall t \in [0, T],$$

then $S^{(1)}_t \leq_{\mathcal{F}} S^{(2)}_t$ for all $t \in [0, T]$. 

Chapter 3

Applications

In this chapter we give several applications and some extensions of the ordering results of Chapters 1 and 2. We focus mainly on results in financial mathematics, especially on comparison of option prices in incomplete market models and on ordering results for path-dependent options. Some of the results are partial extensions of existing results in the literature, we give references in the sections below. In the first two sections we treat ordering problems for European options that arise in incomplete market models. If no additional assumptions (like an optimization criterion or utility approach) are made, the price of a European call option in an incomplete market model is not unique. It lies within the no-arbitrage price interval that is due to no-arbitrage considerations. For some classes of models the no-arbitrage interval coincides with the trivial pricing interval and the no-arbitrage principle is of no help for choosing a meaningful price. For the models with non-trivial no-arbitrage intervals a basic task is to identify a martingale measure under which the option price bounds the other option prices of the model that are evaluated w.r.t. all other martingale measures from above or from below. We obtain non-trivial bounds for European option prices in various types of incomplete models in Section 3.1.

To obtain an option price in an incomplete market model, one has to pick out an equivalent martingale measure out of the set of equivalent martingale measures by an additional criterion. Two general approaches have been suggested in the literature. One approach is to price the option by choosing an martingale pricing measure by some optimization criterion, like minimal risk, minimal distance to the underlying measure and/or fitting of some observed prices. In this way one gets several well established martingale measures like the minimal martingale, the $q$-optimal measure, the minimal entropy measure, the Esscher–measure, or the variance minimal measure. A second general approach is the utility based indifference price and variants
of it, which is in one to one correspondence with the martingale pricing measure obtained by minimizing related $f$-divergence distances. In Section 3.2 we derive orderings of European option prices in incomplete market models w.r.t. several distance minimizing martingale measures. As examples for incomplete market models we consider exponential jump models, jump models, and stochastic volatility models. We compute several martingale measures in these models and then obtain ordering of the corresponding option prices.

In Section 3.3 we extend the previous ordering results for European option prices to orderings of prices of several path-dependent options. These include lookback options, Asian options, American options and barrier options. We show that if the underlyings of these options are ordered w.r.t. the (increasing) convex order, then the corresponding path-dependent option prices are ordered in the case of lookback and Asian options. For Asian options, American options and barrier options we obtain ordering results that are similar to the ordering results of Chapter 1.

Finally, in Section 3.4 we derive increasing convex orderings for univariate $\alpha$-stable processes, $\alpha \in (1, 2)$, and of univariate and multivariate normal inverse Gaussian (NIG) processes. The univariate results are established via domination criteria for the corresponding Lévy measures. The approach for the ordering results for multivariate NIG processes relies on mixing type representations of generalized hyperbolic (GH) distributions.

### 3.1 Non-trivial bounds for European option prices

The models that are leading to the trivial price interval have been characterized completely in Eberlein and Jacod (1997), Jacod (1997), Frey and Sin (1999), Jakubenas (2002), Gushchin and Mordecki (2002). Also some results for models with nontrivial bounds and corresponding comparison results are given in the literature. Nontrivial bounds for univariate diffusion models are given in Bergman, Grundy, and Wiener (1996) and for stochastic volatility models nontrivial bounds are established in El Karoui, Jeanblanc-Picqué, and Shreve (1998), Hobson (1998a), and Frey and Sin (1999). Bellamy and Jeanblanc (2000) (see also Henderson and Hobson (2003)) show that the price of a European call for a diffusion with jumps is bounded below by the corresponding Black–Scholes price and above by the trivial upper price. Nontrivial bounds for exponential Lévy models are characterized in Jakubenas (2002), and in Gushchin and Mordecki (2002). Finally, Gushchin and
Mordecki (2002) and Rueschendorf (2002) establish that a nontrivial upper bound in discrete time models is given by the Cox–Ross–Rubinstein model. We obtain nontrivial bounds for a variety of quasi-leftcontinuous semi-martingale models. Our results are partial extensions of the continuous time results in the literature given above, mainly in two directions. We also consider nontrivial bounds for multivariate models and we obtain bounds for stochastic volatility models that additionally exhibit jumps. Due to the continuity assumption, discrete time models are excluded from our considerations. We establish nontrivial upper and lower bounds for stochastic volatility models, with and without jumps, for Lévy driven diffusions, and for exponential and stochastic exponential Lévy models.

Let $S$ and the candidate model for the upper (resp. lower) bound $S^*$ be models for the discounted values of underlying securities. We call $S = E(X)$ a $d$-dimensional stochastic volatility model, if $X \sim (0, c, 0)$ for a predictable process $c$ on a stochastic basis $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \in [0,T]}, P)$ with values in $M_+(d, \mathbb{R})$. As candidate for the upper and the lower bound, respectively, we consider a non-negative diffusion $S^*$ that is the positive solution of the SDE

$$dS^*_t = \sigma^{S^*}(t, S^*_t) dW^*_t,$$

where $W^*$ is a $d$-dimensional Brownian motion and $\sigma^{S^*} : [0, T] \times \mathbb{R}^d \to M_+(d, \mathbb{R})$. As $S^*$ is positive, the stochastic logarithm $X^* = \Log(S^*) \sim (0, c^*, 0)$ exists and by Jacod and Shiryaev (2003, Theorem III.2.26) and Lemma A.1.2,

$$c^{*ij}(t, S^*_t(\omega)) = \sum_{k \leq d} \frac{\sigma^{S^*ik}(t, S^*_t(\omega)) \sigma^{S^*jk}(t, S^*_t(\omega))}{S^*_t(\omega)}.$$

If the diffusion coefficient $c^*$ dominates the stochastic volatility $c$ appropriately, Theorems 1.1.3 and 1.2.2 imply that the diffusion $S^*$ is an upper bound for the stochastic volatility model $S$.

**Theorem 3.1.1 (Upper bound for multivariate SV model).** Let $g : \mathbb{R}^d \to \mathbb{R}$, $g \in \mathcal{F}_{cx}$. Assume that $S = E(X)$ is a $d$-dimensional stochastic volatility model, $X \sim (0, c, 0)$, and that $S^*$ is a $d$-dimensional diffusion with diffusion coefficient function $c^*$ given by (3.1). Let $S, S^*$ satisfy Assumptions MG and BIC($g$) (or CD($g$)) and assume that $S^*$ satisfies Assumption SC($g$) and that the Euler scheme $\tilde{S}^*_K$ of $S^*$ satisfies Assumption AP($g$). Furthermore, assume that the Kolmogorov backward equation $T_{\Log}G(t, S_t(\omega)) = 0$ is satisfied $\lambda \times P$-a.e.. If $\sigma^{S^*}(t, \cdot)$ is convex, then the comparison of the differential characteristics of the stochastic logarithms

$$c_t(\omega) \leq_{psd} c^*(t, S_t(\omega)), \quad \lambda \times P$$. a.e.,
implies
\[ E_g(S_T) \leq E^\ast g(S^\ast_T). \]

**Remark 3.1.2 (PII bound).** The previous result hinges on the approximation property \( \text{AP}(g) \) for the Euler scheme \( \tilde{S}^\ast_K \) of \( S^\ast \). If either \( c^\ast \) or \( \sigma^S \) is independent of \( S^\ast \), then \( X^\ast \) or \( S^\ast \) is a PII and Theorem 3.1.1 holds true without assumption \( \text{AP}(g) \), cp. Lemma 1.2.11. Additionally, in the first case convexity of \( \sigma^S(t, \cdot) \) does not have to be required and in the second case this condition is redundant. This also holds true for the corresponding lower bound results that are given in Theorem 3.1.5 below.

As corollary we obtain an upper bound for univariate stochastic volatility models. Following Hofmann, Platen, and Schweizer (1992), a general class of one-dimensional stochastic volatility models is given by
\[
\frac{dS_t}{S_t} = \sigma(t, S_t, v_t) dW_t,
\]
\[
dv_t = b(t, S_t, v_t) dt + \eta_1(t, S_t, v_t) dW_t + \eta_2(t, S_t, v_t) dB_t,
\]
where \( W \) and \( B \) are independent Brownian motions. This includes the well-known stochastic volatility models of Hull and White (1987), Wiggins (1987), Scott (1987), Stein and Stein (1991), and Heston (1993), see also Frey (1997) for an overview over stochastic volatility models. If (3.2) has a positive solution, then \( X = \log(S) \sim (0, \sigma^2(t, S_t, v_t), 0) \) exists, and the upper bound candidate is given by the positive solution \( S^\ast \) of the generalized Black–Scholes model
\[
\frac{dS^\ast_t}{S^\ast_t} = \sigma^\ast(t, S^\ast_t) dW^\ast_t,
\]
where \( W^\ast \) is a one-dimensional Brownian motion, and \( X^\ast = \log(\sigma^\ast(t, S_t), 0) \).

**Corollary 3.1.3 (Upper bound for univariate SV model).** Let \( g : \mathbb{R} \to \mathbb{R}, \ g \in F_{cx}, \) and assume that \( S \) is the positive solution of the univariate stochastic volatility model (3.2), and that \( S^\ast \) is the positive solution of the diffusion model (3.3). Let \( S, S^\ast \) satisfy Assumptions MG and BIC\((g)\) (or (CD\((g)\))). Additionally, assume that \( S^\ast \) satisfies Assumption SC\((g)\), and that the Euler scheme \( \tilde{S}^\ast_K \) of \( S^\ast \) satisfies assumption \( \text{AP}(g) \). Furthermore, assume that the Kolmogorov backward equation \( T_{\log G}(t, S_{t_1}(-\omega)) = 0 \) is satisfied \( \lambda \times P\text{-a.e.} \). If \( f_t(s) := s \sigma^\ast(t, s) \) is convex and
\[
\sigma(t, S_t(\omega), v_t(\omega)) \leq \sigma^\ast(t, S_t(\omega)),
\]
\( \lambda \times P\text{-a.e.}, \) then the generalized Black–Scholes model \( S^\ast \) is an upper bound for the stochastic volatility model \( S \).
Remark 3.1.4. Corollary 3.1.3 generalizes the upper bound result of Frey and Sin (1999). El Karoui, Jeanblanc-Picqué, and Shreve (1998) obtain this result with different assumptions on \( f_t \). In Hypothesis 5.1, they assume that \( D_s f_t(s) \) is smooth and bounded and, additionally, that \( \sigma^* : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous and bounded above.

Next, we obtain lower bounds for multivariate stochastic volatility models. The result is parallel to Theorem 3.1.1 and follows from Theorems 1.1.5 and 1.2.2.

**Theorem 3.1.5 (Lower bound for multivariate SV model).** Let \( g : \mathbb{R}^d \to \mathbb{R}, g \in \mathcal{F}_{cx} \). Assume that \( S = \mathcal{E}(X), X \sim (0, c, 0) \), is a positive \( d \)-dimensional stochastic volatility model, and that \( S^* = \mathcal{E}(X^*), X^* \sim (0, c^*, 0) \), is a \( d \)-dimensional diffusion with diffusion coefficient function \( c^* \) given by (3.1). Let \( S, S^* \) satisfy Assumptions MG and CD\((g)\) and assume that \( S^* \) satisfies Assumption SC\((g)\) and that the Euler scheme \( \tilde{S}_K \) of \( S^* \) satisfies AP\((g)\). Furthermore, assume that the Kolmogorov backward equation \( T_{\log G}(t, S_t(\omega)) = 0 \) is satisfied \( \lambda \times P \)-a.e..

If \( \sigma^{S^*}(t, \cdot) \) is convex, then the comparison of the differential characteristics of the stochastic logarithms

\[
c^*(t, S_t(\omega)) \leq \text{psd} \ c_t(\omega), \quad \lambda \times P \text{-a.e.,}
\]

implies

\[
E^* g(S^*_T) \leq E g(S_T).
\]

Remark 3.1.6 (Lower bound for univariate SV model). Theorem 3.1.5 implies a lower bound result for univariate stochastic volatility models that is similar to Corollary 3.1.3. If the volatility function \( \sigma^* \) of the generalized Black–Scholes model \( S^* \) is dominated by the stochastic volatility \( \sigma \) of the stochastic volatility model \( S \) in an appropriate way, then the Black–Scholes model is a lower bound for the stochastic volatility model.

Multivariate diffusions are also candidate models for lower bounds for stochastic volatility with jumps models \( S = \mathcal{E}(X) \) with \( X \sim (0, c, K) \), where \( c \) is predictable with values in \( M_+(d, \mathbb{R}) \) and \( K \) is a stochastic Lévy kernel.

**Theorem 3.1.7 (Lower bound for multivariate SV with jumps model).** Let \( g : \mathbb{R}^d \to \mathbb{R}, g \in \mathcal{F}_{cx} \). Assume that \( S = \mathcal{E}(X), X \sim (0, c, K) \), is a \( d \)-dimensional stochastic volatility with jumps model, and that \( S^* \) is a \( d \)-dimensional diffusion with diffusion coefficient function \( c^* \) given by (3.1). Let \( S, S^* \) satisfy Assumptions MG and CD\((g)\) and assume that \( S^* \)
satisfies Assumption SC\((g)\) and that the Euler scheme \(\tilde{S}_k^*\) of \(S^*\) satisfies AP\((g)\). Furthermore, assume that the Kolmogorov backward equation 
\[ T_{\log} \mathcal{G}(t, S_{t-}(\omega)) = 0 \]
is satisfied \(\lambda \times P\text{-}a.e..\) If \(\sigma^*(t, \cdot)\) is convex, then the comparison of the differential characteristics of the stochastic logarithms
\[ c^*(t, S_{t-}(\omega)) \leq_{psd} c_t(\omega), \quad \lambda \times P\text{-}a.e., \]
implies
\[ E^*g(S^*_T) \leq Eg(S_T). \]

**Remark 3.1.8 (Lower bound for univariate SV with jumps model).**
Parallel to Corollary 3.1.3, from the previous theorem a lower bound result for univariate stochastic volatility with jumps models follows. Here, we also have to assume that \(f_t(s) = s\sigma^*(t, s)\) is convex for all \(t \in [0, T]\). Bellamy and Jeanblanc (2000) establish that a univariate diffusion with jumps model \(S\) is bounded below by a univariate diffusion model \(S^*\) that has the same diffusion coefficient as \(S\) under different conditions. Similar to El Karoui, Jeanblanc-Picqué, and Shreve (1998) they assume in Hypothesis H1 that \(D_s f_t(s)\) is smooth and bounded, and that \(\sigma^*: [0, T] \times \mathbb{R}_+ \to \mathbb{R}\) is continuous and bounded.

Next, we establish an upper bound for a stochastic volatility with jumps model \(S = \mathcal{E}(X), X \sim (0, c, K)\). As candidate for the bounding model we consider a \(d\)-dimensional diffusion with jumps \(S^*\) that is the positive solution of the SDE
\[ dS^*_t = \sigma^*(t, S^*_t) dW^*_t + \phi^*(t, S^*_t, z)(p^*(dt, dz) - q^*(dt, dz)), \]
where \(W^*\) is a \(d\)-dimensional Brownian motion and \(p\) is a Poisson random measure with intensity measure \(q^*(dt, dy) = dt \otimes \lambda^*(dz)\), cp. Jacod and Shiryaev (2003, §III.2c). In this case the characteristics of \(X^* = \log(S^*)\) are of the form
\[ c^*_{ij}(t, S^*_t(\omega)) = \sum_{k \leq d} \frac{\sigma^*_{ik}(t, S^*_t(\omega)) \sigma^*_j(t, S^*_t(\omega))}{S^*_t(\omega)} \]
\[ K^*_i(s, G) = \int 1_{G \setminus \{0\}} \left( \frac{\phi^*(t, s, y)}{s} \right) \lambda^*(dy), \quad G \in \mathcal{B}^d, \]
which follows from Jacod and Shiryaev (2003, Theorem III.2.26) and Lemma A.1.2.

We recall that if \(S^*\) is a diffusion with jumps and neither \(S^*\) nor \(X^* = \log(S^*)\) has independent increments, in Subsection 1.2.2 we established the propagation of convexity property under the following additional assumptions on the jump component of \(S^*\).
3.1. Non-trivial bounds for European option prices

(J1) \( \phi^* : [0, T] \times \mathbb{R}_+^d \times E \rightarrow \mathbb{R}^d \) factorizes into \( \phi^*(t, s, y) = \varphi^*(t, s) \psi^*(t, y) \), where \( \varphi^* : [0, T] \times \mathbb{R}_+^d \rightarrow \mathbb{R}_+^d \), \( \varphi^*(t, \cdot) \) is convex, \( \forall t \in [0, T] \), and \( \psi^* : [0, T] \times E \rightarrow \mathbb{R}^d \).

(J2) \( \phi^* : [0, T] \times \mathbb{R}_+^d \times E \rightarrow \mathbb{R}^d \) factorizes into \( \phi^*(t, s, y) := \varphi^*(t, s) \psi^*(t, y) \), where \( \varphi^* : [0, T] \times \mathbb{R}_+^d \rightarrow \mathbb{R}_+^d \), \( \varphi^*(t, \cdot) \) is affine-linear, \( \forall t \in [0, T] \), and \( \psi^* : [0, T] \times E \rightarrow \mathbb{R} \).

(J3) \( d = 1, E = \mathbb{R}, N \geq 0 \) and \( \phi^* : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+ \) factorizes into \( \phi^*(t, s, y) := \sum_{i=1}^{m} \varphi_i^*(t, s) \psi_i^*(t, y) \), where \( \varphi_i^* : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is convex, \( \forall t \in [0, T] \), \( i \leq m \) and \( \psi_i^* : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+ \) is non-decreasing, \( \forall i \leq m \).

The upper bound result for a stochastic volatility with jumps model is the following corollary of Theorems 1.1.3 and 1.2.6.

**Theorem 3.1.9 (Upper bound for multivariate SV with jumps model).** Let \( g : \mathbb{R}^d \rightarrow \mathbb{R}, g \in \mathcal{F}_{\text{ex}} \). Assume that \( S = \mathcal{E}(X) \) is a \( d \)-dimensional stochastic volatility with jumps model, \( X \sim (0, c, K) \), and that \( S^* \) is a \( d \)-dimensional diffusion with jumps that is the positive solution of (3.4). Let \( S, S^* \) satisfy Assumptions MG and BIC\((g)\) (or CD\((g)\)). Additionally, assume that \( S^* \) satisfies Assumption SC\((g)\) and that the Euler scheme \( \tilde{S}^* \) of \( S^* \) satisfies Assumption AP\((g)\). Furthermore, assume that the Kolmogorov backward equation \( T_{t, \omega} \mathcal{G}(t, S_{t-}^*(\omega)) = 0 \) is satisfied \( \lambda \times P \)-a.e. If for all \( t \in [0, T] \) the diffusion coefficient function \( \sigma_S(t, \cdot) \) is convex, one of the conditions (J1)–(J3) is satisfied, and if for the differential characteristics of the stochastic logarithms \( X, X^* \) it holds true that

\[
\mathcal{E}_t(\omega) \leq \mathcal{E}^*(t, S_{t-}^*(\omega)),
\]

\[
\int_{(-1, \infty)^d} f(t, S_{t-}, x) K_{\omega,t}(dx) \leq \int_{(-1, \infty)^d} f(t, S_{t-}, x) K^*(t, S_{t-}^*(\omega), dx), \tag{3.5}
\]

\( \lambda \times P \)-a.e., for all \( f : [0, T] \times \mathbb{R}_+^d \times (-1, \infty)^d \rightarrow \mathbb{R}_+ \) with \( f(t, s, \cdot) \in \mathcal{F}_{\text{ex}} \) such that the integrals exist, then

\[
\mathcal{E}g(S_T) \leq \mathcal{E}^*g(S_T^*).
\]

**Remark 3.1.10 (Lower bound for multivariate SV with jumps model).** Similarly, the result for the lower bound follows with reversed inequalities and under the appropriate integrability condition CD\((g)\).
From the previous result we obtain an upper and a lower bound for a univariate stochastic volatility with jumps model that is the positive solution \( S \) of the SDE
\[
\frac{dS_t}{S_t} = \sigma_t dW_t + \phi_t(y)(N(dt, dy) - \lambda(dy)dt),
\] (3.6)
where \( \sigma \) is an adapted process, \( \phi \) is a \( \tilde{P} \)-measurable, and \( X = \text{Log}(S) \sim (0, \sigma^2, K) \), with \( \mathbb{E} = \mathbb{R}, \lambda(\mathbb{E}) < \infty \) and \( K_t(\omega, G) = \int 1_{\{G > 0\}}(\phi_t(\omega, y))\lambda(dy) \).

As candidate for the bounding diffusion with jumps model we consider the positive solution \( S^* \) of the diffusion SDE
\[
\frac{dS^*_t}{S^*_t} = \sigma^*(t, S^*_t) dW_t^* + \phi^*(t, S^*_t, y)(N^*(dt, dy) - \lambda^*(dy)dt). \tag{3.7}
\]
In this case we have \( K_t^*(s, G) = \int 1_{G \setminus \{0\}}(\phi^*(t, s, y))\lambda^*(dy) \), hence \( X^* \sim 0, (\sigma^*(t, S^*_t))^2, \lambda^* \phi^*(t, S^*_t, \cdot) \).

**Corollary 3.1.11 (Bounds for univariate SV with jumps model).**

Let \( g : \mathbb{R} \to \mathbb{R}, g \in F_{cx} \). Assume that \( S \) is the positive solution of (3.6) and \( S^* \) is the positive solution of (3.7). Let \( S, S^* \) satisfy Assumption MG and assume that \( S^* \) satisfies Assumption SC(\( g \)) and that the Euler scheme \( S\tilde{K}^* \) of \( S^* \) satisfies Assumption AP(\( g \)). Furthermore, assume that the Kolmogorov backward equation \( T_{\text{Log}\mathcal{G}}(t, S_t(\omega)) = 0 \) is satisfied \( \lambda \times P\)-a.e.. Additionally, assume that \( f_t(s) := s \sigma^*(t, s) \in F_{cx} \), for all \( t \in [0, T] \), and that the jump part \( \phi^S(t, s, y) = s \phi^*(t, s, y) \) of \( S^* \) satisfies condition (J1) with \( d = 1 \) or condition (J3).

1. **(Upper bound)** Let Assumption BIC(\( g \)) or CD(\( g \)) be satisfied. If
\[
\sigma_t(\omega) \leq \sigma^*(t, S_t(\omega)),
\]
\[
\int_{(-1, \infty)} f(t, S_{t-}(\omega), \phi_t(\omega, y))\lambda(dy) \leq \int_{(-1, \infty)} f(t, S_{t-}(\omega), \phi^*(t, S_{t-}(\omega), y))\lambda^*(dy)
\]
\( \lambda \times P\)-a.e., for all \( f : [0, T] \times \mathbb{R}_+ \times (-1, \infty) \to \mathbb{R}_+ \) with \( f(t, s, \cdot) \in F_{cx} \) such that the integrals exist, then
\[
Eg(S_T) \leq E^*g(S^*_T).
\]

2. **(Lower bound)** Let Assumption CD(\( g \)) be satisfied. If
\[
\sigma_t(\omega) \geq \sigma^*(t, S_t(\omega)),
\]
\[
\int_{(-1, \infty)} f(t, S_{t-}(\omega), \phi_t(\omega, y))\lambda(dy) \geq \int_{(-1, \infty)} f(t, S_{t-}(\omega), \phi(t, S_{t-}(\omega), y)\lambda^*(dy))
\]
3.1. Non-trivial bounds for European option prices

\[ \lambda \times P\text{-a.e., for all } f : [0, T] \times \mathbb{R}_+ \times (-1, \infty) \to \mathbb{R}_+ \text{ with } f(t, s, \cdot) \in \mathcal{F}_{cx} \text{ such that the integrals exist, then} \]

\[ E g(S_T) \geq E^* g(S^*_T). \]

**Remark 3.1.12 (PII bounds).** Similar to the second point of Remark 3.1.2 the conditions in Theorem 3.1.9 and Corollary 3.1.11 simplify if \( S^* \) or \( X^* = \log(S^*) \) is a PII. In this case, one does neither need assumption AP(\( g \)), nor the convexity assumption for \( f \), nor assumptions (J1) and (J3).

As corollary of Corollary 3.1.11 we obtain bounds for the positive solution \( S \) of a one-dimensional Lévy driven SDE

\[ \frac{dS_t}{S_t} = \gamma(t, S_t) dL_t, \quad S_0 = 1, \]

where \( L \sim (0, c, F)_{id} \) is a Lévy process and \( \gamma : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \). As candidate for the upper bound process we consider the positive solution \( S^* \) of another Lévy driven SDE

\[ \frac{dS^*_t}{S^*_t} = \gamma^*(t, S^*_t) dL^*_t, \quad S^*_0 = 1, \]

with a Lévy process \( L^* \sim (0, c^*, F^*)_{id} \) and \( \gamma^* : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \).

**Corollary 3.1.13 (Upper bound for univariate Lévy driven diffusion).** Let \( g : \mathbb{R} \to \mathbb{R}, g \in \mathcal{F}_{cx} \) and assume that \( S, S^* \) are positive solutions of the one-dimensional Lévy driven SDEs given above. Let \( S, S^* \) satisfy Assumptions MG and BIC(\( g \)) (or CD(\( g \))). Let further \( S^* \) satisfy Assumption SC(\( g \)) and assume that the Euler scheme \( \tilde{S}_K^* \) of \( S^* \) satisfies Assumption AP(\( g \)). Furthermore, assume that the Kolmogorov backward equation \( T_{\log} G(t, S_{t-}(\omega)) = 0 \) is satisfied \( \lambda \times P\text{-a.e..} \)

If \( s \mapsto s\gamma^*(t, s) \in \mathcal{F}_{cx}, \| F \| = \| F^* \| < \infty \) and if it holds true that

\[ c \leq c^*, \]

\[ \gamma(t, s) \leq \gamma^*(t, s), \text{ for all } (t, s) \in [0, T] \times \mathbb{R}_+, \]

\[ F \leq_{cx} F^* \quad \text{and} \quad \int x F(dx) = 0 \quad \text{or} \quad \int x F^*(dx) = 0, \]

then \( E g(S_T) \leq E^* g(S^*_T). \)

**Proof.** \( S \) is Markovian with \( \sigma_t(\omega) = \sqrt{\gamma(t, S_t(\omega))} \) and \( \phi_t(\omega, x) = x\gamma(t, S_{t-}(\omega)) \). The coefficient functions of \( S^* \) are similarly given by \( \sigma^*(t, s) = \sqrt{\gamma^*(t, s)} \) and \( \phi(t, s, x) = x\gamma(t, s) \). Convexity of \( s \mapsto s\gamma^*(t, s) \) implies that
\( \phi_{S^*} \) satisfies the jump condition (J1) and that \( f_\varepsilon(s) = s\sqrt{c^*}\gamma^*(t, s) \) is convex. From \( c \leq c^* \) and \( \gamma(t, s) \leq \gamma^*(t, s) \) it follows that
\[
\sigma_1(\omega) = \sqrt{c^*}\gamma(t, S_1(\omega)) \leq \sqrt{c^*}\gamma^*(t, S_1(\omega)) = \sigma^*(t, S_1(\omega)).
\]

If \( \int xF^*(dx) = 0 \), then \( \gamma(t, s) \leq \gamma^*(t, s) \) and \( F \leq_{cx} F^* \) imply for \( f \in F_{cx} \)
\[
\int f(x\gamma(t, s))F(dx) \leq \int f(x\gamma^*(t, s))F^*(dx) \leq \int f(x\gamma^*(t, s))F^*(dx),
\]
as for a random variable \( X \) with \( EX = 0 \) and \( a \leq b \) it holds true that \( aX \leq_{cx} bX \). The case \( \int xF^*(dx) = 0 \), is similarly treated. Hence the result follows from Corollary 3.1.11.

Now we derive upper and lower bounds for positive multivariate pure jump stochastic exponential Lévy models \( S \). Firstly, we treat the particular case where \( X = \log(S) \sim (0, 0, F)_{id} \) is a compound Poisson process. In this case, the corresponding Lévy measure is a finite measure on \((-1, \infty)^d\). As candidate for a bounding process we consider another compound Poisson process \( X^* \sim (0, 0, F^*)_{id} \) with finite Lévy measure on \((-1, \infty)^d\). We obtain the following nontrivial bounds for European options on \( d \)-dimensional underlyings \( S \) that have (directionally) convex payoff functions. We define \( \lambda := \|F\| \) and \( R(dx) = \frac{1}{\lambda}F(dx) \), similar for \( F^* \).

**Theorem 3.1.14 (Bounds for stochastic exponential compound Poisson model).** For \( F \in \{F_{cx}, F_{dex}\} \), let \( g \in F \), and let \( S = \mathcal{E}(X) \), \( S^* = \mathcal{E}(X^*) \) be stochastic exponentials of compound Poisson processes \( X \sim (0, 0, \lambda R)_{id} \), \( X^* \sim (0, 0, \lambda^* R^*)_{id} \), with \( \text{supp}(R) \subset (-1, \infty)^d \), \( \text{supp}(R^*) \subset (-1, \infty)^d \). Let \( S^* \) satisfy Assumption SC\((g)\) and for \( g \in F_{dex} \) additionally assume that \( \int_{[0,1]} \int_{(-1,\infty)^d} \left( \left| \Lambda g \right| (u, S_{u-}, x) \right) \lambda R(dx)du \in \mathcal{A}_{loc}^{\infty} \). Furthermore, assume that the Kolmogorov backward equation \( T_{\log\mathcal{G}}(t, S_{t-}(\omega)) = 0 \) is satisfied \( \lambda \times P\text{-a.e.} \).

1. (Upper bound) Let \( S, S^* \) satisfy Assumption BIC\((g)\).
   (a) If \( \lambda = \lambda^* \) and \( R \leq_{F} R^* \), then \( S_T \leq_{F} S_T^* \).
   (b) If \( \lambda \leq \lambda^* \) and \( R \leq_{cx} R^* \) with \( ER = 0 \), then \( S_T \leq_{cx} S_T^* \).

2. (Lower bound) Let \( S, S^* \) satisfy Assumption CD\((g)\).
   (a) If \( \lambda = \lambda^* \) and \( R \leq_{F} R^* \), then \( S_T^* \leq_{F} S_T \).
   (b) If \( \lambda^* \leq \lambda \) and \( R \leq_{cx} R^* \) with \( ER = 0 \), then \( S_T^* \leq_{cx} S_T \).
1. Under the stated conditions it follows from Theorem 3.1.14 that $S_n \leq \lambda^* R$.

Theorem 3.1.14 implies an ordering result for stochastic exponential Lévy processes with infinite Lévy measures by approximation. For infinite Lévy measure $F$ we recall the definition of truncated Lévy measures $F_n(dx) := 1_{(\bar{\varepsilon}_n, \varepsilon_n)}(x)F(dx)$, $\bar{\varepsilon}_n \uparrow 0$, $\varepsilon_n \downarrow 0$, which are finite Lévy measures. Ordering of the corresponding stochastic exponential compound Poisson processes $S_n = \mathcal{E}(X_n)$, $S_n^* = \mathcal{E}(X_n^*)$, where

$$X_n \sim (0, 0, F_n)_{id}, \quad X_n^* \sim (0, 0, F_n^*)_{id}$$

implies ordering of the limit models $S, S^*$.

**Corollary 3.1.15 (Bounds for stochastic exponential pure jump Lévy models with infinite Lévy measures).** Let $\mathcal{F}_0 \subset \mathcal{F}_{\text{cx}} \cap C^2$, be a generating class of the convex order. Let $F, F^*$ be Lévy measures with infinite total mass that satisfy $\text{supp}(F) \subset (-1, \infty)^d$, $\text{supp}(F^*) \subset (-1, \infty)^d$ and $\int |x|^2 F(dx) < \infty$, $\int |x|^2 F^*(dx) < \infty$, and let $S = \mathcal{E}(X)$ and $S^* = \mathcal{E}(X^*)$, $S_0 = S_0^* = 1$, where $X \sim (0, 0, F)_{id}$ and $X^* \sim (0, 0, F^*)_{id}$. Let $\bar{\varepsilon}_n \uparrow 0$, $\varepsilon_n \downarrow 0$ and $\bar{\varepsilon}_n^* \uparrow 0$, $\varepsilon_n^* \downarrow 0$ be sequences s.t. the following conditions for the truncated Lévy measures $F_n$, $F_n^*$ and $S_n = \mathcal{E}(X_n)$, $S_n^* = \mathcal{E}(X_n^*)$ with $X_n \sim (0, 0, F_n)_{id}$, $S_{n,0} = S_{n,0}^* = 1$ and $X_n^* \sim (0, 0, F_n^*)_{id}$ are satisfied.

1. $S_n, S_n^*$ satisfy Assumption BIC($\mathcal{F}_0$) and $S_n^*$ satisfies Assumption SC($\mathcal{F}_0$).

Furthermore, the Kolmogorov backward equation $T_{\log \mathcal{G}}(t, S_{n,t-})$ is satisfied $\lambda \times P$-a.e. for all $g \in \mathcal{F}_0$.

2. $\|F_n\| = \|F_n^*\|$ and $F_n \leq \text{cx} F_n^*$, for all $n \in \mathbb{N}$.

Then $S \leq \text{cx} S^*$.

**Proof.** Under the stated conditions it follows from Theorem 3.1.14 that $S_n \leq \text{cx} S_n^*$. Corollary 2.2.2 implies $X_n \overset{\mathcal{L}}{\rightarrow} X$. As all $X_n$ and the limit $X$ are Lévy processes, the sequence $(X_n)$ is predictably uniformly tight (P-UT, cp. Jacod and Shiryaev (2003, Definition VI.6.1)), hence it follows from Theorem IX.6.9 of that book, that $(X_n, S_n) \overset{\mathcal{L}}{\rightarrow} (X, S)$, thus $S_n \overset{\mathcal{L}}{\rightarrow} S$. As $ES_n = ES = 1$ and $ES_n^* = ES^* = 1$, the result follows. \qed
Finally, we obtain a nontrivial upper bounds for exponential Lévy models \( S = e^{X} \) in the case when the support of the Lévy measure is bounded above, \( \text{sup} \supp(F) = \kappa < \infty \). The upper bound candidate is given by an exponential Lévy process \( S^* = e^{X^*} \), where the Lévy measure of \( X^* \) is some constant times the Dirac measure \( \delta_{\{\kappa\}} \).

**Theorem 3.1.16 (Upper bound for exponential Lévy model).** Let \( g : \mathbb{R}_+ \to \mathbb{R}_+ \), \( g \in \mathcal{F}_{cx} \), and assume that there is a real \( K > 0 \) s.th. \(|g(x)| \leq K|x|\). Let \( \bar{F} \) be a one-dimensional Lévy measure that satisfies \( \sup \supp(\bar{F}) \subset \mathbb{R}_+ \), assume that \( \sup \supp(F) = \kappa < \infty \), and let \( \int_{x \in (0,1]} x \bar{F}(dx) < \infty \). For Lévy processes \( \bar{X} \sim ( - \int (e^x - 1 - x) \bar{F}(dx), 0, \bar{F})_{id} \) and \( \bar{X}^* \sim (\bar{\lambda}^*(\kappa + 1 - e^{\kappa}), 0, \bar{\lambda}^* \delta_{\{\kappa\}})_{id} \), where \( \bar{\lambda}^* = \frac{1}{\kappa} \int x \bar{F}(dx) \), let \( S = e^{\bar{X}} \) and \( S^* = e^{\bar{X}^*} \). Assume that \( S^* \) satisfies Assumption SC\((g)\), and that the Kolmogorov backward equation \( T \bar{G}(t, S_{t-}) = 0 \) is satisfied \( \lambda \times P\text{-a.e.} \). Then

\[
Eg(S_T) \leq E^*g(S^*_T).
\]

**Proof.** We apply Theorem 1.1.7 in combination with Remark 1.1.8 and cut criteria from Chapter 2. As \( \bar{b}(id) = - \int (e^x - 1 - x) \bar{F}(dx) \) and \( \bar{b}^*(id) = \bar{\lambda}^*(\kappa + 1 - e^{\kappa}) \), it follows from Lemma A.1.1 and Kallsen (2000, Lemma 4.4) that \( S, S^* \) are martingales. From boundedness of the support of the Lévy measure \( \bar{F} \), \( \int e^x \bar{F}(dx) < \infty \) follows, hence the first moments of \( S_t, S^*_t \) exist for all \( t \in [0, T] \). Due to the assumptions on \( g \), it follows that \( E\bar{G}(t, S_t) = \int \int g(s + s^*) P^S\tilde{\kappa}^{-\varepsilon}(s^*) P^S(t, ds) < \infty \), hence Assumption BIC\((g)\) is satisfied.

Let \( \|\bar{F}\| < \infty \). From \( \int x \bar{F}(dx) = \bar{\lambda}^* \kappa \) and \( \sup \supp(\bar{F}) = \kappa \) it follows that \( \|\bar{F}\| = \int_0^\kappa 1 \bar{F}(dx) \geq \int_0^\kappa \frac{\kappa}{\kappa} \bar{F}(dx) = \bar{\lambda}^* \), thus the cut criterion of Theorem 2.1.5 implies \( \bar{F} \leq_{cx} \bar{\lambda}^* \delta_{\{\kappa\}} \), where \( \bar{F} \) is the modified Lévy measure of \( \bar{F} \). As \( \bar{X} \sim ( - \int (e^x - 1 - x) \bar{F}(dx), 0, \bar{F})_{id} \), it follows from Lemma A.1.1 and A.1.2 that \( S \sim (0, 0, K^S(s))_{id} \) and \( S^* \sim (0, 0, K^{S^*}(s))_{id} \), where

\[
K^S(s; dx) = \bar{F}^{x-s(e^x-1)}(dx), \quad K^{S^*}(s; dx) = \bar{\lambda}^* \delta_{\{\kappa\}}^{x-s(e^x-1)}(dx).
\]

As \( S, S^* \) only have positive jumps, it suffices to establish (1.8) for all \( f : [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) that are increasing and convex in the last component, cp. Remark 1.1.8. For such \( f \), the mapping \( x \mapsto f(t, s, (e^x - 1)) \) is increasing and convex, hence (1.8) follows from the ordering \( \bar{F} \leq_{cx} \bar{\lambda}^* \delta_{\{\kappa\}} \).

If \( \bar{F} \) has finite total mass, a cut criterion that is similar to Theorem 2.2.8, cp. also Remark 2.2.7. For \( \varepsilon_n \downarrow 0 \) let \( N \in \mathbb{N} \) be big enough s.th. \( \|\bar{F}_n\| > \bar{\lambda}^* \). Then \( \bar{F}_n \leq_{cx} \bar{\lambda}^* \delta_{\{\kappa\}} + \left(\|\bar{F}_n\| - \bar{\lambda}^*\right) \delta_{\{0\}} =: \bar{F}^* \), and the result follows. \( \square \)
**Remark 3.1.17.** Based on the work of Eberlein and Jacod (1997), Jakubenas (2002) completely characterizes trivial and non-trivial bounds in univariate exponential Lévy models, which include the ordering result of the previous theorem under milder conditions.

### 3.2 Comparison of martingale measures

As rendered in the introduction of this chapter, one approach to obtain a European option price in an incomplete market model out of the price interval is to choose a martingale measure by an optimization criterion. Examples for well established martingale measures that are derived by such optimization criteria are the minimal martingale, the $q$-optimal measure, the minimal entropy measure, the Esscher measure, or the variance minimal measure (see Schweizer (1996), Delbaen and Schachermayer (1996), Keller (1997), Chan (1999), Frittelli (2000), Goll and Rüschendorf (2001)). The second general approach to pricing of contingent claims, the utility based indifference price and variants of it due to Hodges and Neuberger (1989) and Davis (1997) (see also Kallsen (1998)), is in one to one correspondence with the martingale pricing measure obtained by minimizing related $f$-divergence distances (see Goll and Rüschendorf (2001)). In a complete market model, the diverse optimization criteria lead to the same martingale measure and one obtains the arbitrage-free option price by risk neutral pricing with this measure. In incomplete models, different optimization criteria typically yield different martingale measures, hence different arbitrage-free prices for contingent claims. In this section we compute several martingale measures in different incomplete market models, where incompleteness is due to jumps or stochastic volatility, and derive ordering of European option prices with convex payoff functions that are computed w.r.t. these measures.

In the literature, there are some ordering results for European option prices under various martingale measures for different incomplete market models. Chan (1999), Henderson and Hobson (2003), and Möller (2004) compute martingale measures in different types of jump models. Chan (1999) considers a Lévy driven diffusion model. In the special case with constant diffusion parameter $\sigma = 1$ and a driving compound Poisson process with Lévy measure $F(dx) = \delta_{\{-\frac{1}{2}\}}(dx) + \delta_{\{\frac{1}{2}\}}(dx)$, he states a numerical result that gives rise to the assumption that in this model the price of a European option w.r.t. the minimal martingale measure is smaller than its price w.r.t. the measure that minimizes relative entropy, which in turn is smaller than the price w.r.t. the Merton measure. He also numerically derives that in a stochastic exponential gamma model with $\sigma = 1$ and where the driving
pure jump Lévy process has Lévy density \( f(x) = \frac{1}{2}e^{-x}I_{[0,\infty)}(x) \), the minimal martingale measure yields smaller European call option prices than the Merton measure. Henderson and Hobson (2003) compute several distance minimizing martingale measures in a diffusion with jumps model of stochastic exponential type. They first parameterize the set of equivalent martingale measures in terms of the Girsanov jump parameter. Then they obtain ordering of European option prices w.r.t. this parameter, and identify the previously obtained distance minimizing martingale measures with the parameter to obtain ordering of the corresponding prices. Møller (2004) establishes ordering of reinsurance contracts with convex payoff w.r.t. the minimal martingale measure, the minimal entropy martingale measure, and some other martingale measures in a compound Poisson model with positive jumps. In the appendix, he gives an extension of his results to a PII model with finite, deterministic Lévy kernels. Henderson (2005) and Henderson, Hobson, Howison, and Kluge (2003) obtain orderings of European option prices w.r.t. \( q \)-optimal martingale measures in stochastic volatility models with diffusion type volatility process. They first establish ordering of prices in the drift of the diffusion volatility and then show that the drift of the volatility process is ordered in the parameter \( q \). Henderson (2005) considers the case where the volatility process is driven by a Brownian motion that is independent of the driving Brownian motion of the underlying. As corollary she obtains ordering of the minimal martingale measure, the minimal entropy martingale measure and the variance-optimal martingale measure, as these measures can be identified with values 0, 1 and 2, respectively, of the parameter \( q \). Henderson, Hobson, Howison, and Kluge (2003) partially extend these results to the case where the driving Brownian motions of the underlying and the volatility process are correlated.

Throughout this section we assume that \( S \) and \( S^* \) model the discounted values of an underlying security under different martingale measures \( P \) and \( P^* \), respectively. In Subsection 3.2.1 we derive and compare various distance minimizing martingale measures in incomplete jump models. We first consider a diffusion with jumps model, that includes the model in Chan (1999) and the model in the “deterministic case” of Henderson and Hobson (2003). We give a theoretical verification of the numerical results of Chan (1999). From our comparison results of Chapters 1 and 2, we also derive the orderings of European option prices with convex payoff functions that are computed w.r.t. distance minimizing martingale measures that are given in Henderson and Hobson (2003). Then we derive ordering results in the compound Poisson model of Møller (2004), and also in the extended PII model. In Subsection 3.2.2 we treat the ordering of \( q \)-optimal martingale measures.
within the notions of semimartingale characteristics and derive the orderings that are obtained in Henderson (2005), and Henderson, Hobson, Howison, and Kluge (2003).

### 3.2.1 Jump models

The jump models that are considered in Chan (1999) and Henderson and Hobson (2003) are of stochastic exponential type and are special cases of positive solutions $S$ of the SDE

$$dS_t = S_t(b(t)dt + \sigma(t)dW_t + \int_{\mathbb{R}} \phi(t, y)(p(dt, dy) - K(t; dy)dt)$$

(3.9)

where $b : [0, T] \to \mathbb{R}_+$, $\sigma : [0, T] \to \mathbb{R}_+$ and $\phi : [0, T] \times \mathbb{R} \to (-1, \infty)$, $W$ is a one-dimensional Brownian motion under the original measure $P$, and $p$ is a Poisson random measure with deterministic compensator $dtK(t, dx)$ w.r.t. $P$. The stochastic logarithm $X = \log(S) \sim (b(t), \sigma^2(t), K_x \rightarrow \phi(t, x)(t, dx))_{id}$ has independent increments under $P$. Chan (1999) considers the special case of (3.9), where $S$ is a Lévy driven diffusion with SDE

$$dS_t = S_t(\Sigma(t)dL_t + \gamma(t)dt),$$

(3.10)

with $\gamma : [0, T] \to \mathbb{R}$, $\Sigma : [0, T] \to \mathbb{R}_+$ and a Lévy process $L \sim (a, c, F)_{id}$. In this case, $b(t) = \gamma(t) + a\Sigma(t)$, $\sigma(t) = \sqrt{\Sigma(t)}$, $\phi(t, y) = y\Sigma(t)$, $p$ is the jump measure $\mu^L$ of $L$ with deterministic and time-independent compensator $K(t; dy) = F(dy)$ under $P$. To ensure non-negativity of the underlying $S$, the support of the Lévy measure $F$ is assumed to be in $[-d_1, d_2]$, $d_i \geq 0$, where at least one $d_i$ is finite, and the volatility coefficient function $\sigma$ is assumed to be bounded by $-\frac{1}{d_2} \leq \sigma(t) \leq \frac{1}{d_1}$, for all $t \in [t, T]$. Henderson and Hobson (2003) consider a “deterministic case”, which is the special case of (3.9) with

$$K(t; dy) = \lambda(t)\lambda|[0, 1](dy), \lambda(t) : [0, T] \to \mathbb{R}_+$$

and

$$\phi : [0, T] \times [0, 1] \to (-1, \infty).$$

(3.11)

In the “non-deterministic case”, they consider (3.9) with jumps that are due to a counting process $N_t(\omega) = \int_0^t p(\omega; du, [0, 1])$ that counts the total number of jumps of $S$ up to time $t$ and has a Markovian, non-deterministic compensator $\int_0^t \lambda(u, S_{u-}(\omega))du$ under $P$.

In the sequel, we assume that the set of locally equivalent martingale measures

$$\mathcal{M}_{e,loc}^{X,PH} = \{ Q \sim P : \log(X) = S \text{ is a } Q\text{-martingale, } X \text{ is in the class } PH \}$$
is not empty. From Girsanov’s Theorem (cp. Jacod and Shiryaev (2003, Theorem III.3.24) it follows that the differential characteristics of $X$ under a probability measure $Q^{(i)}_{\text{loc}} \ll P$ are of the form

$$b^Q_t = b(t) + \sigma^2(t)\beta^Q_t + \int \phi(t, x) \left( Y^Q(t, \phi(t, x), \cdot) - 1 \right) K(t; dx),$$

$$c^Q_t = \sigma^2(t), \quad K^Q_t(dx) = Y^Q(t, x, \cdot)K(x\rightarrow\phi(t,x))(t; dx),$$

with Girsanov parameters $\beta^Q, Y^Q$, where $\beta^Q$ is a predictable process with values in $\mathbb{R}$ and $Y^Q$ is a non-negative $\tilde{P}$-measurable function. It turns out that the Girsanov parameters $\beta^Q, Y^Q$ of the distance minimizing martingale measures $Q$ that we consider are deterministic, which implies that $X$ is also a PII for any of these measures. If the martingale equation $b^Q_t = 0$ is satisfied, then $X$ is a local martingale under $Q$, hence also $S$ is a local $Q$-martingale. If $S$ is a true martingale under $Q$, then $Q \in \mathcal{M}_{\text{e,loc}}^\pi$. Generally, this is satisfied if $S$ is a process of class (D) under $Q$. If $\bar{X} = \log(S)$ is a Lévy process under $Q$, Lemma A.1.1 and Kallsen (2000, Lemma 4.4) imply that $S$ is a true martingale under $Q$. The density process $Z^Q_t = \frac{dQ}{dP}|_A$, that corresponds to $Q$ is given by $Z^Q = \mathcal{E}(N^Q)$, where

$$N^Q_t = \int_0^t \beta^Q(u)\sigma(u)dW_u + \int_0^t \int \mathbb{R} \left( Y^Q(u, \phi(u, x)) - 1 \right) (p(du, dx) - K(u, dx)du),$$

(3.12)

cp. Jacod and Shiryaev (2003, Theorem III.5.35). We compute the Girsanov jump parameter $Y^Q$ for the distance minimizing martingale measure $Q$ we consider, and then apply the comparison theorems of Chapters 1 and 2 in order to obtain ordering of the prices of European options that are computed w.r.t. these measures.

Under the Merton measure $Q^{\text{Merton}}$, the jump part remains unchanged and the Girsanov parameter $\beta^{\text{Merton}}$ is chosen s.th. the martingale equation $b^{\text{Merton}}_t = 0$ is satisfied, hence $Y^{\text{Merton}}(t, y) \equiv 1$ and $\beta^{\text{Merton}}(t) = -\frac{b(t)}{\sigma^2(t)}$. Chan (1999) calls the pricing w.r.t. the Merton measure $Q^{\text{Merton}}$ pricing by martingale decompositions.

The density $Z^{\text{min}}$ of the minimal martingale measure $Q^{\text{min}}$ is the solution of the SDE

$$dZ^{\text{min}}_t = Z^{\text{min}}_t \eta(t)\sigma(t)dW_t + \int \mathbb{R} \phi(t, y)(p(dt, dy) - K(t; dy)dt).$$

Identification of the coefficients with the coefficients in (3.12) implies that the corresponding Girsanov parameters are given by $\beta^{\text{min}}(t) = \eta^*(t)$ and $Y^{\text{min}}(t, y) = y\eta^*(t) + 1$, where $\eta^* : [0, T] \rightarrow \mathbb{R}$, $\eta^*(t) = -b(t)(\sigma^2(t) + \int \mathbb{R} \phi(t, y)(p(dt, dy) - K(t; dy)dt).$
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\[ \int_\mathbb{R} \varphi^2(t,x) K(t; dx) \] 

is the deterministic solution of the martingale equation \( b_{\text{min}} = 0 \). Observe that we have to assume that \( Y_{\text{min}} > 0 \) s.th. the minimal martingale measure exists. Hence, in the typical situation \( b > 0 \), we have to require additional bounding conditions on the support of the Lévy kernels \( K(t; \cdot) \).

The density corresponding to the \textit{Esscher measure} requires the notions of Laplace cumulant processes and exponentially special semimartingales. We refer to Kallsen and Shiryaev (2002), see also Jacod and Shiryaev (2003, Chapter III, §7). Chan (1999) and Henderson and Hobson (2003) compute different versions of Esscher measures. The Esscher measure that is considered in Chan (1999) is computed w.r.t. \( X = \log(S) \), whereas Henderson and Hobson (2003) derive the Esscher measure w.r.t. \( \bar{X} = \log(S) \). The corresponding densities are of the form

\[ Z^\theta_t(X) = \exp\{\theta \cdot X_t - \tilde{K}^X(\theta)_t\}, \]
\[ Z^\theta_t(\bar{X}) = \exp\{\theta \cdot \bar{X}_t - \tilde{K}^\bar{X}(\theta)_t\}, \]

with \textit{Laplace cumulant processes} \( \tilde{K}^X(\theta)_t = \int^t_0 \tilde{\kappa}^X(\theta)_u du \) and \( \tilde{K}^\bar{X}(\theta)_t = \int^t_0 \tilde{\kappa}^\bar{X}(\theta)_u du \), where

\[ \tilde{\kappa}^X(\theta)(t) = \vartheta(t)b(t) + \frac{1}{2} \vartheta^2(t)\sigma^2(t) + \int_\mathbb{R} (e^{\vartheta(t)\phi(t,x)} - 1 - \vartheta(t)\phi(t,x)) K(t; dx) \]

and \( \vartheta : [0,T] \to \mathbb{R} \) are deterministic. \( \tilde{\kappa}^\bar{X}(\theta) \) is similarly defined w.r.t. the differential characteristics of \( \bar{X} \). We assume that \( \vartheta \cdot X \) and \( \theta \cdot \bar{X} \), respectively, are exponentially special s.th. the densities are well-defined. Additionally, we assume that \( Z^\theta_t(X) \) and \( Z^\theta_t(\bar{X}) \), respectively, are uniformly integrable. Kallsen and Shiryaev (2002) establish several criteria for uniform integrability of (stochastic) exponential local martingales.

We compute both versions of Esscher martingale measures and call the Esscher martingale measure w.r.t. \( X = \log(S) \) the \textit{minimal entropy martingale measure} \( Q^{\text{entr}} \), as this measure minimizes the relative entropy between the original measure \( P \) and the set of equivalent martingale measures, cp. Miyahara (1999), Chan (1999) and Henderson and Hobson (2003). By Jacod and Shiryaev (2003, Theorem III.7.28) the corresponding Girsanov parameters are given by \( \beta^{\text{entr}}(t) = \vartheta^*(t) \) and \( Y^{\text{entr}}(t,x) = e^{\vartheta^*(t)x} \), where \( \vartheta^*(t) \) is the solution of the martingale equation

\[ 0 = b^{\text{entr}}(t) = b(t) + \sigma^2(t)\vartheta(t) + \int_\mathbb{R} \phi(t,x)(e^{\vartheta(t)\phi(t,x)} - 1) K(t; dx), \]

for all \( t \in [0,T] \).
The Esscher measure w.r.t. $\bar{X} = \log(\mathcal{E}(X))$ is derived as follows. From Lemma A.1.1 we obtain $\bar{X} \sim (\bar{b}(t), \bar{c}(t), \bar{K}(t; \cdot))_{\text{id}}$, where

\[
\begin{align*}
\bar{b}(t) &= b(t) - \frac{\sigma^2(t)}{2} + \int (\log(1 + x) - x) K^{x \rightarrow \phi(t,x)}(t; dx), \\
\bar{c}(t) &= \sigma^2(t), \\
\bar{K}(t; dx) &= K^{x \rightarrow \log(1 + \phi(t,x))}(t; dx),
\end{align*}
\]

under the original measure $P$. Due to Jacod and Shiryaev (2003, Theorem III.7.18), $e^X$ is a local martingale under the corresponding measure $Q^\theta$, iff the martingale equation

\[
b(t) + \theta(t)\sigma^2(t) + \int \phi(t, x)((1 + \phi(t, x))^{\theta(t)} - 1) K(t; dx) = 0 \quad (3.13)
\]

is satisfied. The Girsanov parameters in this case are given by $Y^{\text{Esscher}}(t, x) = (1 + x)^{\theta^*(t)}$ and $\beta^{\text{Esscher}}(t) = \theta^*(t)$, where $\theta^*$ solves the martingale equation (3.13).

As first example for orderings of European option prices w.r.t. the different martingale measures given above we consider the Lévy driven diffusion model $S$ in (3.10) of Chan (1999) with coefficients

\[
\Sigma \equiv \gamma \equiv \sigma \equiv c = 1, \quad a = 0, \quad \text{and} \quad F(dx) = \delta_{\{-\frac{1}{2}\}}(dx) + \delta_{\{\frac{1}{2}\}}(dx). \quad (3.14)
\]

In this case $X = \log(S) \sim (1, 1, \delta_{\{-\frac{1}{2}\}}(dx) + \delta_{\{\frac{1}{2}\}}(dx))_{\text{id}}$ is a Lévy process under the original measure $P$. The option prices given in Chan (1999, Table 1) give rise to the assumption that the minimal martingale measure yields smaller prices than the minimal entropy martingale measure that in turn gives smaller prices than the Merton measure. The first ordering is demonstrable by our techniques, whereas the second ordering is not, cp. Example 3 on page 77. By $S^{\min}$ and $S^{\text{entr}}$ we denote $S$ w.r.t. the minimal martingale measure and the minimal entropy martingale measure, respectively, similar for the other martingale measures that occur in the sequel.

**Theorem 3.2.1.** Let $g \in \mathcal{F}_{\infty}$ and let the coefficients of the Lévy driven SDE (3.10) be given by (3.14). Then

\[
E g(S_T^{\text{min}}) \leq E g(S_T^{\text{entr}}).
\]

**Proof.** In this model the Girsanov parameter $Y^{\min}$ of the minimal martingale measure is time-independent and given by $Y^{\min}(x) = 1 - \frac{2}{5}x$. To compute the Girsanov parameter $Y^{\text{entr}}$ corresponding to the minimal entropy martingale measure we first solve numerically the martingale equation for $\theta$ that is of the form

\[
1 + \vartheta = \sinh \left( -\frac{\vartheta}{2} \right).
\]
The solution is approximately given by \( \vartheta \approx -0.6626 \). Thus
\[
Y^{\min}(-\frac{1}{2}) = \frac{4}{3} < 1.3928 \approx Y^{\text{entr}}(-\frac{1}{2}),
\]
\[
Y^{\min}(\frac{1}{2}) = \frac{2}{3} < 0.7180 \approx Y^{\text{entr}}(\frac{1}{2}).
\]
As the Lévy measure of \( X = \log(S) \) is given by \( F(dx) = \delta_{\{-\frac{1}{2}\}}(dx) + \delta_{\{\frac{1}{2}\}}(dx) \) and \( Q^{\min} \) and \( Q^{\text{entr}} \) are martingale measures, the result follows from Theorem 1.1.3 and Lemma 1.2.11.

**Remark 3.2.2.** The ordering results of Chapter 2 do not apply in the previous theorem. The natural approach is to compute the characteristics of \( X \) under the martingale measures \( Q^{\min} \) and \( Q^{\text{entr}} \) from the characteristics of \( X \) under these measures, then establish ordering \( X^{\min} \leq_{\text{lex}} X^{\text{entr}} \) and then use Remark 2.2.16 to obtain \( S_T^{\min} \leq_{\text{lex}} S_T^{\text{entr}} \). The drift and jump characteristics of \( X^{\min} \) and \( X^{\text{entr}} \) are approximately given by
\[
\bar{b}^{\min}(0) = -\frac{1}{2} - \int x\left(\frac{4}{3}\delta_{\{-\frac{1}{2}\}}(dx) + \frac{2}{3}\delta_{\{\frac{1}{2}\}}(dx)\right),
\]
\[
\bar{b}^{\text{entr}}(0) = -\frac{1}{2} - \int x\left(1.3928\delta_{\{-\frac{1}{2}\}}(dx) + 0.7180\delta_{\{\frac{1}{2}\}}(dx)\right),
\]
\[
\bar{F}^{\min}(dx) = \frac{4}{3}\delta_{\{\log(\frac{1}{2})\}}(dx) + \frac{2}{3}\delta_{\{\log(\frac{1}{2})\}}(dx),
\]
\[
\bar{F}^{\text{entr}}(dx) = 1.3928\delta_{\{\log(\frac{1}{2})\}}(dx) + 0.7180\delta_{\{\log(\frac{1}{2})\}}(dx).
\]
We obtain \( \int x\bar{F}^{\min}(dx) \approx -0.6539 > -0.6743 \approx \int x\bar{F}^{\text{entr}} \) and \( \bar{b}^{\min}(0) \approx -0.1667 < -0.1626 \approx \bar{b}^{\text{entr}} \).

A similar result as in the previous remark also holds true for comparison of the minimal entropy martingale measure and the Merton measure in this model. We give an example to show that in this case also the approach of Chapter 1 does not yield the ordering \( S_T^{\text{entr}} \leq_{\text{lex}} S_T^{\text{Merton}} \). Hence the theoretical verification of the numerical result of Chan (1999) for \( Q^{\text{entr}} \) and \( Q^{\text{Merton}} \) remains open.

**Example 3.** For \( D_+ > 0 \) and \( D_- < 0 \) let \( g(x) := (x - D_+)^+ + (D_- - x)^+ \). \( g \) is convex, \( g(0) = 0 \), and \( g \) is decreasing on the negative half axis and increasing on the positive half axis. For \( D_+ > \frac{1}{2} \) and \( D_- \in (-\frac{1}{2}, 0) \) it holds true that
\[
\int g(x)Y^{\text{entr}}(x)\left(\delta_{\{-\frac{1}{2}\}}(dx) + \delta_{\{\frac{1}{2}\}}(dx)\right)
\]
\[
\approx 0.7180\left(\frac{1}{2} - D_+\right)^+ + 1.3928(D_- + \frac{1}{2})^+ = 1.3928(D_- + \frac{1}{2})^+ \quad (3.15)
\]
\[
> (D_- + \frac{1}{2})^+ = \int g(x)Y^{\text{Merton}}(x)\left(\delta_{\{-\frac{1}{2}\}}(dx) + \delta_{\{\frac{1}{2}\}}(dx)\right),
\]
whereas for $D_+ \in (0, \frac{1}{2})$ and $D_- < -\frac{1}{2}$ the reversed inequality in (3.15) holds true. Hence, for $f(t, s, \cdot) = g(\cdot)$, for all $(t, s) \in [0, T] \times \mathbb{R}_+$, the ordering condition on the jump kernels in Theorem 1.1.3 (in combination with Remark 1.1.4) is not satisfied.

We obtain ordering of the Hellinger and the minimal entropy martingale measure in this model. For $X \sim (1, 1, \delta_{\frac{-1}{2}}(dx) + \delta_{\frac{1}{2}}(dx))$ (under the original measure $P$), the Hellinger measure $Q_{\text{Hellinger}}$ is the special case $p = -1$ of local martingale measures $Q^p$ that minimize the $f_p$-divergence for $f_p(x) := -\frac{p-1}{p} x^{\frac{p}{p-1}}$ and that have density process

$$
\frac{dQ^p}{dP} = \mathcal{E} \left( (p-1)\xi X^c + ((1 + \xi x)^{p-1} - 1) * (\mu^X - \nu^X) \right),
$$

hence $Y^p(x) = (1 + \xi x)^{p-1}$, with $\nu^X(du, dx) = F(dx)du$, $F(dx) = \delta_{\frac{-1}{2}}(dx) + \delta_{\frac{1}{2}}(dx)$, and where $\xi \in \mathbb{R}$ is the solution of the martingale equation

$$1 + (p-1)\xi + \frac{1}{2} \left( \left( 1 + \frac{\xi}{2} \right)^{p-1} - \left( 1 - \frac{\xi}{2} \right)^{p-1} \right) = 0, \quad (3.16)$$

cp. Goll and Rüschendorf (2001). The special case $Q^{-1}$ is the martingale measure $Q_{\text{Hellinger}}$ that minimizes the Hellinger distance (cp. Grandits (1999)).

**Theorem 3.2.3.** Let $g \in \mathcal{F}_{\text{ex}}$ and let the coefficients of the Lévy driven SDE (3.10) be given by (3.14). Then

$$
E g(S_T^{\text{entr}}) \leq E g(S_T^{\text{Hellinger}}).
$$

**Proof.** The solution to the martingale equation (3.16) for $p = -1$ is given by $\xi^* \approx 0.3273$. As $Y_{\text{Hellinger}}(x) = (1 + \xi^* x)^{-2}$, it follows that

$$
Y_{\text{Hellinger}}\left(-\frac{1}{2}\right) \approx 1.4295 > 1.3928 \approx Y_{\text{entr}}\left(-\frac{1}{2}\right), \\
Y_{\text{Hellinger}}\left(\frac{1}{2}\right) \approx 0.7385 > 0.7180 \approx Y_{\text{entr}}\left(\frac{1}{2}\right),
$$

and the result follows from Theorem 1.1.3 and Lemma 1.2.11.

\[\square\]

Next, we give the theoretical verification of the numerical result of Chan (1999, Table 2) for an exponential gamma model. This model is a specification of the Lévy driven diffusion (3.10) with coefficients

$$
\Sigma \equiv \gamma \equiv a = c = 1, \quad \text{and} \quad F(dx) = \frac{1}{x} e^{-x} \mathbb{1}_{[0, \infty)}(x) dx \quad (3.17)
$$
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under the original measure $P$, hence $X \sim (2, 1, \frac{1}{x}e^{-x}\mathbb{1}_{[0,\infty)}(x)dx) \text{id}$ under this measure. The numerical results of Chan (1999, Table 2) give rise to the assumption that the minimal entropy martingale measure yields smaller prices for European call options than the Merton measure. We derive the theoretical verification of this assertion for European options with convex payoff functions.

**Theorem 3.2.4.** Let $g \in \mathcal{F}_{cx}$ and let the coefficients of the Lévy driven diffusion with SDE (3.10) be given by (3.17). Then

$$Eg(S_T^{\text{entr}}) \leq Eg(S_T^{\text{Merton}}).$$

**Proof.** We recall that the Girsanov jump parameters in this stochastic exponential Lévy model are given by $Y^{\text{Merton}} \equiv 1$ and $Y^{\text{entr}}(x) = e^{\vartheta^*x}$, where $\vartheta^*$ is the solution of the corresponding martingale equation

$$0 = 2 + \vartheta + \int_{\mathbb{R}} (e^{\vartheta y} - 1)e^{-y}dy.$$

We assume $\vartheta < 1$ s.th. the integral is finite, which corresponds to the assumption that $\vartheta \cdot X$ is exponentially special. Then, $\vartheta^* = -\sqrt{2}$, hence $Y^{\text{entr}}(x) = e^{-\sqrt{2}x} \leq 1 = Y^{\text{Merton}}(x)$, for all $x \geq 0$, and the result follows from Theorem 1.1.3 and Lemma 1.2.11.

Next, we obtain an ordering result which is a slight generalization of the ordering results of the “deterministic case” in Henderson and Hobson (2003).

**Theorem 3.2.5.** For $g : \mathbb{R} \to \mathbb{R}_+$, $g \in \mathcal{F}_{cx}$, and the diffusion with jumps model given by (3.9) with $b(\cdot) > 0$ assume that the martingale measures that are given above exist and are in $\mathcal{M}_{e,\text{loc}}^X$. Additionally assume that $S$ satisfies the integrability and smoothness conditions of Theorem 1.1.3 under each of these measures.

1. If $\phi(t, x) > 0$, for all $(t, x) \in [0, T] \times \mathbb{R}$, then $Eg(S_T^{\min}) \leq Eg(S_T^{\text{Merton}})$,
   $Eg(S_T^{\text{entr}}) \leq Eg(S_T^{\text{Merton}})$ and $Eg(S_T^{\text{Esscher}}) \leq Eg(S_T^{\text{Merton}})$.

2. If $\phi(t, x) < 0$, for all $(t, x) \in [0, T] \times \mathbb{R}$, then $Eg(S_T^{\text{Merton}}) \leq Eg(S_T^{\min})$,
   $Eg(S_T^{\text{Merton}}) \leq Eg(S_T^{\text{entr}})$ and $Eg(S_T^{\text{Merton}}) \leq Eg(S_T^{\text{Esscher}})$.

3. If $\phi$ is constant, then $Eg(S_T^{\min}) \leq Eg(S_T^{\text{entr}}) \leq Eg(S_T^{\text{Esscher}})$.

**Proof.** We consider the first case. By $\phi(t, x) > 0$ it holds that $Y^{\text{min}}(t, \phi(t, x)) = \phi(t, x)\eta^*(t) + 1 < 1$, as $b(\cdot) > 0$ implies $\eta^*(\cdot) < 0$. Hence for $f : [0, T] \times
$\mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$, $f(t, s, \cdot) \in \mathcal{F}_{cx}$, it follows that

$$
\int f(t, S_{t-}, x) Y^\text{min}(t, x) K^{x-\phi(t, x)}(t, dx)
= \int f(t, S_{t-}, \phi(t, x)) Y^\text{min}(t, \phi(t, x)) K(t, dx)
\leq \int f(t, S_{t-}, \phi(t, x)) K(t, dx)
= \int f(t, S_{t-}, x) Y^{\text{Merton}}(t, x) K^{x-\phi(t, x)}(t, dx),
$$

(3.18)

thus Theorem 1.1.3 and Lemma 1.2.11 imply $Eg(S_T^\text{min}) \leq Eg(S_T^{\text{Merton}})$. Henderson and Hobson (2003) establish that also the orderings $Y^{\text{entr}}(t, \phi(t, x)) \leq Y^{\text{Merton}}(t, \phi(t, x))$ and $Y^{\text{Esscher}}(t, \phi(t, x)) \leq Y^{\text{Merton}}(t, \phi(t, x))$ hold, which implies similar ordering as in (3.18), hence the other orderings in 1 follow from Theorem 1.1.3 and Lemma 1.2.11. If $\phi(t, x) < 0$ the inequality in (3.18) is reversed and the results of 2 follow. For constant $\phi(t, x) \equiv \phi$ Henderson and Hobson (2003) obtain ordering $Y^\text{min}(t, \phi) \leq Y^{\text{entr}}(t, \phi) \leq Y^{\text{Esscher}}(t, \phi)$, which implies ordering in 3 by Theorem 1.1.3 and Lemma 1.2.11.

Møller (2004) studies a compound Poisson model $S \sim (b(0), 0, F)_0$ under the original measure $P$, where the Lévy measure $F$ satisfies $\text{supp}(F) \subset \mathbb{R}_+$, has finite total mass $\lambda$, and where $-b(0) > \int x F(dx)$. The corresponding martingale equation is of the form

$$
\int_{(0, \infty)} y Y(y) F(dy) = -b(0),
$$

(3.19)

for a deterministic Girsanov jump parameter $Y : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Besides the minimal martingale measure and the minimal entropy martingale measure, Møller (2004) derives three other martingale measures $Q^{(i)}$, $i \in \{1, 2, 3\}$, for this model, under which the Poisson intensity $\lambda$ or, alternatively, the claim size distribution $R(dx) = \frac{1}{\lambda} F(dx)$ changes. These measures are introduced in Delbaen and Haezendonck (1989). The corresponding Girsanov jump parameters are listed in the following table.
3.2. Comparison of martingale measures

<table>
<thead>
<tr>
<th>mg meas</th>
<th>Girsanov jump parameter $Y^{(i)}(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q^{(1)}$</td>
<td>$-\frac{b(0)}{\int yF(dy)} \in (0,1)$</td>
</tr>
<tr>
<td>$Q^{(2)}$</td>
<td>$1 + \theta^* \left( y - \int xR(dx) \right)$, where $\theta^* = -\frac{b(0) + \int yF(dy)}{\int y^2F(dy) - \lambda^*(\int yF(dy))^2}$</td>
</tr>
<tr>
<td>$Q^{(3)}$</td>
<td>$e^{\rho^* y} \left( \int e^{\rho^* x} R(dx) \right)^{-1}$, where $\rho^<em>$ solves $\int ye^{\rho^</em> y} R(dy) = b(0)$</td>
</tr>
<tr>
<td>$Q^{\min}$</td>
<td>$1 - \alpha^* y$, where $\alpha^* = \frac{b(0) + \int xF(dx)}{\int x^2F(dx)}$</td>
</tr>
<tr>
<td>$Q^{\entr}$</td>
<td>$e^{\eta^* y}$, where $\eta^<em>$ solves $\int ye^{\eta^</em> y} F(dy) = -b(0)$</td>
</tr>
</tbody>
</table>

From the assumption $-b(0) > \int xF(dx)$ it follows that $\alpha^* < 0$, hence the minimal martingale measure $Q^{\min}$ exists. If $b(0) > \frac{\int x^2F(dx)}{\int xR(dx)}$, then the martingale measure $Q^{(2)}$ also exists, as $Y^{(2)}(y) > 0$ holds true under this drift condition.

In the compound Poisson model $S \sim (b(0),0,F)_0$, supp($F$) $\subset \mathbb{R}_+$, Møller (2004) derives ordering of option prices that are computed w.r.t. the martingale measures of the previous table. We obtain the corresponding ordering result for the finite-dimensional distributions of $S$ under these measures. By $S^{(i)}$ we denote $S$ w.r.t. the martingale measure $Q^{(i)}$, $i = 1,2,3$.

**Theorem 3.2.6.** Under the original measure $P$ let $S \sim (b(0),0,F)_0$, with supp($F$) $\subset \mathbb{R}_+$, be a one-dimensional compound Poisson model and assume that $-b(0) > \int xF(dx)$. Then

$$(S_t^{(1)}) \leq_{cx} (S_t^{\min}) \leq_{cx} (S_t^{\entr}) \leq_{cx} (S_t^{(3)}).$$

If, additionally, $Q^{(2)}$ is well-defined, then

$$(S_t^{(1)}) \leq_{cx} (S_t^{\min}) \leq_{cx} (S_t^{(2)}) \leq_{cx} (S_t^{(3)}).$$

**Proof.** In Møller (2004, Theorem 6.4) it is established that the ordering conditions of the second part of Theorem 2.1.5 are satisfied, cp. also Remark 2.1.6.

An extension of the previous result for processes that have independent increments under $P$ and that additionally have a continuous martingale part holds true. The corresponding Girsanov parameters are deterministic, hence $S$ is also a PII under the considered martingale measures.
Theorem 3.2.7. Let $g \in \mathcal{F}_{cx}$. Under $P$ let $S \sim (b(t; 0), c(t), K(t; \cdot))_0$, be a one-dimensional PII with a finite deterministic Lévy kernel $K$ that satisfies $\text{supp}(K(t, \cdot)) \subset \mathbb{R}_+$ for all $t \in [0, T]$. Assume that $-b(t, 0) > \int xK(t; dx)$, for all $t \in [0, T]$ and assume that the conditions of Theorem 1.1.7 are satisfied under the minimal martingale measure and the minimal entropy martingale measure. Then it holds true that

$$Eg(S^\text{min}_T) \leq Eg(S^\text{entr}_T).$$

Proof. We make use of an extension of the cut criteria in Theorems 2.1.4 and 2.1.5 to the PII case and then use Theorem 1.1.7 and Lemma 1.2.11. The Girsanov parameters of $Q^{\text{min}}$ and $Q^{\text{entr}}$ are given by $Y^{\text{min}}(t, y) = 1 - \alpha^*(t)y$ and $Y^{\text{entr}}(t, y) = e^{\eta^*(t)y}$, respectively, where $\alpha^*$ and $\eta^*$ correspond to a martingale equation for the PII case with continuous martingale part, similarly to the previous compound Poisson case. Møller (2004) establishes that $0 < \eta^*(t) < -\alpha^*(t)$, hence there is a $k_r(t)$ with $Y^{\text{entr}}(t, y) \leq Y^{\text{min}}(t, y)$ for all $y < k_r$ and $Y^{\text{entr}}(t, y) \geq Y^{\text{min}}(t, y)$ for all $y > k_r$. If $K^{\text{min}}(t, \mathbb{R}_+) \leq K^{\text{entr}}(t, \mathbb{R}_+)$, then an extension of Theorem 2.1.4 to the PII case implies that the corresponding modified Lévy kernels are ordered as $\tilde{K}^{\text{min}}(t, \cdot) \leq_{\text{icx}} \tilde{K}^{\text{entr}}(t, \cdot)$. If $K^{\text{min}}(t, \mathbb{R}_+) \leq K^{\text{entr}}(t, \mathbb{R}_+)$, then an extension of Theorem 2.1.5 to the PII case implies this result. As $b^{\text{min}}(t, \text{id}) = b^{\text{entr}}(t, \text{id}) = 0$ and $c^{\text{min}}(t) = c^{\text{entr}}(t)$, for all $t \in [0, T]$, the result follows from Theorem 1.1.7, Remark 1.1.8 and Lemma 1.2.11.

Remark 3.2.8. 1. This result is proved in the appendix of Møller (2004) by a different approach.

2. For $c = 0$ the assertion of the previous theorem as well as the corresponding orderings under $Q^{(i)}$, $i = 1, 2, 3$, follow directly from an extension of Theorem 2.1.5 to the PII case, as in this case it holds true that $\int xK^{\text{min}}(t; dx) = \int xK^{\text{entr}}(t; dx)$. For $c > 0$, Møller (2004) shows that $\int xK^{\text{min}}(t; dx) < \int xK^{\text{entr}}(t; dx)$, hence $b^{\text{min}}(t; 0) > b^{\text{entr}}(t; 0)$ due to the martingale property of $S$ under $Q^{\text{min}}$ and $Q^{\text{entr}}$, respectively. In this case the results of Section 2.3 do not apply directly.

### 3.2.2 Stochastic volatility models

Henderson (2005) and Henderson, Hobson, Howison, and Kluge (2003) establish ordering of option prices w.r.t. $q$-optimal martingale measures in a stochastic volatility model $(S, V)$ with $P$-dynamics

$$\frac{dS_t}{S_t} = V_t(\mu(t, V_t)dt + dB_t),$$

$$dV_t = \eta(t, V_t)dt + \sigma(t, V_t)dW_t,$$

(3.20)
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where $B$ and $W$ are Brownian motions under the original measure $P$ that have constant correlation $\rho = \frac{1}{t} \langle B, W \rangle_t$, the volatility process $V$ of the underlying $S$ is assumed to have a positive solution, and $\sigma : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$. The $q$-optimal martingale measure is the martingale measure that minimizes the distance $H_q(P, Q)$ over all equivalent martingale measures $Q$, where for $Z_T = \frac{dQ}{dP}|_{A^T}$ the distance $H_q(P, Q)$ is defined by

$$H_q(P, Q) = \begin{cases} E\left(\frac{q^q}{q-1} Z_T^q\right), & q \in \mathbb{R}\{0, 1\}.
\end{cases}$$

This distance is a special case of $f$-divergences, cp. Liese and Vajda (1987). Hobson (2004) computes the $q$-optimal martingale measures in the model (3.20). The dynamics of $(S, V)$ under the $q$-optimal martingale measure $Q^q$ are given by the system

$$\frac{dS_t^q}{S_t^q} = V_t^q dB_t^q,$n$$

$$dV_t^q = \eta^q(t, V_t^q) dt + \sigma(t, V_t^q) dW_t^q,$n$$

where $B_t^q, W_t^q$ are $Q^q$-Brownian motions with correlation $\rho = \frac{1}{t} \langle B_t^q, W_t^q \rangle_t$, and $\eta^q(t, v) = \eta(t, v) - \rho \mu(t, v) \sigma(t, v) - (1 - \rho^2) \sigma_2(t, v) \beta^q(t, v)$, where $\beta^q(t, v)$ is the Girsanov parameter that corresponds to the $q$-optimal measure. The differential semimartingale characteristics of the two-dimensional Markov process $(S^q, V^q)$ are given by $b_t^q(\omega) = b^q(t, (S_t^q(\omega), V_t^q(\omega)))$ and $c_t^q(\omega) = c^q(t, (S_t^q(\omega), V_t^q(\omega)))$, with

$$b^q(t, (s, v)) = \begin{pmatrix} 0 & \eta^q(t, v) \\
\end{pmatrix}, \quad c^q(t, (s, v)) = \begin{pmatrix} s^2 v^2 & \rho s v \sigma(t, v) \\
\rho s v \sigma(t, v) & \sigma^2(t, v) \\
\end{pmatrix},$$

and it turns out that the diffusion coefficient function is independent of $q$.

For $\rho = 0$, Henderson (2005) uses coupling arguments to derive orderings of European option prices with convex payoffs w.r.t. the parameter $q$. For general $\rho \in [-1, 1]$, Henderson, Hobson, Howison, and Kluge (2003) establish similar comparison results by a PDE argument. In both papers it is first established that European option prices are ordered w.r.t. drift coefficient functions $\eta^q$ of the volatility processes. Then it is shown that, depending on the monotonicity of the drift coefficient function $\mu$ of $S$, the drift parameters $\eta^q$ are ordered in $q$.

Our typical approach for establishing convex ordering of $S^q_T$ in $q$ fails in this situation. As $S^q(i), i = 1, 2$, are computed with respect to equivalent
martingale measures, the diffusion coefficient functions under different $Q^{(i)}$ are identical, and, as
\[ c^{(1)}_t(\omega) = c(t, (S^{(1)}_t(\omega), V^{(1)}_t(\omega))) = c^{(2)}(t, (S^{(1)}_t(\omega), V^{(1)}_t(\omega))), \]
the required ordering of the diffusion coefficients in Theorem 1.1.7 is not satisfied. Therefore, we follow a different approach that is parallel to the derivation of the results of Section 1.1, cp. also Henderson, Hobson, Howison, and Kluge (2003). In the model (3.20), the price of a European option with payoff function $g : \mathbb{R}_+ \to \mathbb{R}_+$ w.r.t. the $q^{(i)}$-optimal measure at time $t$ is given by
\[ G^{q(i)}_t(t, (s, v)) = E\left(g(S^{q(i)}_T) \mid (S^{q(i)}_t, V^{q(i)}_t) = (s, v)\right), \]
\[ G^{q(i)}_t(T, s, v) = g(s). \]

**Theorem 3.2.9.** Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ and let $(S^{q(i)}, V^{q(i)}), i = 1, 2,$ be solutions of the system (3.21). Assume that $(S^{q(i)}, V^{q(i)}), i = 1, 2,$ satisfy Assumptions MG and BIC$(g)$ and that $(S^{q(2)}, V^{q(2)})$ satisfies Assumption SC$(g)$. If
\[ \eta^{(1)}(t, v) \leq \eta^{(2)}(t, v), \]
for all $(t, v) \in [0, T] \times \mathbb{R}_+$, then $Eg(S^{q(1)}_T) \leq Eg(S^{q(2)}_T)$.

**Proof.** We sketch the proof which is similar to the proofs of the theorems in Section 1.1. For notational convenience we substitute superscript $q^{(i)}$ by superscript $(i)$ in this proof, e.g. $(S^{(i)}, V^{(i)} := (S^{q(i)}, V^{q(i)})$. We establish that $\mathcal{G}^{(2)}(t, (S^{(1)}_t, V^{(1)}_t))$ is a supermartingale under $Q^{(2)}$, then
\[ Eg(S^{(1)}_T) = Eg^{(2)}(T, (S^{(1)}_T, V^{(1)}_T)) \leq \mathcal{G}^{(2)}(0, (S^{(1)}_0, V^{(1)}_0)) = Eg(S^{(2)}_T). \]
$\mathcal{G}^{(2)}(t, (s, v))$ satisfies the Kolmogorov backward equation
\[ D_t\mathcal{G}^{(2)}(t, (s, v)) + \eta^{(2)}(t, v)D_v\mathcal{G}^{(2)}(t, (s, v)) + \frac{1}{2}\sigma^2(t, v)D_{vv}\mathcal{G}^{(2)}(t, (s, v)) \]
\[ + \frac{1}{2}sv^2D_{sv}\mathcal{G}^{(2)}(t, (s, v)) + \rho sv\sigma(t, v)D_{sv}\mathcal{G}^{(2)}(t, (s, v)) = 0, \]
hence
\[ \mathcal{G}^{(2)}(t, (S^{(1)}_t, V^{(1)}_t)) = \mathcal{G}^{(2)}(0, (S^{(1)}_0, V^{(1)}_0)) + M_t \]
\[ + \int \left( \eta^{(1)}(u, V^{(1)}_u) - \eta^{(2)}(u, V^{(1)}_u) \right) D_v\mathcal{G}^{(2)}(u, (S^{(1)}_u, V^{(1)}_u)) du, \]
where $M_t$ is a local martingale under $Q^{(2)}$. Romano and Touzi (1997, Theorem 3.1) establish $D_v\mathcal{G}^{(2)}(t, (s, v)) \geq 0$, hence it follows from $\eta^{(1)}(t, v) \leq \eta^{(2)}(t, v)$. 

$\eta^{(2)}(t,v)$ that $G^{(2)}(t, (S^{(1)}_t, V^{(1)}_t))$ is a (local) supermartingale. To establish that it is a true supermartingale is similarly done as in the proof of Theorem 1.1.3.

For $\rho = 0$, Henderson (2005, Theorem 4.2) establishes that $\beta^q(t,v)$ is increasing in $q$ iff $\mu^2(t,v)$ is increasing in $v$. Together with Theorem 3.2.9 this implies Theorem 4.4 of that paper.

**Corollary 3.2.10.** Let the SV model (3.20) with $\rho = 0$ be given and let the Assumptions of the previous theorem be satisfied.

1. If $\mu^2(t, \cdot)$ is increasing for all $t \in [0,T]$, then for $q^{(1)} \leq q^{(2)}$ it follows that
   \[ E_g(S^{q^{(1)}}_T) \geq E_g(S^{q^{(2)}}_T) \]

2. If $\mu^2(t, \cdot)$ is decreasing for all $t \in [0,T]$, then for $q^{(1)} \leq q^{(2)}$ it follows that
   \[ E_g(S^{q^{(1)}}_T) \leq E_g(S^{q^{(2)}}_T) \]

3. If $\mu(t, \cdot)$ is constant for all $t \in [0,T]$, then the $q$-optimal prices are identical for all $q \in \mathbb{R}$.

**Remark 3.2.11.** As in this model the minimal martingale measure corresponds to the parameter $q = 0$, the minimal entropy martingale measure corresponds to $q = 1$ and the variance-optimal martingale measure $Q^{\text{var}}$ to $q = 2$, the previous corollary yields

\[ S_T^{\text{min}} \geq_{\text{cx}} S_T^{\text{entr}} \geq_{\text{cx}} S_T^{\text{var}}, \]

if $\mu^2(t, \cdot)$ is increasing for all $t \in [0,T]$. If $\mu^2(t, \cdot)$ is decreasing for all $t \in [0,T]$, the inequalities are reversed, cp. Henderson (2005, Theorem 4.4).

For general $\rho$, Henderson, Hobson, Howison, and Kluge (2003) obtain ordering results that are less explicit. In Theorem 2 they establish that in this model $q\mu^2(t, \cdot)$ is increasing iff $\beta^q \geq 0$. As also in this model the minimal martingale measure corresponds to $q = 0$, this implies the following result, cp. Henderson, Hobson, Howison, and Kluge (2003, Corollary 3).

**Corollary 3.2.12.** Let the conditions of Theorem 3.2.9 be satisfied.

1. If $\mu^2(t, \cdot)$ is increasing for all $t \in [0,T]$, then $E_g(S^q_T) \leq E_g(S^{\text{min}}_T)$ for all $q > 0$ and $E_g(S^q_T) \geq E_g(S^{\text{min}}_T)$ for all $q < 0$.

2. If $\mu^2(t, \cdot)$ is decreasing for all $t \in [0,T]$, then $E_g(S^q_T) \geq E_g(S^{\text{min}}_T)$ for all $q > 0$ and $E_g(S^q_T) \leq E_g(S^{\text{min}}_T)$ for all $q < 0$. 
3.3 Comparison of path-dependent options

The ordering results of Chapters 1 and 2 are used in Sections 3.1 and 3.2 to obtain nontrivial bounds and orderings for martingale measures, respectively, in incomplete market models. In this section we establish extension of the results to several option types with path-dependent payoffs. In Subsection 3.3.1 we derive orderings for lookback options, whose payoffs depend on the running maximum of the underlying process. Subsection 3.3.2 is concerned with ordering results for Asian options with continuous averaging. In Subsection 3.3.3 we extend the results of Chapter 1 to American options, and finally in Subsection 3.3.4 we derive orderings for single-barrier options without rebate. References on the literature on orderings of path-dependent option prices are given in the corresponding subsections. The main result of Subsection 3.3.1 is model independent. In Subsection 3.3.2 we assume that the drift of the underlying model under a martingale measure equals zero, whereas we consider models that have non-zero drift components under a martingale measure in Subsections 3.3.3 and 3.3.4.

3.3.1 Lookback options

In this subsection we consider lookback options with continuous averaging that have path-dependent payoffs of the form

\[ g(\max_{t \leq T}(S_t)). \]

There are few results on orderings of lookback options in the literature. Hobson (1998a) derives model independent upper and lower bounds for the price of a lookback option with payoff \( L_T = \max_{t \leq T} S_t \), where \( S \) is some adapted process. The universal upper bound is then given by the first moment of the Hardy-Littlewood transform of a probability measure with unit mean that is in one-to-one correspondence to the price of the corresponding European option in some sense, cp. Hobson (1998a, Lemma 2.3). The universal lower bound is given directly in terms of the European call option. Additionally, Hobson (1998a) gives trading strategies under which these bounds are attained.

We establish that finite-dimensional ordering \((S_t^{(1)}) \leq_{\text{lex}} (S_t^{(2)})\) implies corresponding ordering of the maxima \(\max_{t \leq T}(S_t^{(1)}) \leq_{\text{lex}} \max_{t \leq T}(S_t^{(2)})\), which follows from Franken and Kirstein (1977, Satz 3.1). As corollary we obtain ordering of lookback options that have increasing convex payoff functions in exponential Lévy models.

**Theorem 3.3.1.** Let \(\{S_t^{(i)}\}_{t \in [0,T]}, \ i = 1, 2,\) be one-dimensional càdlàg processes with \(S_t^{(i)} \geq 0\) and \(E \sup_{t \leq T}(S_t^{(i)}) < \infty\). If \((S_t^{(1)}) \leq_{\text{lex}} (S_t^{(2)})\), then

\[
\max_{t \leq T}(S_t^{(1)}) \leq_{\text{lex}} \max_{t \leq T}(S_t^{(2)}).
\]

**Proof.** For \(m \in \mathbb{N}\) and \(0 \leq t_1 < \cdots < t_m \leq T\) the ordering \((S_t^{(1)}) \leq_{\text{lex}} (S_t^{(2)})\) implies \((S_{t_1}^{(1)}, \ldots, S_{t_m}^{(1)}) \leq_{\text{lex}} (S_{t_1}^{(2)}, \ldots, S_{t_m}^{(2)})\). For fixed \(\omega\) let \(S_n^{(i)}(\omega)\) be step functions given by

\[
S_n^{(i)}(\omega) := \begin{cases} 
S_{t_{n,j}}^{(i)}(\omega), & \text{if } t \leq t_{n,j+1}, \ j < k_n, \\
S_T(\omega), & \text{if } t = t_{n,k_n} = T.
\end{cases}
\]

The claim follows from Franken and Kirstein (1977, Satz 3.1), if

\[
\max_{t \leq T}(S_n^{(i)}) \xrightarrow{\mathcal{L}} \max_{t \leq T}(S_t^{(i)}), \quad \text{and} \quad E \max_{t \leq T}(S_n^{(i)}) \to E \max_{t \leq T}(S_t^{(i)}), \quad n \to \infty.
\]

(3.22)

(3.23)

The functional \(\max : \mathbb{D}_{[0,T]} \to \mathbb{R}^+\) is continuous, thus for (3.22) it suffices to show \(S_n^{(i)} \xrightarrow{\mathcal{L}} S^{(i)}\), which follows from construction. As \(\max_{t \leq T}(S_n^{(i)}) \leq \max_{t \leq T}(S_t^{(i)})\)

\[
\int_{\{\max(S_n^{(i)}(\omega)) > a\}} \max_{t \leq T}(S_n^{(i)}) \, dP \leq \int_{\{\max(S_t^{(i)}(\omega)) > a\}} \max_{t \leq T}(S_t^{(i)}) \, dP,
\]

for all \(n \in \mathbb{N}\), the sequences \(S_n^{(i)}\) are uniformly integrable and (3.23) follows.

Theorem 3.3.1 implies ordering for prices of lookback options in exponential Lévy models.

**Corollary 3.3.2 (Ordering of lookback options in exponential Lévy models).** Let \(X^{(i)}_0, X_t^{(i)} = 0, i = 1, 2,\) be Lévy processes with \(E \sup_{t \leq T}(e^{X_t^{(i)}}) < \infty\). If \((\bar{X}_t^{(1)}) \leq_{\text{lex}} (\bar{X}_t^{(2)})\), then

\[
\max_{t \leq T}(S_t^{(1)}) \leq_{\text{lex}} \max_{t \leq T}(S_t^{(2)}).
\]
Proof. This follows from the previous theorem and from \( \exp \in \mathcal{F}_{\text{cx}} \) and 
\[
\exp(\max_{t \leq T}(\bar{X}_t^{(i)})) = \max_{t \leq T}(\exp(\bar{X}_t^{(i)})).
\]
\( \square \)

**Remark 3.3.3.** Various criteria that yield finite-dimensional ordering 
\((\bar{X}_{t}^{(1)}) \leq_{\text{cx}} (\bar{X}_{t}^{(2)})\) are given in Chapters 1 and 2.

### 3.3.2 Asian options

In this subsection we derive orderings for Asian options with continuous 
averaging on univariate and multivariate underlyings \( S \). This type of Asian 
options with averaging time interval \([T - \theta, T]\), \( \theta \in (0, T] \), yields a terminal 
payoff

\[
g\left(\frac{1}{\theta} \int_{T-\theta}^{T} S_u du\right),
\]

for a payoff function \( g \in \mathcal{F}_{\text{cx}} \). We assume that \( S \) is the discounted value of 
an underlying w.r.t. a martingale measure.

There are few ordering results for prices of Asian options with continuous 
averaging in the literature. El Karoui, Jeanblanc-Picqué, and Shreve (1998) establish that Asian option prices in a univariate diffusion model are 
bounded below by the corresponding European option prices. If 
\( S_t \leq_{\text{cx}} S_T \), for all \( t \leq T \), this is a consequence of Jensen’s inequality, which for 
\( g \in \mathcal{F}_{\text{cx}} \) implies

\[
E g\left(\frac{1}{\theta} \int_{T-\theta}^{T} S_u du\right) \leq \frac{1}{\theta} \int_{T-\theta}^{T} E g(S_u) du \leq E g(S_T).
\]

Bellamy and Jeanblanc (2000) establish that the lower bound of an Asian 
option price in a univariate diffusion with jumps model is given by the 
Asian option price under the corresponding Black-Scholes model with the 
same diffusion coefficient.

We derive two different ordering results for Asian options with continuous 
averaging. At first we establish that Asian option prices are decreasing in 
the length of the averaging interval \([T - \theta, T] \). The highest price is given for 
\( \theta \to 0 \), i.e. for European type options, cp. (3.24). In the proof we make use 
of the value process of the average with averaging interval \([T - \theta, T] \),

\[
A_t^\theta := E \left( \frac{1}{\theta} \int_{T-\theta}^{T} S_u du | A_t \right), \quad 0 \leq t \leq T,
\]

which also appears in El Karoui, Jeanblanc-Picqué, and Shreve (1998).

Next, we establish that Asian option prices are ordered, if the characteristics 
of the underlying \( S \) are ordered in an appropriate sense. For notational 
convenience we obtain the results for \( \theta = T \), i.e. the averaging interval is
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[0, T]. We give two different versions of the proof, both are parallel to the proofs of the ordering results of Section 1.1. The first approach uses the value process \( A^T \), the second approach is derived in terms of the averaging process

\[
A_t := \int_0^t S_u \, du. \tag{3.26}
\]

Remark 3.3.4. Another approach to tackle ordering of Asian options is similar to the proof of the ordering result of lookback options. If the finite-dimensional distributions of two underlying processes \( S_i(t) \), \( i = 1, 2 \), are ordered, \( (S_i(t)) \leq_{cx} (S_j(t)) \), then, under appropriate integrability conditions, it follows from Franken and Kirstein (1977, Satz 3.1), that

\[
\int_0^T S_i(t) \, dt \leq \int_0^T S_j(t) \, dt,
\]

as the integral is a linear functional. This approach especially is useful in Lévy models, due to the ordering criteria of Chapter 2.

At first, we compute the characteristics of the value process (3.25).

**Lemma 3.3.5 (Characteristics of the value process \( A_\theta^\theta \)).** Let \( S_t \sim (0, c^S_t, K^S_t) \) be a martingale. Then the value process \( A_\theta^\theta \sim (b_\theta^\theta, c_\theta^\theta, K_\theta^\theta) \) in (3.25) has characteristics

\[
b_\theta^\theta = 0,
\]

\[
c_\theta^\theta = c^S_t \left( I_{[0,T-\theta]}(t) + \left( \frac{T-t}{\theta} \right)^2 I_{[T-\theta,T]}(t) \right),
\]

\[
K_\theta^\theta(G) = \int_{\mathbb{R}^d} I_G(\alpha(t)y) K^S_t(dy), \quad G \in \mathcal{B}^d,
\]

where \( \alpha(t) = I_{[0,T-\theta]}(t) + \frac{T-t}{\theta} I_{[T-\theta,T]}(t) \).

**Proof.** The value process \( A_\theta^\theta \) is a martingale by definition, thus \( b_\theta^\theta = 0 \). From Fubini’s Theorem and the martingale property of \( S \) it follows that \( A_\theta^\theta \) has representation

\[
A_\theta^\theta = \frac{1}{\theta} \int_{T-\theta}^T E(S_u|A_t) \, du
\]

\[
= S_t I_{[0,T-\theta]}(t) + \frac{1}{\theta} \left( \int_{T-\theta}^t S_u \, du + (T-t)S_t \right) I_{[T-\theta,T]}(t).
\]

As \( \int_{T-\theta}^t S_u \, du \) is continuous and of finite variation it follows that the quadratic characteristic of \( A_\theta^\theta \) is given by \( \langle A_\theta^\theta \rangle_t = \langle S \rangle_t I_{[0,T-\theta]}(t) + \left( \frac{T-t}{\theta} \right)^2 \langle S \rangle_t I_{[T-\theta,T]}(t) \), hence the differential quadratic characteristic \( c_\theta^\theta \) is of the stated form. As the jumps of \( A^\theta \) are of the form \( \Delta A^\theta_t = \alpha(t) \Delta S_t = \Delta(\alpha \cdot S)_t \), it
follows that the jump compensator $\nu^{A^\theta}$ of $A^\theta_t$ is given by $\nu^{A^\theta}(\omega; [0, t] \times G) = \int_{[0,t] \times \mathbb{R}^d} \mathbb{I}_G(\alpha(u)y)\nu^{S}(\omega; du, dy)$, hence the differential jump characteristic $K^\theta_t$ is of the stated form. □

**Remark 3.3.6.** From the previous lemma it is seen that if $S$ is Markovian and its paths have dependent increments, then $A^\theta$ is not Markovian, as in this case $c^\theta$ and $K^\theta$ depend on $S_{t-}$, if the paths of $S$ do not have independent increments. Generally, there are two possibilities to circumvent this problem. If $S$ is assumed to be a PII, i.e. its characteristics are deterministic functions of time, then also $A^\theta$ is a PII. The second idea is to enlarge the space of underlyings by $A^\theta$, as $(S, A^\theta)$ is Markovian.

For the next result we assume that $S \sim (0, c^S(t), K^S(t, \cdot))$ is a $d$-dimensional PII martingale, hence the value process $A^T \sim (0, c^{AT}(t), K^{AT}(t, \cdot))$ of the Asian option with averaging interval $[0, T]$ and convex payoff function $g$ also is a PII martingale. If the backward function for the Asian option,

$$G^{AT}(t, a) = E\left(g(A^T_T) \mid A^T_t = a\right),$$

satisfies the Kolmogorov backward equation

$$D_tG^{AT}(t, a) + \frac{1}{2} \sum_{i,j \leq d} D^2_{ij}G^{AT}(t, a)c^{AT}(t) + \int (\Lambda G^{AT})(t, a, y)K^{AT}(t, dy) = 0,$$

where $(\Lambda G^{AT})(t, a, y) = G(t, a + y) - G(t, a) - \sum_{i \leq d} D_iG(t, a)y^i$, then we obtain the following ordering result.

**Theorem 3.3.7 (Ordering of Asian option prices in length of the averaging interval).** Let $g : \mathbb{R}^d \to \mathbb{R}$, $g \in \mathcal{F}_{cx}$, and assume that $S \sim (0, c^S(t), K^S(t, \cdot))_{1d}$ is a positive martingale with independent increments and Lévy kernels $K^S(t, \cdot)$ that satisfy $\int yK^S(t, dy) = 0$, for all $t \in [0, T]$. Assume that $G^{AT} \in C^{1,2}([0, T] \times \mathbb{R}^d)$ and that $G^{AT}(t, a)$ satisfies the Kolmogorov backward equation. Then for $\theta \in (0, T]$ it holds true that

$$Eg\left(\frac{1}{T} \int_0^T S_u du\right) \leq Eg\left(\frac{1}{\theta} \int_{T-\theta}^T S_u du\right).$$

**Proof.** Similar to the proofs of the theorems in Section 1.1 we establish that $G^{AT}(t, A^\theta_t)$ is a supermartingale. From Itô’s formula and the Kolmogorov backward equation it follows that $G^{AT}(t, A^\theta_t) = G^{AT}(t, A^\theta_0) + M_t + V_t^{AT}$, where $M_t$ is a local martingale and

$$V_t^{AT} = \int_0^t \left\{ \frac{1}{2} \sum_{i,j \leq d} D^2_{ij}G^{AT}(u, A^\theta_u)(c^{A^\theta}(u) - c^{AT}(u)) + \int_{\mathbb{R}^d} (\Lambda G^{AT})(u, A^\theta_{u-}, y)(K^{A^\theta}(u, dy) - K^{AT}(u, dy)) \right\} du.$$
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As $A^\theta$ and $A^T$ are PII, convexity of $g$ is propagated to $G^{AT}(t, \cdot)$, cp. Lemma 1.2.11, hence it remains to establish suitable ordering for the characteristics of $A^\theta$ and $A^T$. From Lemma 3.3.5 it follows that

$$c^{AT}(t) = \left(\frac{T-t}{T}\right)^2 c^{S}(t)$$

$$\leq_{psd} c^{S}(t)I_{[0,T-\theta]}(t) + \left(\frac{T-t}{\theta}\right)^2 c^{S}(t)I_{[T-\theta,T]}(t)$$

$$= c^{A^\theta}(t).$$

Lemma 3.3.5 also implies

$$\int f(t,y)K^{AT}(t,dy) = \int f(t,T-ty)K^{S}(t,dy)$$

and

$$\int f(t,y)K^{A^\theta}(t,dy) = \begin{cases} \int f(t,y)K^{S}(t,dy), \quad t < T - \theta, \\ \int f(t,T-ty)K^{S}(t,dy), \quad t \geq T - \theta. \end{cases}$$

As for a random vector $X$ with $EX = 0$ and for $\alpha \in [0,1]$ it holds true that $\alpha X \leq cx$, it follows from the assumption $\int yK^{S}(t,dy) = 0$ that

$$\int f(t,y)K^{AT}(t,dy) \leq \int f(t,y)K^{A^\theta}(t,dy), \quad \forall t \in [0,T],$$

for all $f : [0,t] \times \mathbb{R}^d \to \mathbb{R}$ s.th. $f(t, \cdot) \in \mathcal{F}_{cx}$. \hfill \Box

Next, we establish ordering of Asian options on underlyings $S^{(i)}$, $i = 1, 2$, with value processes

$$A^{(i)}_t := \frac{1}{T}E\left(\int_0^T S^{(i)}_u du \mid A_t\right)$$

(3.27)

in the characteristics of $S^{(i)}$, $i = 1, 2$. Again, we consider the multivariate PII case. The Kolmogorov backward equation for the backward function $G^{(2)}(t, a) = E(g(A^{(2)}_t) \mid A^{(2)}_t = a)$ is of the form

$$D_t G^{(2)}(t, a) + \frac{1}{2} \sum_{i,j \leq d} D^2_{ij} G^{(2)}(t, a)c^{A^{(2)}(t)} + \int (\Lambda G^{(2)})(t, a, y)K^{A^{(2)}(t,dy)} = 0.$$

Theorem 3.3.8 (Comparison of Asian option prices in the characteristics of the underlying). Let $S^{(i)} \sim \left(0, c^{S^{(i)}(t)}(t), K^{S^{(i)}(t, \cdot)}\right)_{id}$, $i = 1, 2$, be $d$-dimensional PII martingales and $g : \mathbb{R}^d \to \mathbb{R}$, $g \in \mathcal{F}_{cx}$. Assume that $G^{(2)} \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and that $G^{(2)}(t, a)$ satisfies the Kolmogorov backward equation. If

$$c^{S^{(1)}}(t) \leq_{psd} c^{S^{(2)}}(t) \quad \text{and} \quad \int f(t,y)K^{S^{(1)}}(t,dy) \leq \int f(t,y)K^{S^{(2)}}(t,dy),$$
for all \( t \) and all \( f : [0, T] \times \mathbb{R}^d \to \mathbb{R} \), s.th. \( f(t, \cdot) \in \mathcal{F}_{\text{cx}} \), then
\[
Eg\left(\frac{1}{T} \int_0^T S_u^{(1)} du\right) \leq Eg\left(\frac{1}{T} \int_0^T S_u^{(2)} du\right).
\]

**Proof.** The proof is similar to the proof of the previous theorem. Due to Lemma 3.3.5 it follows from \( cS^{(1)}(t) \leq \text{psd} cS^{(2)}(t) \) that
\[
cA^{(1)}(t) = cS^{(1)}(t) \left(\frac{T - t}{T}\right)^2 \leq \text{psd} cS^{(2)}(t) \left(\frac{T - t}{T}\right)^2 = cA^{(2)}(t).
\]

For the Lévy kernels \( K^{A^{(i)}} \), Lemma 3.3.5 implies
\[
\int f(t, y) K^{A^{(i)}}(t, dy) = \int f(t, y\frac{T-t}{T}) K^{S^{(i)}}(t, dy),
\]
and for \( g(t, y) := f(t, y\frac{T-t}{T}) \) it follows that \( g(t, \cdot) \in \mathcal{F}_{\text{cx}} \).

We derive a similar ordering result by a different approach which makes use of the averaging process \( A_t \) in (3.26). We establish the ordering for univariate models and consider as upper bound candidate a Markovian model \( S^* \) that is not necessarily PII. For martingales \( S, S^* \), we consider the processes \( A_t := \frac{1}{t} \int_0^t S_u du \) and \( A^*_t := \frac{1}{t} \int_0^t S^*_u du \), respectively, which are continuous, have paths of finite variation and satisfy
\[
dA_t = \frac{1}{t} (S_t - A_t) dt, \quad dA^*_t = \frac{1}{t} (S^*_t - A^*_t) dt.
\]
Hence \( A_t, A^*_t \) are not Markovian, even if \( S^* \) is Markovian. To obtain a Markovian upper bound candidate, we augment \( A \) and \( A^* \) by the underlyings \( S \) and \( S^* \), respectively. Then \((S^*, A^*)\) is Markovian as \( S^* \) is assumed to be Markovian, cp. also Remark 3.3.6. The value process for the Asian option with payoff \( g(A^*_T) \) is of the form
\[
\mathcal{G}^{A^*}(t, s, a) = E^*\left(g(A^*_T) \mid S^*_t = s, A^*_t = a\right) \tag{3.28}
\]
and satisfies the Kolmogorov-backward equation
\[
0 = D_t \mathcal{G}^{A^*}(t, s, a) + D_a \mathcal{G}^{A^*}(t, s, a) \frac{1}{t} (s - a) + \frac{1}{2} D^2_{ss} \mathcal{G}^{A^*}(t, s, a) c^{S^*}(t) \mathcal{G}^{A^*}(t, s, a) + \int_{\mathbb{R}} \left( \mathcal{G}^{A^*}(t, s + y, a) - \mathcal{G}^{A^*}(t, s, a) - D_y \mathcal{G}^{A^*}(t, s, a)y \right) K^{S^*}(t, s, dy),
\]

cp. Barraquand and Pudet (1996) for the continuous case. We obtain the following variant of the result of the previous theorem.

**Theorem 3.3.9 (Comparison of Asian option prices in the characteristics of the underlying).** Let \( S \sim (0, cS, K^S)_{\text{id}}, S^* \sim (0, cS^* (t, s), \)
3.3. Comparison of path-dependent options

$K^{S^*}(t, s, \cdot)$ be one-dimensional martingales, and additionally assume that $S^*$ is Markovian. Let $g : \mathbb{R} \to \mathbb{R}$, $g \in \mathcal{F}_{cx}$, and for $\mathcal{G}^A(t, s, a) \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R})$, and $\mathcal{G}^A(t, \cdot, a) \in \mathcal{F}_{cx}$. If

$$c^S_t(\omega) \leq c^{S^*}(t, S_t - (\omega)) \quad \text{and}$$

$$\int f(t, S_t - (\omega), z) K^{S^*}_{\omega,t}(dz) \leq \int f(t, S_t - (\omega), z) K^S_{\omega,t}(S_t - (\omega), dz),$$

$\lambda \times Q$-a.e., for all $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ with $f(t, s, \cdot) \in \mathcal{F}_{cx}$ such that the integrals exist, then

$$E g\left(\frac{1}{T} \int_0^T S^{(1)}_u du\right) \leq E g\left(\frac{1}{T} \int_0^T S^{(2)}_u du\right).$$

Proof. We establish that $\mathcal{G}^A(t, S_t, A_t)$ is a supermartingale. Due to Itô’s formula and the form of the Kolmogorov-backward equation the crucial equation is

$$\mathcal{G}^A(t, S_t, A_t) = \mathcal{G}^A(0, S_0, A_0) + M_t + \int_0^t \frac{1}{2} D^2 ss \mathcal{G}^A(t, S_{u-}, A_{u-})(c^S_u - c^{S^*}(u, S_{u-})) du$$

$$+ \int_0^t \int_{\mathbb{R}} \left( \mathcal{G}^A(u, S_{u-} + y, A_{u-}) - \mathcal{G}^A(u, S_{u-}, A_{u-}) - D_s \mathcal{G}^A(u, S_{u-}, A_{u-})y \right) \left(K^{S^*}_u - K^{S^*}(u, S_{u-}, dy)\right) du,$$

hence convexity of $\mathcal{G}^A(t, \cdot, a)$ and the orderings of the characteristics of $S$ and $S^*$ imply the result. □

3.3.3 American options

In this subsection we obtain orderings for American option prices in a univariate diffusion with jumps model. Parallel to the European case, the orderings are formulated in terms of the differential characteristics of the stochastic logarithms of the underlying. We consider models that have constant positive or negative drift components $b, b^*$ under a respective martingale measure. In an equity market model, a positive drift component corresponds to the case where the interest rate is higher than the continuously paid dividend yield; in a foreign exchange market model, positive drift is due to a domination of the foreign interest rate by the domestic interest rate. Negative drift of the underlying corresponds to reversed orderings of interest rate and dividends, or domestic and foreign interest rate. If the drift
of the underlying model under a martingale measure is positive, the prices of American and European call options on this underlying coincide, whereas American put options yield higher prices than their European counterparts. Similarly, if the underlying model has negative drift under a martingale measure, then American and European put option prices are identical, whereas American call options yield higher prices than their European counterparts.

There are several ordering results for American options in the literature, most of which are derived as extensions of ordering results for European option prices in the same papers. Pham (1997) derives properties of American option prices in a jump-diffusion model, where the jump part is driven by a homogeneous Poisson random measure with finite intensity $\lambda$. He characterizes American option prices by a parabolic integro-differential free-boundary problem, and then obtains monotonicity of American put option prices in $\lambda$ and in the Girsanov jump parameter. El Karoui, Jeanblanc-Picqué, and Shreve (1998) establish upper and lower bounds for American put option prices in univariate stochastic volatility models. They use a variational inequality approach that is parallel to the stochastic calculus approach that is used to establish orderings for European type options, cp. Chapter 1. Hobson (1998b) also considers continuous models. For a univariate diffusion he derives convexity of American put option prices in the underlying, and also obtains option price monotonicity in the diffusion coefficient. Parallel to the approach for European type options of that paper, he uses coupling arguments to derive the orderings. Bellamy and Jeanblanc (2000) establish that the lower bound of an American option price w.r.t. to a diffusion with jumps model is given by the price of an American option w.r.t. the corresponding generalized Black–Scholes model. As in the case of European options, the result is obtained by the stochastic calculus approach. Henderson and Hobson (2003) obtain ordering of American option prices in a diffusion with jumps model. They establish ordering in the Girsanov parameter by coupling techniques and then derive ordering of American option prices w.r.t. well-established martingale measures. The result is parallel to the result they obtain for European option prices. In a univariate diffusion model, Ekström (2004) considers American options with payoff functions that satisfy certain growth conditions, which are especially satisfied by decreasing functions. To establish convexity of American option prices in the underlying, he uses the notion of volatility time, a time change introduced in Janson and Tysk (2003), and obtains monotonicity and continuity of the prices in the volatility.

We derive ordering between an American option price on a univariate underlying $S^{(2)}$ and a European option price on another univariate underlying
S^{(1)} in the case when the differential characteristics of \(X^{(2)} = \log(S^{(2)})\) dominate those of \(X^{(1)} = \log(S^{(1)})\) in an appropriate sense. For notational convenience we consider equity models \(S^{(i)}, i = 1,2\), with zero dividends and constant interest rate \(r > 0\), where \(S^{(i)}\) (under respective martingale measures) are the solutions of

\[
\frac{dS^{(i)}_t}{S^{(i)}_t} = r dt + \sigma^{(i)} dW_t + \int_{(-1,\infty)} x(p^{(i)}(dt, dx) - F^{(i)}(dx)dt),
\]

where \(\sigma^{(i)} > 0\), \(W_t\) is a univariate Brownian motion, and \(p^{(i)}\) is a homogeneous Poisson random measure with finite compensator \(F^{(i)}(dx)dt\). Hence \(X^{(i)} = \log(S^{(i)})\) is a PIIS with characteristics \(X^{(i)} \sim (r, \sigma^{(i)}^2, F^{(i)})_{id}\). We assume that a riskless security \(B_t = e^{rt}\) is given in the market model.

The arbitrage-free price of an American option with payoff function \(g\) and underlying model \(S^{(i)}\) is given by

\[
G^{(i)}_{Am}(t, s) = \sup_{\tau \in \mathcal{T}_{t, T}} E\left(e^{-(r-t)\tau}g(S^{(i)}_\tau) \mid S^{(i)}_t = s\right).
\]

Instead of the Kolmogorov backward equation in the case of European options, we assume that \(G^{(2)}_{Am}\) satisfies the Hamilton–Jacobi–Bellman equation

\[
\max \left\{ \int \left( G^{(2)}_{Am}(t, s(1 + x)) - G^{(2)}_{Am}(t, s) - sxD_sG^{(2)}_{Am}(t, s) \right) F^{(2)}(dx) + D_tG^{(2)}_{Am}(t, s) + rsD_sG^{(2)}_{Am}(t, s) + \frac{\sigma^{(2)^2} s^2}{2} D_{ss}G^{(2)}_{Am}(t, s) - rG^{(2)}_{Am}(t, s), \right. \\
g(s) - G^{(2)}_{Am}(t, s) \right\} = 0,
\]

on \([0, T] \times \mathbb{R}_+\) and \(G^{(2)}_{Am}(T, s) = g(s)\). Zhang (1994, Proposition 3.5) establishes that this condition is satisfied in terms of the logarithm \(\hat{X}^{(2)} = \log(S^{(2)})\), iff \(G^{(2)}_{Am}\) is the solution to the corresponding variational inequality, cp. also Cont and Tankov (2004, §12.1.3).

We obtain the following ordering result.

**Theorem 3.3.10.** Let \(g : \mathbb{R}_+ \to \mathbb{R}_+\) be convex, and \(S^{(i)} = \mathcal{E}(X^{(i)})\), with \(X^{(i)} \sim (r, \sigma^{(i)}^2, F^{(i)})_{id}\) and \(S^{(i)}_0 = s, i = 1,2\). Assume that \(G^{(2)}_{Am} \in C^{1,2}([0, T] \times \mathbb{R}^d), EG^{(2)}_{Am}(t, S^{(1)}_t) < \infty\) and that the Hamilton–Jacobi–Bellman equation holds true. If

\[
\sigma^{(1)} \leq \sigma^{(2)}, \\
\int f(x)F^{(1)}(dx) \leq \int f(x)F^{(2)}(dx),
\]

(3.29)
for all non-negative \( f \in \mathcal{F}_{cx} \), then
\[
e^{-rT}E^{(1)}g(S_{T}^{(1)}) \leq \mathcal{G}_{Am}^{(2)}(0, s).
\]

**Proof.** Parallel to the proofs of the ordering results in Chapter 1 we establish that \( e^{-rT}\mathcal{G}_{Am}^{(2)}(t, S_{t}^{(1)}) \) is an \((\mathcal{A}_{i}^{(1)})\)-supermartingale under \( P^{(1)} \), then
\[
\sup_{\tau \in T_{T, s}} E^{(2)}(e^{-rt}g(S_{\tau}^{(2)})) = \mathcal{G}_{Am}^{(2)}(0, s) \geq e^{-rT}\mathcal{G}_{Am}(T, S_{T}^{(1)}) = e^{-rT}E^{(1)}g(S_{T}^{(1)}).
\]

For \( \mathcal{G}_{Am} \in C^{1,2}([0, T] \times \mathbb{R}+) \), Itô’s formula implies \( e^{-rt}\mathcal{G}_{Am}(t, S_{t}^{(1)}) = \mathcal{G}_{Am}(0, s) + \int_{0}^{t} \frac{1}{2} H_{u} du + M_{t} \), where \( M_{t} \) is a local \((\mathcal{A}_{i}^{(1)})\)-martingale under \( P^{(1)} \) and
\[
H_{t} = D_{t}\mathcal{G}_{Am}(t, S_{t}^{(1)}) + D_{s}\mathcal{G}_{Am}(t, S_{t}^{(1)}) rS_{t}^{(1)} - \mathcal{G}_{Am}(u, S_{u}^{(1)}) rS_{t}^{(1)} - S_{t}^{(1)}[D_{t}\mathcal{G}_{Am}(t, S_{t}^{(1)})]
\]
\[
\quad + \int \left( \mathcal{G}_{Am}(t, S_{t}^{(1)}(1 + x)) - \mathcal{G}_{Am}(t, S_{t}^{(1)})
\quad \quad - S_{t}^{(1)}x D_{t}\mathcal{G}_{Am}(t, S_{t}^{(1)}) \right) F^{(1)}(dx) + \frac{\sigma^{(1)2}}{2}S_{t}^{(1)2}D_{ss}\mathcal{G}_{Am}(t, S_{t}^{(1)}).
\]

From the Hamilton–Jacobi–Bellman equation it follows that
\[
H_{t} \leq \frac{1}{2} \left( \sigma^{(1)2} - \sigma^{(2)2} \right) S_{t}^{(1)2}D_{ss}\mathcal{G}_{Am}(t, S_{t}^{(1)})
+ \int \left( \mathcal{G}_{Am}(t, S_{t}^{(1)}(1 + x)) - \mathcal{G}_{Am}(t, S_{t}^{(1)})
\quad \quad - S_{t}^{(1)}x D_{t}\mathcal{G}_{Am}(t, S_{t}^{(1)}) \right) \left( F^{(1)}(dx) - F^{(2)}(dx) \right).
\]

Pham (1997) establishes that \( \mathcal{G}_{Am}(t, \cdot) \in \mathcal{F}_{cx} \). Due to the orderings of the differential characteristics of \( X^{(i)} \), the right-hand side of the last equation is not bigger than zero, hence the supermartingale property of \( e^{-rt}\mathcal{G}_{Am}(t, S_{t}^{(1)}) \) follows. \( \square \)

### 3.3.4 Barrier options

In this subsection we obtain comparison results for single-barrier options of European type without rebate in univariate models. The terminal payoff of a **knock-out type barrier option** on an underlying \( S^{*} \) with barrier \( \beta : [0, T] \rightarrow \mathbb{R}_{+} \) and payoff function \( g \) is given by
\[
g(S_{T}^{*})I_{\{S_{T}^{*} \geq \eta \beta(t), \forall t \in [0, T]\}}, \quad \eta \in \{-1, 1\}.
\]

(3.30)

For \( \eta = 1 \), (3.30) is the terminal payoff of a **down-and-out** barrier option, for \( \eta = -1 \) it is the terminal payoff of an **up-and-out** barrier option. We denote the value function of a down-and-out barrier option with barrier \( \beta(t) \) by
\[
\mathcal{G}_{out}(t, s) = E^{*} \left( g(S_{T}^{*})I_{\{S_{T}^{*} \geq \beta(t), \forall u \in [t, T]\}} | S_{t}^{*} = s \right),
\]
and the value function of a up-and-out barrier option with barrier $\beta(t)$ by

$$G^{\text{out}}(t, s) = E^\star(g(S^*_T)1_{\{S^*_T < \beta(\omega), \forall \omega \in [t, T]\}|S^*_t = s}).$$

Similarly, a knock-in type barrier option has terminal payoff

$$g(S^*_T)1_{\{\eta S^*_T < \eta \beta(t), \exists t \in [0, T]\}}, \quad \eta \in \{-1, 1\},$$

where $\eta = 1$ corresponds to a down-and-in barrier option, and $\eta = -1$ to an up-and-in barrier option with corresponding value functions $G^{\text{in}}(t, s)$ and $G^{\text{in}}(t, s)$. A barrier option is said to be regular, if $g(\beta(T)) = 0$. For down-type barrier options we assume $S_0 > \beta(0)$ and for up-type barrier options we assume $S_0 < \beta(0)$. Finally, we distinguish between a constant barrier $\beta(t) = \beta > 0$, for all $t \in [0, T]$, and a barrier with constant value $\beta(t) = \beta e^{-b(T-t)}$, $\beta > 0$, where $b$ is the drift of the underlying $S$ under an equivalent martingale measure. In an equity model, if $S$ is a stock with continuously compounded annual dividend yield $d$, the drift is $b = r - d$, where $r$ is the risk-free rate of interest. In foreign exchange markets, where $S$ is an exchange rate between a domestic and a foreign currency, $b = r_{\text{dom}} - r_{\text{for}}$, where $r_{\text{dom}}$ and $r_{\text{for}}$ denote the domestic and the foreign interest rate, respectively.

In a one-dimensional diffusion model, Eriksson (2004, 2005) establishes monotonicity in the diffusion coefficient for the various types of barrier options that are given above. These ordering results depend on the drift coefficient of the underlying $S$. For $b = 0$ it is shown that for regular down-and-out and up-and-out contracts and down-and-in and up-and-in contracts this monotonicity result holds true.

We assume that the value functions of the barrier options with underlying $S^*$ satisfy a PIDE that is similar to the Kolmogorov-backward equation. Additionally to the terminal boundary condition, in the case of barrier option a boundary condition in the space variable occurs. In the sequel we discuss the case $b = 0$, the cases $b \neq 0$ are treated similarly. As example we consider comparison of down-and-out barrier options on one-dimensional underlyings $S, S^*$ in a market model with zero interest rate. As in section 1.1 we assume that $S, S^*$ satisfy Assumption MG, i.e. we assume that $S \sim (0, e^{S}, K^S)_{\text{id}}$, $S^* \sim (0, e^{S^*}(t, S^*_t), K^{S^*}(t, S^*_t, \cdot))_{\text{id}}$ are positive martingales and that $S^*$ is Markovian. $S$ need not be Markovian. We assume that the value function $G^{\text{out}}(t, s)$ satisfies the Kolmogorov backward equation

$$D_t G^{\text{out}}(t, s) + \frac{1}{2} D^2_{ss} G^{\text{out}}(t, s)e^{S^*}(t, s) + \int (\Lambda G^{\text{out}})(t, s, y)K^{S^*}(t, s, dy) = 0,$$

(3.31)
on \((\beta, \infty)\), subject to boundary conditions
\[
\begin{aligned}
\mathcal{G}_{\text{out}}(t, s) &= 0, \quad s \leq \beta \text{ for all } t \in [0, T], \\
\mathcal{G}_{\text{out}}(T, s) &= g(s), \quad s > \beta,
\end{aligned}
\]
we refer to Cont, Tankov, and Voltchkova (2004) and Cont and Voltchkova (2005) for conditions that ensure the validity of this PIDE. We obtain the following ordering result for down-and-out barrier options.

**Theorem 3.3.11 (Upper bound for barrier option price).** Let \(S \sim (0, e^S, K^S)_{t \in \mathbb{R}_+}, S^* \sim (0, e^{S^*}(t, s), K^{S^*}(t, s, \cdot))_{t \in \mathbb{R}_+}\) be one-dimensional positive martingales, \(S^*\) Markovian. For a constant barrier \(\beta\) and a payoff function \(g : \mathbb{R}_+ \to \mathbb{R}_+\), s.th. \(g\) is convex on \((\beta, \infty)\) assume \(\mathbb{E} \mathcal{G}_{\text{out}}(t, S_t) < \infty\), \(\mathcal{G}_{\text{out}}(t, \cdot) \in C^{1,2}(0, T] \times \mathbb{R}_+\) and that \(\mathcal{G}_{\text{out}}(t, \cdot)\) satisfies the Kolmogorov backward equation (3.31). If \(\mathcal{G}_{\text{out}}(t, \cdot) \in F_{\text{cx}}, and the differential characteristics of \(S, S^*\) are ordered as
\[
c_t^S(\omega) \leq c_t^{S^*}(t, S_{t-}(\omega)),
\]
\[
\int_{(-1, \infty)^d} f(t, S_{t-}(\omega), x) K_t^S(dx) \leq \int_{(-1, \infty)^d} f(t, S_{t-}(\omega), x) K_t^{S^*}(S_{t-}(\omega), dx),
\] for all \(f : [0, T] \times \mathbb{R}_d \times (-1, \infty)^d \to \mathbb{R}\) with \(f(t, \cdot, \cdot) \in F_{\text{dx}}\) such that the integrals exist, then the down-and-out barrier option prices are ordered as
\[
\mathbb{E}(g(S_T)I\{S_t > \beta(t), \forall t \in [0, T]\}) \leq \mathbb{E}(g(S^*_T)I\{S^*_t > \beta(t), \forall t \in [0, T]\}).
\]

**Proof.** We sketch the proof, which is similar to the proofs of the comparison results of Section 1.1. Let \(\tau_\beta := \inf\{t > 0 : S_t \leq \beta\}\) denote the time at which \(S\) first crosses the barrier \(\beta\). Then it follows from Itô’s formula and the Kolmogorov backward equation (3.31) that
\[
\begin{aligned}
\mathcal{G}_{\text{out}}(t \land \tau_\beta, S_{t \land \tau_\beta}) &= \mathcal{G}_{\text{out}}(0, S_0) + M_{t \land \tau_\beta} \\
+ \int_0^{t \land \tau_\beta} \left\{ D_{uu}^2 \mathcal{G}_{\text{out}}(u, S_{u-})(c_u^S - c_u^{S^*}(u, S_{u-})) \\
+ \int_0^u (\Delta \mathcal{G}_{\text{out}})(u, S_{u-y})(K_u^S(dy) - K_u^{S^*}(u, S_{u-y}, dy)) \right\} du
\end{aligned}
\]
On \(\{t > \tau_\beta\}\) it holds true that \(\mathcal{G}_{\text{out}}(t \land \tau_\beta, S_{t \land \tau_\beta}) = 0\) and for \(\{t \leq \tau_\beta\}\) the ordering (3.32) implies the result. \(\square\)
Eriksson (2004, 2005) establishes $G(t, \cdot) \in \mathcal{F}_{\text{cx}}$ in the model above for several types of single-barrier options. Similar to our results he uses this property to derive upper and lower bounds of barrier option prices that are computed w.r.t. stochastic volatility models. The result for down-and-out barrier options in this type of models is the following corollary of Theorem 3.3.11.

**Corollary 3.3.12 (Upper bound of barrier option price w.r.t. SV model).** Let $S \sim (0, c^S, 0)_{\text{id}}$, $S^* \sim (0, c^{S^*}(t, s), 0)_{\text{id}}$ be one-dimensional positive martingales, $S^*$ a diffusion. For a constant barrier $\beta$ and a payoff function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, s.th. $g$ is convex on $(\beta, \infty)$ assume $E_{\text{out}}(t, S_t) < \infty$, $G_{\text{out}}(t, s) \in C^{1,2}([0, T] \times \mathbb{R}^+)$ and that $G_{\text{out}}(t, S_t -)$ satisfies the Kolmogorov backward equation (3.31). If the stochastic volatility $c^S$ and the diffusion part $c^{S^*}(t, s)$ of $S^*$ are ordered as

$$c^S_t(\omega) \leq c^{S^*}(t, S_t - (\omega)),$$

$\lambda \times Q$-a.e., for all $f : [0, T] \times \mathbb{R}^d \times (-1, \infty)^d \rightarrow \mathbb{R}$ with $f(t, s, \cdot) \in \mathcal{F}_{\text{dcx}}$ such that the integrals exist, then the down-and-out barrier option price w.r.t. the diffusion model is an upper bound for the down-and-out barrier option price w.r.t. the SV model.

### 3.4 Ordering results for $\alpha$-stable and NIG processes

In this section we derive increasing convex orderings of finite-dimensional distributions of univariate $\alpha$-stable processes for $\alpha \in (1, 2)$ and of univariate and multivariate normal inverse Gaussian (NIG) processes. The ordering criteria are formulated in terms of parameters of the models. For the proofs of the univariate ordering results we apply the cut criteria for Lévy measures of Chapter 2. To derive orderings for multivariate NIG processes we make use of a mixing type representation of GH distributions. These are variance mixtures of multivariate normal distributions with a generalized inverse Gaussian (GIG) distribution as mixing distribution. We first establish likelihood ratio ordering of GIG random variables, from which we derive increasing convex ordering of GH distributions. Convolution stability of NIG processes then yields increasing convex ordering also for the processes. We start with the ordering result for univariate $\alpha$-stable processes.

**Theorem 3.4.1 (Comparison of $\alpha$-stable processes).** For $c^{(i)}, d^{(i)} > 0$, and $\alpha^{(i)} \in (1, 2)$, $i = 1, 2$, let $S^{(i)} \sim (ES^{(i)}, 0, (c^{(i)} x^{-1-\alpha^{(i)}} \mathbb{1}_{\mathbb{R}_+}(x) +$
be one-dimensional \(\alpha\)-stable Lévy processes. If \(\alpha(2) \leq \alpha(1)\) and \(ES(1) \leq ES(2)\), then

\[
(S_t^{(1)}) \leq_{\text{icx}} (S_t^{(2)}).
\]

**Proof.** For \(x < 0\) it holds true that \(d(1)|x|^{-1-\alpha(1)} < d(2)|x|^{-1-\alpha(2)}\) if and only if \(x < \left(\frac{d(2)}{d(1)}\right)^{\frac{1}{\alpha(2)-\alpha(1)}}\) and for \(x > 0\) we have \(c(1)x^{-1-\alpha(1)} < c(2)x^{-1-\alpha(2)}\) if and only if \(x > \left(\frac{c(2)}{c(1)}\right)^{\frac{1}{\alpha(2)-\alpha(1)}}\). Hence the cut criterion of Theorem 2.2.6 is satisfied with \(k^{(1)} = -\left(\frac{d(2)}{d(1)}\right)^{\frac{1}{\alpha(2)-\alpha(1)}}\) and \(k^{(2)} = \left(\frac{c(2)}{c(1)}\right)^{\frac{1}{\alpha(2)-\alpha(1)}}\). It remains to verify the conditions stated in the second part of that theorem. We choose the sequences \(\varepsilon_n^{(i)} \uparrow 0, \pi_n^{(i)} \downarrow 0\) s.t.

\[
\int_{\varepsilon_n^{(1)}} x F^-(x) \, dx = \int_{\varepsilon_n^{(2)}} x F^-(x) \, dx \quad \text{and} \quad (3.33)
\]

This is possible as \(\int_A x F^i(x) \, dx = \infty\) for \(A = (-1, 0)\) and \(A = (0, 1)\). From (3.33) we obtain

\[
\varepsilon_n^{(1)} = \left(\frac{d(2)}{d(1)}\right)^{\frac{1}{\alpha(2)-\alpha(1)}} |x_n^{(2)}|^{\frac{1}{1-\alpha(1)}}, \\
\epsilon_n^{(1)} = \left(\frac{c(2)}{c(1)}\right)^{\frac{1}{\alpha(2)-\alpha(1)}} \pi_n^{(2)}^{\frac{1}{1-\alpha(1)}}.
\]

As \(\frac{1-\alpha(2)}{1-\alpha(1)} > 1\) and \(|\varepsilon_n^{(2)}|, \pi_n^{(2)} \downarrow 0\), it follows that there are \(N, \overline{N} \in \mathbb{N}\), s.t.

\(\varepsilon_n^{(1)} < \varepsilon_n^{(2)}\), for all \(n \geq N\) and \(|\varepsilon_n^{(1)}| < |\varepsilon_n^{(2)}|\), hence \(\varepsilon_n^{(1)} > \varepsilon_n^{(2)}\), for all \(n \geq N\). Therefore, the conditions of the second part of Theorem 2.2.6 are satisfied and it follows that \((S_t^{(1)}) \leq_{\text{icx}} (S_t^{(2)}).\)

\[\square\]

**Remark 3.4.2.** Despite the fact that we can establish the domination criterion also for the Lévy measures of \(\alpha\)-stable processes with stability parameters \(0 < \alpha(1) \leq \alpha(2) < 1\), an analogous ordering result does not hold true in this case. This is due to the fact that the first moments do not exist, cp. Sato (1999, Proposition 3.14.5).

Next, we establish ordering of one-dimensional normal inverse Gaussian processes in the shape and scaling parameters \(\alpha\) and \(\delta\). The Lévy density of an NIG=NIG(\(\alpha, \beta, \delta, \mu\)) distributed random variable \(S\) is given by

\[
f_{\alpha,\beta,\delta}(x) = \frac{\delta \alpha K_1(\alpha |x|) e^{\beta x}}{\pi |x|}, \quad (3.34)
\]
where $K_1$ denotes the modified Bessel function of third kind with index 1, $\alpha > 0$, $0 \leq |\beta| \leq \alpha$ and $\delta > 0$ and $S$ has expectation $ES = \mu + \frac{\delta \beta}{\sqrt{\alpha^2 - \beta^2}}$.

**Theorem 3.4.3 (Ordering of NIG processes).** Let $S^{(i)} \sim (ES^{(i)}, 0, f^{(i)}(x)dx)_{id}$, $i = 1, 2$, be one-dimensional NIG processes.

1. **Ordering in $\alpha$.** Let $f^{(i)}(x) := f_{\alpha^{(i)}, \beta^{(i)}, \delta^{(i)}}(x)$, $i = 1, 2$. If $|\beta| \leq \alpha^{(2)} \leq \alpha^{(1)}$ and $ES^{(1)} \leq ES^{(2)}$, then $(S^{(1)}_t) \leq_{icx} (S^{(2)}_t)$.

2. **Ordering in $\delta$.** Let $f^{(i)}(x) := f_{\alpha^{(i)}, \beta, \delta^{(i)}}(x)$, $i = 1, 2$. If $\delta^{(1)} \leq \delta^{(2)}$ and $ES^{(1)} \leq ES^{(2)}$, then $(S^{(1)}_t) \leq_{icx} (S^{(2)}_t)$.

**Proof.**

1. Let $\delta > 0$ and $\beta, |\beta| \leq \alpha^{(2)}$ be given. For fixed $x > 0$ and $\alpha \geq \alpha^{(2)}$ we consider $g(\alpha) := f_{\alpha, \beta, \delta}(x)$. As $g'(\alpha) = -\frac{4\delta \beta}{\pi} \alpha K_0(\alpha x) \leq 0$, it follows that $f^{(1)}(x) \leq f^{(2)}(x)$, for all $x \in \mathbb{R}_+$. For fixed $x < 0$ we similarly obtain $g'(\alpha) = -\frac{4\delta \beta}{\pi} \alpha K_0(-\alpha x) \leq 0$, thus $f^{(1)}(x) \leq f^{(2)}(x)$, for all $x \in \mathbb{R}_-$. As NIG processes have paths of infinite variation, Theorem 2.2.11 implies $(S^{(1)}_t) \leq_{icx} (S^{(2)}_t)$.

2. As $\delta^{(1)} \leq \delta^{(2)}$ implies $f^{(1)}(x) \leq f^{(2)}(x)$, for all $x \in \mathbb{R}$, Theorem 2.2.11 yields $(S^{(1)}_t) \leq_{icx} (S^{(2)}_t)$.

**Remark 3.4.4.** If in the previous theorem it holds true that $\mu^{(1)} = \mu^{(2)}$, then the ordering condition on the expectations is satisfied.

For some cases of interest it is possible to obtain comparison results by using mixing type representations. We apply this approach to the class of GH distributions. Further classes of particular interest for financial mathematical models are multivariate $t$-distributions and elliptically contoured distributions (see Bingham, Kiesel, and Schmidt (2003)). GH distributions are variance mixtures of multivariate normal distributions with a generalized inverse Gaussian as mixing distribution. For $\mu^{(i)}, \beta^{(i)} \in \mathbb{R}^d$, $\Delta^{(i)} \in \mathcal{M}(d, \mathbb{R})$ with $\det(\Delta^{(i)}) = 1$ $i = 1, 2$, and $N^{(i)} \sim \mathcal{N}(0, \Delta^{(i)})$ we consider the $d$-dimensional random variable

$$S^{(i)} = \mu^{(i)} + X^{(i)} \Delta^{(i)} \beta^{(i)} + \sqrt{X^{(i)}} N^{(i)}, \tag{3.35}$$

where $X^{(i)}$ are generalized inverse Gaussian random variables with densities

$$d_{GIG(\lambda, \delta, \gamma)}(x) := \left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{1}{2K_\lambda(\delta \gamma)} e^{-\frac{1}{2}(\frac{\gamma^2}{\delta^2} + \gamma x)} I_{\mathbb{R}_+}(x), \tag{3.36}$$

where $\delta \geq 0$, $\alpha^2 > \beta \Delta \beta^T$ and $\gamma = \sqrt{\alpha^2 - \beta \Delta \beta^T}$. Then $S$ is generalized hyperbolic distributed with parameters $d, \lambda, \alpha, \beta, \delta, \mu$ and covariance matrix $\Delta$ and we write $GH(d, \lambda, \alpha, \beta, \delta, \mu, \Delta)$ (cp. Barndorff-Nielsen (1977)).
The following lemma states a comparison result for GIG distributions with respect to the likelihood ratio order $\leq_{lr}$, if the parameters $\lambda, \delta, \gamma$ are ordered.

**Lemma 3.4.5 (Likelihood ratio ordering of GIG random variables).**

Let $X^{(i)}$ be GIG distributed with density $d_{\text{GIG}}(\lambda^{(i)}, \delta^{(i)}, \gamma^{(i)})(x)$. If $\lambda^{(1)} \leq \lambda^{(2)}$, $\delta^{(1)} \leq \delta^{(2)}$, and $\gamma^{(1)} \geq \gamma^{(2)}$, then $X^{(1)} \leq_{lr} X^{(2)}$.

**Proof.** We consider the likelihood ratio

$$g(x) := \frac{d_{\text{GIG}}(\lambda^{(1)}, \delta^{(1)}, \gamma^{(1)})(x)}{d_{\text{GIG}}(\lambda^{(2)}, \delta^{(2)}, \gamma^{(2)})(x)} = K x^{\lambda^{(1)}-\lambda^{(2)}} e^{\frac{1}{2}(\delta^{(2)}-\delta^{(1)})^2 x + (\gamma^{(2)}-\gamma^{(1)})^2 x} 1_{\mathbb{R}_+}(x),$$

with $K > 0$. The first derivative of $g$ is of the form

$$g'(x) = K_1 x^{\lambda^{(1)}-\lambda^{(2)}} + K_2 x^{\delta^{(1)}-\delta^{(2)}} + (\gamma^{(2)}-\gamma^{(1)})^2 x,$$

where $K_1, K_2 \geq 0$, hence it follows from the orderings on the parameters that $g'(x) \leq 0$.

In the following theorem we establish increasing convex ordering of multivariate GH distributions with mixing type representation (3.35). We consider the following three cases for $\beta^{(i)}$ and $\Delta^{(i)}$:

$$0 \leq \beta^{(1)} \leq \beta^{(2)}, \quad \Delta^{(i)} = I,$$  \hspace{1cm} (3.37)

$$\beta^{(i)} = 0, \quad \Delta^{(1)} \leq_{\text{psd}} \Delta^{(2)}.$$  \hspace{1cm} (3.38)

$$0 \leq \beta^{(1)} \leq \beta^{(2)}, \quad \Delta^{(1)} \leq_{\text{psd}} \Delta^{(2)}, \quad 0 \leq \Delta^{(i)}_{ij} \leq \Delta^{(2)}_{ij}, \forall i, j \leq d.$$  \hspace{1cm} (3.39)

**Theorem 3.4.6 (Increasing convex comparison of GH distributions).** Let $S^{(i)}$ be GH$(d, \lambda^{(i)}, \alpha^{(i)}, \beta^{(i)}, \delta^{(i)}, \mu^{(i)}, \Delta^{(i)})$ distributed. If

$$\lambda^{(1)} \leq \lambda^{(2)}, \quad \delta^{(1)} \leq \delta^{(2)}, \quad \alpha^{(1)} \geq \alpha^{(2)}, \quad \mu^{(1)} \leq \mu^{(2)}$$

and one of the cases (3.37)–(3.39) holds true for $\beta^{(i)}$ and $\Delta^{(i)}$, then $S^{(1)} \leq_{\text{icx}} S^{(2)}$.

**Proof.** First we prove that the conditions on the parameters of $X^{(i)}$ imply $X^{(1)} \leq_{\text{st}} X^{(2)}$. For $\gamma^{(i)} = \sqrt{\alpha^{(i)2} - \beta^{(i)} \Delta^{(i)}(\beta^{(i)})^T}$ it follows from the conditions on $\alpha^{(i)}, \Delta^{(i)}$, and $\beta^{(i)}$ in all three cases (3.37)–(3.39) that $\gamma^{(1)} \geq \gamma^{(2)}$. Due to Lemma 3.4.5 the ordering conditions on $\lambda^{(i)}$ and $\delta^{(i)}$ imply $X^{(1)} \leq_{lr} X^{(2)}$ and this yields $X^{(1)} \leq_{\text{st}} X^{(2)}$. 

3.4. Ordering results for \( \alpha \)-stable and NIG processes

1. Let \( \beta^{(i)} \) and \( \Delta^{(i)} \) satisfy (3.37). For \( f \in \mathcal{F}_{\text{cx}} \) we define \( g(x) := Ef(\mu^{(1)} + \beta^{(1)}x + \sqrt{x}N) \), with \( N \sim \mathcal{N}(0, I) \), \( i = 1, 2 \). Let \( 0 \leq x^{(1)} \leq x^{(2)} \). As \( \beta^{(1)}x + \sqrt{x}N \sim \mathcal{N}(\beta^{(1)}x, I) \), \( \beta^{(1)}x^{(1)} \leq \beta^{(1)}x^{(2)} \) and \( x^{(1)}I \leq_{\text{psd}} x^{(2)}I \), it follows from Corollary 1.2.12 that \( g \) is increasing. Then \( X^{(1)} \leq_{\text{st}} X^{(2)} \) implies

\[
Ef(\mu^{(1)} + \beta^{(1)}X^{(1)} + \sqrt{X^{(1)}}N) = Eg(X^{(1)}) \\
\leq Eg(X^{(2)}) = Ef(\mu^{(1)} + \beta^{(1)}X^{(2)} + \sqrt{X^{(2)}}N).
\]

As \( f \) is increasing and \( X^{(2)} \) is non-negative, it follows from \( \mu^{(1)} \leq \mu^{(2)} \) and \( \beta^{(1)} \leq \beta^{(2)} \), that

\[
Ef(\mu^{(1)} + \beta^{(1)}X^{(2)} + \sqrt{X^{(2)}}N) = E^{X^{(2)}}f(\mu^{(1)} + \beta^{(1)}x^{(2)} + \sqrt{x^{(2)}}N) \\
\leq E^{X^{(2)}}f(\mu^{(2)} + \beta^{(2)}x^{(2)} + \sqrt{x^{(2)}}N) = Ef(S^{(2)}).
\]

2. Let \( \beta^{(i)} \) and \( \Delta^{(i)} \) satisfy (3.38) and assume that \( f \in \mathcal{F}_{\text{cx}} \) and \( x^{(1)} \geq 0 \).

From \( \Delta^{(1)} \leq_{\text{psd}} \Delta^{(2)} \) it follows that \( x^{(1)}\Delta^{(1)} \leq_{\text{psd}} x^{(1)}\Delta^{(2)} \) and, therefore, Corollary 1.2.12 implies \( Ef(S^{(1)}) \leq Ef(\mu^{(2)} + \sqrt{X^{(2)}}N^{(2)}) \). For \( z \geq 0 \) let \( g(z) := Ef(\mu^{(2)} + zN^{(2)}) \). As \( \Delta^{(2)} \) positive semidefinite, it follows for \( z^{(1)} \leq z^{(2)} \) that \( z^{(1)}\Delta^{(2)} \leq_{\text{psd}} z^{(2)}\Delta^{(2)} \) and Corollary 1.2.12 implies that \( g \) is increasing. From \( X^{(1)} \leq_{\text{st}} X^{(2)} \) it follows that

\[
Ef(\mu^{(2)} + \sqrt{X^{(1)}}N^{(2)}) = Eg(\sqrt{X^{(1)})} \\
\leq Eg(\sqrt{X^{(2)})} = Ef(\mu^{(2)} + \sqrt{X^{(2)}}N^{(2)}).
\]

3. Let \( \beta^{(i)} \) and \( \Delta^{(i)} \) satisfy (3.39). Condition \( \Delta^{(i)}_{ij} \geq 0, i, j \leq d \), implies that \( \beta^{(1)}\Delta^{(i)}\beta^{(1)^T} \leq (\beta^{(2)}\Delta^{(i)}\beta^{(2)^T}) \), for \( 0 \leq \beta^{(1)} \leq \beta^{(2)} \). The result follows similar to the previous ones.

\[ \square \]

In the case \( \lambda = -\frac{1}{2} \), \( S^{(i)} \) is normally inverse Gaussian distributed. NIG distributed random variables are stable under convolutions,

\[
NIG(d, \alpha, \beta, \delta, \mu, \Delta; t) = NIG(d, \alpha, \beta, t\delta, t\mu, \Delta; 1).
\]

Therefore, Theorem 3.4.6 also implies increasing convex comparison of the finite-dimensional distributions of NIG processes. These have mixing type representation

\[
S^{(i)}_t := \mu^{(i)} t + X^{(i)}_t \Delta^{(i)} \beta^{(i)} + \sqrt{X^{(i)}_t N^{(i)}},
\]

with \( X^{(i)}_t \sim GIG\left(-\frac{1}{2}, t\delta^{(i)}, \gamma^{(i)}\right) \).
Corollary 3.4.7 (Increasing convex comparison of NIG processes).

Let \( S_t^{(i)} \) be \( \text{NIG}(d, \alpha^{(i)}, \beta^{(i)}, \delta^{(i)}, \mu^{(i)}, \Delta^{(i)}; t) \) processes. If

\[
\delta^{(1)} \leq \delta^{(2)}, \quad \alpha^{(1)} \geq \alpha^{(2)}, \quad \mu^{(1)} \leq \mu^{(2)}
\]

and one of the cases (3.37)–(3.39) holds true for \( \beta^{(i)} \) and \( \Delta^{(i)} \), then

\[
(S_t^{(1)}) \leq_{\text{icx}} (S_t^{(2)}).
\]

Proof. As in Theorem 3.4.6 it follows from the ordering conditions that

\[
S_t^{(1)} \leq_{\text{icx}} S_t^{(2)}, \quad \forall t \in [0, T],
\]

hence finite-dimensional ordering \( (S_t^{(1)}) \leq_{\text{icx}} (S_t^{(2)}) \). \( \Box \)
Appendix A

Appendix

A.1 Stochastic analysis

In Lemma A.1.1 the relationship between stochastic and ordinary exponential in terms of characteristics is established. This is a multivariate extension of Goll and Kallsen (2000, Lemma A.8) and Jacod and Shiryaev (2003, Theorem II.8.10). For a truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ we denote by $X \sim (B, C, \nu)_h$ that $X$ has drift characteristic $B = B(h)$, that depends on the truncation function $h$, Gaussian characteristic $C$ and jump characteristic $\nu$ (cp. Jacod and Shiryaev (2003, Definition II.2.6)). Lemma A.1.1 in particular implies that $X = \text{Log}(S)$ is PII iff $\bar{X} = \log(S)$ is PII, as the characteristics of $X$ are deterministic iff the characteristics of $\bar{X}$ are deterministic.

**Lemma A.1.1.** 1. Let $X \sim (B, C, \nu)_h$ be a $d$-dimensional semimartingale and $\bar{X} := \log \mathcal{E}(X)$. Then the characteristics $(\bar{B}, \bar{C}, \bar{\nu})_h$ of $\bar{X}$ are given by

\[
\bar{B} = B - \frac{\text{diag}(C)}{2} + (h(\log(1 + x)) - h(x)) \ast \nu,
\]

\[
\bar{C} = C,
\]

\[
\bar{\nu}([0, t] \times G) = \int_{[0, t] \times (-1, \infty)^d} \mathbf{1}_G(\log(x + 1))\nu(du, dx), \quad G \in \mathcal{B}^d,
\]

where $\text{diag}(C) := (C^{11}, \ldots, C^{dd})$ is the diagonal of the matrix $C$.

2. Let $X \sim (\bar{B}, \bar{C}, \bar{\nu})_h$ be a $d$-dimensional semimartingale and let $X := \log(S)$ be a one-dimensional Brownian motion.
\(\log(e^X)\). Then the characteristics \((B,C,\nu)_h\) of \(X\) are given by

\[
B = \bar{B} + \frac{\text{diag}(\bar{C})}{2} + (h(e^x - 1) - \bar{h}(x)) \ast \bar{\nu},
\]

\[
C = \bar{C},
\]

\[
\nu([0,t] \times G) = \int_{[0,t] \times \mathbb{R}^d} 1_G(e^x - 1)\bar{\nu}(du, dx), \quad G \in \mathcal{B}((-1, \infty)^d).
\]

**Proof.** As the multivariate (stochastic) exponential (resp. logarithm) is defined componentwise, the arguments of the proof of the univariate case apply to every component of the multivariate characteristics and the result follows from

\[
\bar{X}^i = X^i - X_0^i - \frac{1}{2} \langle X^c, X^c \rangle - \frac{1}{2} (\log(1 + x^i) - x^i) \ast \mu^{X^i}.
\]

The appearance of the diagonal of \(C\) in the drift part \(\bar{B}\) is due to that representation. The proof of part 2 is similar to that of the first part. \(\square\)

**Lemma A.1.2.** 1. Let \(X \sim (B, C, \nu)_h\) be a \(d\)-dimensional semimartingale with \(X_0 = 0\) and \(\Delta X > -1\). Then \(S := \mathcal{E}(X) \sim (B^S, C^S, \nu^S)_hS\) is a positive \(d\)-dimensional semimartingale with characteristics

\[
B^S_{t,i} = \int_{[0,t]} S^i_u dB^S_{i,u} + \int_{[0,t] \times (-1, \infty)^d} (h^{S,i}(S_{u-} - x^i) - S^i_{u-} h(x^i)) \nu(du, dx),
\]

\[
C^S_{t,ij} = \int_{[0,t]} S^i_u - S^j_u - dC^S_{ij},
\]

\[
\nu^S([0,t] \times G) = \int_{[0,t] \times (-1, \infty)^d} 1_G(S_{u-} x) \nu(du, dx), \quad G \in \mathcal{B}^d.
\]

2. Let \(S \sim (B^S, C^S, \nu^S)_h\) be a \(d\)-dimensional semimartingale with \(S_0 = 1\) and \(S, S_- > 0\). Then \(X := \log S \sim (B, C, \nu)_h\) is a \(d\)-dimensional semimartingale with \(\Delta X > -1\) and

\[
B^i_t = \int_{[0,t]} \frac{1}{S^i_{u-}} dB^S_{S^i,u} + \int_{[0,t] \times \mathbb{R}^d} \left\{ h^i \left( \frac{s}{S^i_{u-}} \right) - \frac{1}{S^i_{u-}} h^{S,i}(s) \right\} \nu^S(du, ds),
\]

\[
C^{ij}_t = \int_{[0,t]} \frac{1}{S^i_{u-} S^j_{u-}} dC^S_{S^i,u},
\]

\[
\nu([0,t] \times G) = \int_{[0,t] \times \mathbb{R}^d} 1_G \left( \frac{s}{S^i_{u-}} \right) \nu^S(du, ds), \quad G \in \mathcal{B}^d.
\]
Proof. 1. This is proved in Goll and Kallsen (2000, Example 4.3) in the case where \( X \) is a \( d \)-dimensional Lévy process. The proof for semimartingales is similar.

2. As \( S = \mathcal{E}(X) \) is defined componentwise, \( X \) is given as \( X^i = \frac{1}{s^i} \cdot S^i, i \leq d \). For the Gaussian characteristic it follows that \( C^{ij} = \langle X^i, X^j \rangle = \frac{1}{s^i} \cdot \frac{1}{s^j} \cdot C^{S,i,j} \). To compute the jump compensator let \( f : (0, \infty)^d \to \mathbb{R}^d \) be defined by \( f(s) = (\log(s^1), \ldots, \log(s^d))^T \). Itô’s Lemma for characteristics (see e.g. Goll and Kallsen (2000, corollary A.6)) implies \( \mathcal{I}([0, t] \times G) = \int_{[0, t] \times \mathbb{R}^d} \mathbf{1}_G \left( \log \left( \frac{S^{u+} - s_i}{S^{u-}} \right) \right) \nu^S(du, ds) \). From part 2 of Lemma A.1.1 we obtain \( \nu([0, t] \times G) = \int_{[0, t] \times \mathbb{R}^d} \mathbf{1}_G \left( \frac{s_i}{S^{u-}} \right) \nu^S(du, ds) \).

It remains to compute \( B_t \). In the same manner as for the jump compensator we use Itô’s Lemma to obtain for \( G \in \mathcal{B}^d \) and \( i \leq d \)

\[
\tilde{B}^i_t = \int_{[0, t]} \frac{1}{S^{u-}_i} dB^{S,i}_u - \frac{1}{2} \int_{[0, t]} \left( \frac{1}{S^{u-}_i} \right)^2 dC^{S,i,i}_u + \int_{[0, t] \times \mathbb{R}^d} \left\{ \tilde{h}^i \left( \log(S^{u-} + s) - \log(S^{u-}) \right) - \frac{1}{S^{u-}_i} h^{S,i}(s) \right\} \nu^S(du, ds).
\]

Applying part 2 of Lemma A.1.1 this yields after a bit of calculus

\[
B^i_t = \int_{[0, t]} \frac{1}{S^{u-}_i} dB^{S,i}_u + \int_{[0, t] \times \mathbb{R}^d} \left\{ h^i \left( \frac{s}{S^{u-}_i} \right) - \frac{1}{S^{u-}_i} h^{S,i}(s) \right\} \nu^S(du, ds) \quad \square
\]

Proof of Lemma 1.1.1 (Kolmogorov backward equation). Assume that \( |W^*_t|^{S^*} \in \mathcal{A}_{loc}^+ \) and let \( t \in (0, T] \). Similar to the proof of Theorem 1.1.3, Itô’s lemma and Lemma A.1.2 imply that \( \mathcal{H}(t, S^*_t) \) is a semimartingale with evolution \( \mathcal{H}(t, S^*_t) = \mathcal{H}(0, 1) + M^*_t + M^{**,*}_t + A^*_t \), where \( M^*, M^{**} \) are local \((\mathcal{A}_t)\)-martingales under \( Q^* \) and

\[
A^*_t := \int_{[0, t]} \left\{ D^i \mathcal{H}(u, S^*_u) + \frac{1}{2} \sum_{i,j \leq d} D^2_{ij} \mathcal{H}(u, S^*_u) S^{*i}_{u-c^*u}(u, S^*_u) \right\} du + \int_{(-1, \infty)^d} (\Lambda \mathcal{H})(u, S^*_{u-}, x) K_{u}^*(S^*_{u-}, dx) du.
\]

is predictable and of finite variation. As \( \mathcal{H}(t, S^*_t) \) is a local \((\mathcal{A}_t^*)\)-martingale it follows by the uniqueness of the representation of a special semimartingale,
that $A^*_t$ is a predictable local martingale with finite variation starting at zero and, therefore, is indistinguishable from zero. Hence there is a $Q^*$-null set $N^*$, s.th. $A^*_t(\omega^*) = 0$, for all $t \in [0, T]$ and all $\omega^* \in N^c$.

It remains to establish that $\mathcal{H}(t, \cdot) \in \mathcal{F}_{\text{ex}}$ implies the integrability condition $|W^*| \ast \mu^{S^*} \in A^*_{\text{loc}}$. As $\mathcal{H}(t, S^*_n)$ is a local martingale under $Q^*$, it is a special semimartingale. The process

$$\int_{[0,t]} D_t \mathcal{H}(u, S^*_{u-}) du + \frac{1}{2} \sum_{i,j \leq d} \int_{[0,t]} D^2_{ij} \mathcal{H}(u, S^*_{u-}) dC^S_{ij}$$

is of finite variation and predictable and therefore is in $A_{\text{loc}}$. Due to a representation result for special semimartingales (see Jacod and Shiryaev (2003, Proposition I.4.23)) it follows that $W^* \ast \mu^{S^*} \in A_{\text{loc}}$. Convexity of $\mathcal{H}(t, \cdot)$ implies $W^*(\omega^*, t, s) \geq 0$. Therefore, $|W^*| \ast \mu^{S^*} = W^* \ast \mu^{S^*} \in A^*_{\text{loc}}$. \hfill \Box

**Lemma 2.2.1 (Functional weak convergence).** Let $S \sim (b(h), 0, F)_h$ be a $d$-dimensional Lévy process whose Lévy measure $F$ has infinite total mass and for $\varepsilon_n \uparrow 0$, $\tau_n \downarrow 0$ let $F_n$ be the corresponding truncated Lévy measure given in (2.9). If $b_n(h) \to b(h)$ then for the compound Poisson processes $S_n \sim (b_n(h), 0, F_n)_h$ functional weak convergence

$$S_n \xrightarrow{\mathcal{L}} S$$

holds true.

**Proof.** We establish convergence of $\tilde{c}_n(h)$ and $F_n(g)$, $g \in C_2(\mathbb{R}^d)$, where the modified second characteristic $\tilde{c}_n(h)$ is given by $\tilde{c}_n^k(h) = \int h^k(x) h^l(x) F_n(dx)$ and $C_2(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{R} : f \text{ is bounded, continuous and 0 around 0} \}$. Then the result follows from Jacod and Shiryaev (2003, Corollary VII.3.6). From $h^k(x) h^l(x) \mathbb{I}_{\{ |x| < 1 \}}(x) \to h^k(x) h^l(x)$ and $\int_{|x| < 1} |x|^2 F(dx)$ it follows that $\tilde{c}_n \to \tilde{c}$, due to the Lebesgue Theorem. As $g \in C_2(\mathbb{R}^d)$ is zero around the origin, there is a $N \in \mathbb{N}$ s.th. $F_n(g) = \int g(x) \mathbb{I}_{\{ |x| > N \}}(x) F(dx) = \int g(x) F(dx) = F(g)$, for all $n \geq N$. \hfill \Box

**Corollary 2.2.2 (Functional weak convergence).** Let $F$ be a Lévy measure with infinite total mass and for sequences $\varepsilon_n \uparrow 0$, $\tau_n \downarrow 0$ let $F_n$ be the corresponding truncated Lévy measure given in (2.9).

1. If $\int_{|x| > 1} |x|^2 F(dx) < \infty$, then for $S \sim (ES_1, 0, F)_{id}$ and $S_n \sim (ES_1, 0, F_n)_{id}$ it holds true that $S_n \xrightarrow{\mathcal{L}} S$. 

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2. If for a continuous truncation function $h$ it holds true that $\int |h(x)|F(dx) < \infty$, then for $S \sim (b(0), 0, F)_0$ and $S_n \sim (b(0), 0, F_n)_0$ it follows that $S_n \xrightarrow{\text{w}} S$.

Proof. 1. As $b_n(h) = ES_1 + \int (h(x) - x)F_n(dx)$ and $b(h) = ES_1 + \int (h(x) - x)F(dx)$, there is a $N_0 \in \mathbb{N}$ such that $b_n(h) = b(h)$ for all $n \geq N_0$. The result follows from Lemma 2.2.1.

2. An extension of Jacod and Shiryaev (2003, Proposition II.2.24) to a “truncation function” $h' \equiv 0$ implies $b(h) = b(0) + \int h(x)F(dx)$ and $\tilde{C}(h) = \tilde{C}(0) + (h'h^j \ast \nu)$, similar for the truncated characteristics $b_n(h)$, $\tilde{C}_n(h)$. As $\tilde{C}(0) = C = 0$ it remains to establish $b_n(h) \to b(h)$, then the result follows from Lemma 2.2.1. As $b_n(0) = b(0)$ and $h(x)\mathbb{1}_{(\xi_{s,n}, \tau_{s,n})}(x) \to h(x)$ this follows from the Lebesgue Theorem.

Lemma 2.3.4 (Functional weak convergence, PII case). For a continuous truncation function $h$ let $S \sim (b(t; h), 0, K(t; \cdot))_h$ be a $d$-dimensional Lévy process whose Lévy kernel $K(t; \cdot)$ has infinite total mass and for $\tilde{\xi}_{t,n} \uparrow 0$, $\tau_{t,n} \downarrow 0$ let $K_n(t; \cdot)$ be the corresponding truncated Lévy kernel introduced in (2.16). If $\sup_{s,t} |B_{n,s} - B_s| \to 0$, for all $t \in [0, T]$, then for $S_n \sim (b_n(t; h), 0, K_n(t; \cdot))_h$ functional weak convergence

$$S_n \xrightarrow{\text{w}} S$$

holds true.

Proof. Similar to the proof of Lemma 2.2.1 we establish appropriate convergence of the characteristics to obtain the result from Jacod and Shiryaev (2003, Theorem VII.3.4). Let $t \in [0, T]$ and $(x, s) \in \mathbb{R}^d \times [0, t]$. As $\xi_{s,n} \uparrow 0$, $\tau_{s,n} \downarrow 0$, it holds true that $|h^k(x)h^l(x)|\mathbb{1}_{(\xi_{s,n}, \tau_{s,n})}(x) \to 0$, hence it follows from the Lebesgue Theorem that

$$\tilde{C}_{n,t}^{k,l} = \int_0^t \int h^k(x)h^l(x)\mathbb{1}_{(\xi_{s,n}, \tau_{s,n})}(x)K(s; dx)ds$$

$$\to \int_0^t \int h^k(x)h^l(x)K(s; dx)ds = \tilde{C}^{k,l}.$$  

For $g \in C_1(\mathbb{R}^d) := \{f \in C_2(\mathbb{R}^d) : f \geq 0\}$, where the function classes $C_i(\mathbb{R}^d)$ are given in Jacod and Shiryaev (2003, VII.2.7), it holds true that $g$ is zero around the origin. For $t \in [0, T]$ let $(x, s) \in \mathbb{R}^d \times [0, t]$. As $\xi_{s,n} \uparrow 0$, $\tau_{s,n} \downarrow 0$, it holds true that there is a $N \in \mathbb{N}$ s. th. $g(x)\mathbb{1}_{(\xi_{s,n}, \tau_{s,n})}(x) = 0$, hence $g(x)\mathbb{1}_{(\xi_{s,n}, \tau_{s,n})}(x) - g(x) = 0$, for all $n \geq N$. As $g \in C_1(\mathbb{R}^d)$ is bounded, it follows from the Lebesgue Theorem that $g \ast \nu_{n,t} \to g \ast \nu_t$.  

\qed
Corollary 2.3.5 (Functional weak convergence, PII case). Let $K(t; \cdot)$ be a Lévy kernel that has infinite total mass for all $t \in [0, T]$, and for sequences $\varepsilon_{t,n} \uparrow 0$, $\varepsilon_{t,n} \downarrow 0$ let $K_n(t; \cdot)$ be the corresponding truncated Lévy kernel given in (2.16).

1. If $|x|^2 \ast \nu_t < \infty$, then for $S \sim (ES_t, 0, K(t; \cdot))_\text{id}$ and $S_n \sim (ES_t, 0, K_n(t; \cdot))_\text{id}$ it holds true that $S_n \xrightarrow{\mathcal{L}} S$.

2. If for a continuous truncation $h$ it holds true that $\int h(x)K(t; dx) < \infty$, for all $t \in [0, T]$, then for $S \sim (b(t; 0), 0, K(t; \cdot))_0$ and $S_n \sim (b(t; 0), 0, K_n(t; \cdot))_0$ it follows that $S_n \xrightarrow{\mathcal{L}} S$.

Proof. 1. As $B_n(s; \text{id}) = ES_n = B(s; \text{id})$, the result follows similar to the proof of Lemma 2.3.4 from Jacod and Shiryaev (2003, Theorem VII.3.7), which is the square-integrable version of Jacod and Shiryaev (2003, Theorem VII.3.4).

2. The result follows similar to the proof of Corollary 2.2.2. As for the modified second characteristics it holds true that $\tilde{C}_n = \tilde{C} = C = 0$ it remains to establish $\sup_{s \leq t} |B_n(s; h) - B(s; h)| \to 0$, for a continuous truncation function $h$. Observe that $B_n(s; 0) = B(s; 0)$. As for $s \leq t$ it holds true that

$$|B_n(s; h) - B(s; h)| = |B_n(s; 0) + h \ast \nu_{n,s} - B(s; 0) - h \ast \nu_s|$$

$$\leq \int_{[0,s] \times \mathbb{R}^d} |h(x)I_{\{\varepsilon_{n,s} \leq n\}}(x)|\nu(du, dx)$$

$$\leq \int_{[0,t] \times \mathbb{R}^d} |h(x)I_{\{\varepsilon_{n,s} \leq n\}}(x)|\nu(du, dx),$$

the result follows from the Lebesgue Theorem, as $|h| \ast \nu_t < \infty$.

Theorem 2.3.1 (Norberg (1993)). Let $S \sim (0, 0, K(s; dy))_0$ be a d-dimensional PII with $\lambda(s) := K(s; \mathbb{R}^d) < \infty$ and define $\Lambda(t) := \int_0^t \lambda(s)ds$. Let $Y_t$ be a random sum process that is defined by

$$Y_t = \sum_{j=1}^{\tilde{N}_t} \tilde{X}_{t,j}, \quad t \in [0, T],$$

where the extended Poisson process $\tilde{N}_t \sim \mathcal{P}(\Lambda(t))$ is independent of the iid sequence $(\tilde{X}_{t,j}) \sim R_t$, $R_t(dy) = \frac{1}{\Lambda(0)} \int_0^t K(s; dy)ds$.

Then for all $t \in [0, T]$ it holds true that $S_t \overset{d}{=} Y_t$. 

\[\boxdot\]
Proof. For $t \in [0,T]$ we obtain
\[
Ee^{i\langle z,Y_t \rangle} = \sum_{n=0}^{\infty} P(N_t = n)Ee^{i\langle z, \sum_{j=1}^{n} \tilde{X}_{t,j} \rangle} = \frac{1}{n!} e^{-\Lambda(t)} \sum_{n=0}^{\infty} \left( \int_0^t \lambda(s)ds \int e^{i\langle z,x \rangle} \mathcal{R}_t(dx) \right)^n = \exp \left\{ \int_0^t \int (e^{i\langle z,x \rangle} - 1) K(s,dx)ds \right\},
\]
and the result follows from the Lévy-Khintchine type formula for PII in Jacod and Shiryaev (2003, II.4.16).

A.2 Stochastic orders

We list several properties of integral stochastic orders we use in this thesis, cp. Müller and Stoyan (2002, Definitions 2.4.1, 3.2.1)

Property (T) if $U \leq_F U + a$ holds for all r.v. $U$ and all positive $a$.

Property (C) if $P^{(1)} \leq_F P^{(2)}$ implies $P^{(1)} * Q \leq_F P^{(2)} * Q$ for all probability measures $Q$.

Property (MI) if $P_\theta \leq_F Q_\theta$ for all $\theta \in \Theta$ implies $P \leq_F Q$, where $P = \int P_\theta \mu(d\theta)$ and $Q = \int Q_\theta \mu(d\theta)$ for some measure $\mu$ on $\Theta$.

Property (W) if $\leq_F$ is closed with respect to weak convergence.

For random vectors $U, V, U^{(i)}, V^{(i)}$, multivariate orders $\leq_F$ satisfy

Property (MA) if $U \leq_F V$ implies $U_K \leq_F V_K$, for all $K \subset \{1, \ldots, d\}$.

Property (ID) if $U \leq_F V$ implies $(U_K, U_L) \leq_F (V_K, V_L)$ for all $K, L \subset \{1, \ldots, d\}$.

Property (IN) if $U^{(i)} \leq_F V^{(i)}$ for $i = 1, 2$ implies $P^{U^{(1)}} \otimes P^{U^{(2)}} \leq_F P^{V^{(1)}} \otimes P^{V^{(2)}}$.

Remark A.2.1. We formulate the property (IN) in terms of the product measure, whereas Müller and Stoyan (2002, Definition 3.2.1) give a formulation in terms of the joint distributions $P^{U^{(1)} U^{(2)}}, P^{V^{(1)} V^{(2)}}$ with independent $U^{(i)}$ and $V^{(i)}$, respectively. Our definition is more useful for this thesis and in fact is the property that is established in Müller and Stoyan (2002), cp. also Li and Xu (2000, Proposition 2.6).

In the next lemma we list some properties the orders $\leq_F$ that are generated by $\mathcal{F}$ in (1). These results are given e.g. in Müller and Stoyan (2002).
Lemma A.2.2 (Properties of \( \leq_F \)).

The stochastic order \( F_{st} \) satisfies all properties given above.

The convex type orders \( F_{cx}, F_{dcx}, F_{sm} \) satisfy the properties (C), (MI), (MA), (ID) and (IN).

The monotone convex type orders \( F_{icx}, F_{idcx}, F_{ism} \) additionally satisfy property (T).

The next lemma shows that the distribution of a compound Poisson process \( S \) is invariant w.r.t. modification of the corresponding Lévy measure.

**Lemma A.2.3.** Let \( S \sim (b(0), 0, F)_0 \) and \( \tilde{S} \sim (b(0), 0, \tilde{F})_0 \) be \( d \)-dimensional compound Poisson processes, where \( \tilde{F}(dx) := F(dx) + a\delta_{\{0\}}(dx) \), \( a \in \mathbb{R}_+ \), is the modified Lévy measure of the finite Lévy measure \( F \). Then for \( t \in [0, T] \) it holds true that \( S_t \overset{d}{=} \tilde{S}_t \).

**Proof.** For \( t \in [0, T] \) the characteristics functions of \( S_t \) and \( \tilde{S}_t \) are identical:

\[
Ee^{i\langle u, S_t \rangle} = \exp \left\{ t \int (e^{i\langle u, x \rangle} - 1) F(dx) \right\} \\
= \exp \left\{ t \int (e^{i\langle u, x \rangle} - 1) \tilde{F}(dx) - at \int (e^{i\langle u, x \rangle} - 1) \delta_{\{0\}}(dx) \right\} \\
= \exp \left\{ t \int (e^{i\langle u, x \rangle} - 1) \tilde{F}(dx) \right\} = Ee^{i\langle u, \tilde{S}_t \rangle}
\]


