Families of canonically polarized manifolds over log Fano varieties

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Introduction

Shafarevich's conjecture

During the past decades, the study of moduli spaces has been a very active field of research in algebraic geometry. It is a natural question whether non-trivial maps $X \to \mathfrak{M}$ from a given variety X to a moduli space $\mathfrak{M}$ exist. If $\mathfrak{M}$ is a fine moduli space, then by the universal property of $\mathfrak{M}$ this is equivalent to the study of families over X. If on the other hand $\mathfrak{M}$ is only a coarse moduli space, then there exists no universal family, and a map $X \to \mathfrak{M}$ does not necessarily induce a family over X. In this situation, we ask for maps $X \to \mathfrak{M}$ which are induced by families over X.

One famous theorem which describes the geometry of the moduli space of curves of fixed genus $g \geq 2$ was conjectured by Shafarevich in 1962, see [Sha63]. It was proved by Parshin in [Par68], and by Arakelov in [Ara71]. We state one part of this result in the following theorem.

Theorem. *Let $f : \mathfrak{X} \to Y$ be a smooth projective family of complex curves of genus $g \geq 2$. Then all members of the family are isomorphic, if Y is one of the following:*

- *the projective line $\mathbb{P}^1_{\mathbb{C}}$,*
- *the affine line $\mathbb{A}^1_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} \setminus \{point\}$,*
- *the affine line minus one point $\mathbb{A}^1_{\mathbb{C}} \setminus \{point\}$, or*
- *an elliptic curve.* $\square$

In particular, if a map from a rational or elliptic curve to the coarse moduli space of genus g curves is induced by a family, then this map is constant. For a survey of related results, we refer to [Keb11] and [Kov09].

In this thesis, we consider the coarse moduli space of *canonically polarized projective mannifolds* with fixed Hilbert polynomial, see [Vie95]. This space was studied by Viehweg-Zuo in [VZ02], and by Kebekus-Kovács in [KK08] and [KK10].

The result of Kebekus and Kovács [KK10, Theorem 1.2] will be essential in this thesis, but before explaining it explicitly, let us consider the following construction.

Example. Let $f : \mathfrak{X} \to C$ be a smooth family of curves of general type over a curve which is not isotrivial, i.e., the fibers are not all isomorphic. Then the surface $Y := C \times \mathbb{P}^1$ parametrizes a non-isotrivial family, namely the pullback of f along the projection $Y \to C$.

On the other hand, Shafarevich's conjecture directly implies that any smooth family of curves of general type is isotrivial in the $\mathbb{P}^1$-direction, hence a family over Y does not vary "too much".

This example motivates the following definition, we also refer to [Keb11, Definition 2.5] for a more general definition.

Definition. *The* variation *of a family $f : \mathfrak{X} \to Y$ is the dimension of the image of the induced moduli map μ.*

It is clear that the variation takes values between 0 and $\dim Y$. In the example given above, the variation is one, so it is not maximal. We replace the question of whether maps from a variety to the moduli space of canonically polarized manifolds exist by the following one.

> Let $f : \mathfrak{X} \to Y$ be a family of canonically polarized manifolds. Does information about the base Y give information about the variation of f?

Results by Kebekus-Kovács, and Viehweg-Zuo

As mentioned above, an essential step toward an answer to this question has been made by Viehweg and Zuo in [VZ02]. They showed that the existence of families of positive variation is related to the existence of certain sheaves of pluri-log forms.

Kebekus and Kovács use these *Viehweg-Zuo sheaves* in [KK10] to relate the induced moduli map to the minimal model program. As a corollary they conclude that the variation of a smooth projective family of canonically polarized varieties over a quasi-projective manifold Y with $\dim Y \leq 3$ is bounded by its Kodaira-Iitaka-dimension $\kappa(Y)$. More precisely, they distinguish the following two cases:

- If $\kappa(Y) \geq 0$, then $\mathrm{Var}(f) \leq \kappa(Y)$.
- If $\kappa(Y) = -\infty$, then $\mathrm{Var}(f) < \dim(Y)$.

Note that the curves given in Shafarevich's conjecture are exactly the curves which do not have maximal Kodaira-Iitaka dimension. Consequently, this is a generalization of Shafarevich's conjecture.

The main result

In this thesis we focus on the case $\kappa(Y) = -\infty$. For example manifolds of type $Z \times \mathbb{P}^1$ belong to this class for any manifold Z, and therefore, as the example given above shows, the estimate $\mathrm{Var}(f) < \dim(Y)$ is generally optimal in this case. We therefore have to make additional assumptions.

Note that a family restricted to a rational curve is necessarily isotrivial, as Kebekus' and Kovács' result shows; see also [Kov00, 0.2]. Consequently, there is no smooth projective family of canonically polarized varieties with positive variation over a rationally connected manifold. An important class of such manifolds is the class of Fano manifolds. In particular, a family over a Fano manifold is isotrivial.

We will consider the larger class of log Fano varieties. More precisely, if (X, Δ) is a dlt pair with X projective, then we call (X, Δ) *log Fano* if the divisor $-(K_X + \Delta)$ is $\mathbb{R}$-ample. The main result is the following:

Isotriviality Theorem (See Theorem 5.1)**.** *Let (X, Δ) be a dlt pair where Δ is an effective $\mathbb{R}$-divisor, where $-(K_X + \Delta)$ is $\mathbb{R}$-ample, and X is projective. Let $T \subset X$ be a subvariety of $\mathrm{codim}_X(T) \geq 2$ such that $X \setminus (T \cup \mathrm{Supp}\lfloor\Delta\rfloor)$ is smooth. Then any smooth family of canonically polarized varieties over $X \setminus (T \cup \mathrm{Supp}\lfloor\Delta\rfloor)$ is isotrivial.*

It is still an open question if log Fano varieties are rationally connected by curves that intersect Δ in at most two points. Therefore, the short line of argument given above to show that families over Fano manifolds are isotrivial does not apply.

Instead we will prove the Isotriviality Theorem by induction on the dimension of the variety X. As part of the induction we show Kebekus' and Kovács' result [KK10, Theorem 1.2] for varieties of arbitrary dimension and with negative Kodaira-Iitaka-dimension.

The rough idea is the following: Let $\dim X = n$ and consider the induced map from the base $X \setminus (T \cup \mathrm{Supp}\lfloor\Delta\rfloor)$ to the coarse moduli space of canonically polarized varieties. If Kebekus' and Kovács' result holds for n-dimensional vaieties, then a run of the minimal model program for the pair (X, Δ) terminates with a Mori fiber space and factorizes the moduli map birationally. The ampleness of $-(K_X + \Delta)$ implies that there are sufficiently many such minimal model programs, which implies the Isotriviality Theorem for n-dimensional varieties.

On the other hand, the Isotriviality Theorem in dimension n, and the recently proven Bogomolov-Sommese vanishing for lc pairs [GKKP10, Theorem 7.2] imply Kebekus' and Kovács' result for $(n+1)$-dimensional varieties of negative Kodaira-Iitaka-dimension. This completes the proof.

Moving curves

To apply Kebekus' and Kovács' result in a reasonable way, we explore the different types of minimal model programs that lead to a Mori fiber space.

For $\mathbb{Q}$-factorial *klt* pairs, Araujo proved in [Ara10] that to each $(K_X + \Delta)$-negative exposed ray of the cone $\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}$ we can associate a minimal model program that terminates with a Mori fiber space and "contracts" the ray. We prove a generalization to the dlt case.

Moving Cone Theorem (See Theorem 3.5)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair, where Δ is an effective $\mathbb{R}$-divisor, and X is projective. Let R be an exposed ray of the cone $\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}$ that intersects $K_X + \Delta$ negatively. Then there is an irreducible locally closed subset H_R of the Hilbert scheme of curves on X such that*

1. *each closed point of H_R corresponds to a curve that generates R,*

2. *for any closed subset $Z \subset X$ of $\operatorname{codim}_X(Z) \geq 2$, there is a non-empty open subset H_R^Z of H_R such that any curve that corresponds to a closed point of H_R^Z avoids Z,*

3. *there exists a run of the minimal model program with scaling that terminates with a Mori fiber space*

$$\begin{tikzcd} X \arrow[r, dashed, "\lambda_R"] & X_R \arrow[d, "\pi_R"] \\ & B_R, \end{tikzcd}$$

such that any closed point of H_R corresponds to a curve that is contained in the open set $U \subset X$, where λ_R is an isomorphism of U onto its image. Moreover, the image of this curve via λ_R is contained in a fiber of π_R.

$\mathbb{Q}$-factorializations

Our results concerning the minimal model program require the $\mathbb{Q}$-factoriality of the underlying variety X. In order to get rid of this assumption, we must pass to a $\mathbb{Q}$-*factorialization* of X, i.e., a proper birational morphism $f : Y \to X$ such that Y is $\mathbb{Q}$-factorial projective, and f does not contract any divisors. In this situation the groups of Weil divisors are isomorphic, but any $\mathbb{R}$-Weil-divisor on X becomes $\mathbb{R}$-Cartier on Y. Moreover, if we are given a dlt pair (X, Δ), then a $\mathbb{Q}$-factorialization Y of X exists, and if Δ_Y denotes the strict transform of Δ, then the pair (Y, Δ_Y) is dlt again.

Unfortunately, a $\mathbb{Q}$-factorialization of a log Fano variety is generally not log Fano, and this causes a problem in the proof of our main theorem. However, the $\mathbb{Q}$-factorialization still has good properties, which allow us to use flops to obtain new $\mathbb{Q}$-factorializations. Using these flops to pass from one $\mathbb{Q}$-factorialization to another, we eventually obtain a $\mathbb{Q}$-factorialization suitable for our purposes.

Outline of thesis

We will start with a preliminary chapter on the minimal model program. Beside the standard minimal model program which is also discussed in our main reference [KM98] we will prove generalizations for $\mathbb{R}$-divisors.

The second chapter gives an introduction to the minimal model program with scaling. We will generalize a result of [BCHM10] from the klt to the dlt case. In particular, a minimal model program with scaling for $\mathbb{Q}$-factorial dlt pairs (X, Δ) terminates with a Mori fiber space, provided $K_X + \Delta$ is not pseudo-effective.

With the result of the previous chapters at hand we can prove the Moving Cone Theorem 3.5. This will be done in Chapter 3. The proof uses methods from Araujo's proof, in particular the construction of the divisor for which we run the minimal model program with scaling is very similar. However, the argument that this divisor actually gives the right minimal model program is different from Araujo's. A central result from the second chapter asserts that a minimal model program with scaling of H for a specific pair (X, Δ) is indeed a minimal model program with scaling for the pair $(X, \Delta + \varepsilon H)$, for any sufficiently small ε. It turns out that this result already implies the Moving Cone Theorem.

In Chapter 4 we turn our attention to $\mathbb{Q}$-factorializations. For each effective Weil divisor D on a log Fano dlt pair we construct a $\mathbb{Q}$-factorialization (Y, Δ_Y) such that the strict transform of D is not numerically trivial on all $(K_Y + \Delta_Y)$-negative exposed rays of the cone $\overline{\mathrm{NM}}_1(Y) + \overline{\mathrm{NE}}_1(Y)_{K_Y+\Delta\geq 0}$.

The proof of the Isotriviality Theorem is given in Chapter 5 by induction, and the last Chapter 6 shows that the Isotriviality Theorem can be used to obtain a description of the moving cone of $\mathbb{Q}$-factorial dlt pairs.

Notations and conventions

We briefly fix some notation which will regularly be used in this thesis.

- Throughout this thesis we work over the field $\mathbb{C}$ of complex numbers.
- We denote the intersection product of an $\mathbb{R}$-Cartier divisor D with a curve C by $D \cdot C \in \mathbb{R}$.
- Let X be a normal projective variety and $k \in \{\mathbb{Q}, \mathbb{R}\}$. We denote the group of k-Weil divisors by $\mathrm{WDiv}_k(X)$, and of k-Cartier divisors by $\mathrm{Div}_k(X)$. We call X *$\mathbb{Q}$-factorial*, if $\mathrm{WDiv}_{\mathbb{Q}}(X) = \mathrm{Div}_{\mathbb{Q}}(X)$ holds.
- Let $k \in \{\mathbb{Q}, \mathbb{R}\}$. We denote k-linear equivalence of divisors by $\sim_k$.
- The *support* of an $\mathbb{R}$-divisor $D = \sum_i a_i D_i$ is defined as $\operatorname{Supp} D = \bigcup_i D_i$, where D_i are distinct prime divisors, and $a_i \neq 0$.

- The *round down* of an $\mathbb{R}$-divisor $D = \sum_i a_i D_i$ is defined as

$$\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i,$$

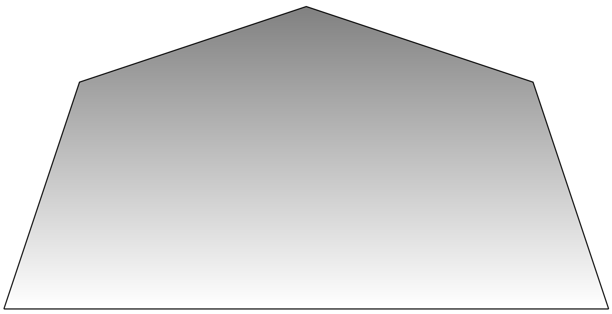
where $\lfloor a_i \rfloor$ denotes the usual round down of a real number, and D_i are distinct prime divisors.

- Given a subset $S \subset V$ of a finite dimensional real vector space V and a linear form $0 \neq \alpha \in V^\vee$, we set

$$S_{\alpha \leq 0} := \{v \in S \,|\, \alpha(s) \leq 0\}, \quad S_{\alpha = 0} := \{v \in S \,|\, \alpha(s) = 0\}, \quad \text{and so forth.}$$

We denote by $\alpha^\perp \subset V$ the hyperplane defined by the kernel of α.

- Pictures of cones, which do not contain a line, always show a cross section of the cone. For instance, the picture



visualizes a cone in $\mathbb{R}^3$ whose origin is located behind the drawing plane.

- Let $f : X \dashrightarrow Y$ be a rational map. We denote pullback and pushforward by f^* and f_*, respectively. It will be clear from the context if we take pulback, respectively pushforward, of (classes of) divisors or classes of curves.

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Chapter 1

The standard minimal model program

In this chapter we will recall some facts which we will often use without mentioning them explicitly. We assume that the reader is familiar with basic algebraic geometry, as it can be found in [Har77]. For a complete introduction to the minimal model program we refer to [KM98].

1.1 Convex geometry

In the sequel, we will have to analyze the geometry of certain convex cones in $N_1(X)$ and $N^1(X)$, respectively. We will therefore need some basic notation and results of convex geometry. We also refer to the book by Barvinok [Bar02]. Recall the following definitions.

Definition 1.1. *Let V be a finite dimensional real vector space. A subset $\emptyset \neq \mathcal{C} \subset V$ is called a* convex cone *if it is closed under vector addition, and under multiplication by non-negative scalars. A cone is called* strict convex *if it does not contain a line.*

Given a cone $\mathcal{C} \subset V$ we define its dual cone *$\mathcal{C}^\vee \subset V^\vee$ as*

$$\{\lambda \in V^\vee \mid \lambda(v) \geq 0, \text{ for any } v \in \mathcal{C}\}.$$

The computations that the dual of a cone is a cone are straight forward, thus we omit them.

Definition 1.2. *Let V be a real vector space and $\mathcal{C} \subset V$ a cone which does not contain a line. A subcone $F \subset \mathcal{C}$ is called an:*

- extremal face*, if $a, b \in \mathcal{C}$ and $a + b \in F$ necessarily implies $a, b \in F$,*
- exposed face*, if there is a linear form $\lambda \in V^\vee$ such that $F = \lambda^\perp \cap \mathcal{C}$ and $\mathcal{C} \subset V_{\lambda \geq 0}$ holds.*

If the subspace spanned by F is one-dimensional, we call F a ray*.*

Remark 1.3. Any exposed face is also extremal, but the converse is generally false.

Lemma 1.4 (The dual of a strict convex cone)**.** *Let V be a finite dimensional real vector space and $\mathcal{C} \subset V$ a closed convex cone. Then $\mathcal{C}$ contains a line iff the interior of its dual $\mathcal{C}^\vee$ is empty.*

Proof. Assume that $\mathcal{C}$ contains a line, say $L = v_0 + L_0$, with $v_0 \in V$, and $L_0 \subset V$ a one-dimensional subspace. Since $\mathcal{C}$ is closed under multiplication by positive scalars, the line $\frac{1}{n}v_0 + L_0$ is contained in $\mathcal{C}$ for any $n \in \mathbb{N}$. Since $\mathcal{C}$ is closed, the line L_0 is contained in $\mathcal{C}$, thus we can assume without loss of generality that $L \subset V$ is a linear subspace.

We show that the interior of $\mathcal{C}^\vee$ is empty. Let $\lambda \in \mathcal{C}^\vee$ be any form, and let $\mu \in V^\vee$ be a form such that there exists a $v \in L$ with $\mu(v) < 0$. Note that λ is trivial on L, thus for any $\varepsilon > 0$, the form $\lambda + \varepsilon\mu$ is not contained in $\mathcal{C}^\vee$. In particular, λ is not contained in the interior of $\mathcal{C}^\vee$.

Now assume that the interior of $\mathcal{C}^\vee$ is empty. Then $\mathcal{C}^\vee$ is contained in a proper subspace of $V^\vee$. Consequently, there exists an element $0 \neq w \in V$ such that $\lambda(w) = 0$ for all $\lambda \in \mathcal{C}^\vee$. In particular, the line $\mathbb{R}w$ is contained in $(\mathcal{C}^\vee)^\vee = \mathcal{C}$, since $\mathcal{C}$ is closed. □

1.2 Cones of divisors and curves

A detailed treatment of intersection theory, and of cycles on varieties, or more generally on schemes, is given in [Ful98]. The definition of $\mathbb{Q}$- and $\mathbb{R}$-divisors can be found in Lazarsfeld's book [Laz04a, Chapter 1.3]. We will mainly work with numerical equivalence classes of divisors. The following definitions and facts are taken from [Laz04a, Chapter 1.1.C].

Definition 1.5 (Numerical equivalence and the Neron-Severi space)**.** *Let X be a normal projective variety, and let $D_1, D_2 \in \mathrm{Div}_{\mathbb{R}}(X)$ be $\mathbb{R}$-Cartier divisors.*

- *D_1 and D_2 are called* numerically equivalent*, denoted*

$$D_1 \equiv D_2,$$

  if for any irreducible curve C the intersection products are equal, i.e.,

$$D_1 \cdot C = D_2 \cdot C, \quad \textit{for any curve } C \subset X.$$

  They are called numerically proportional*, if there exists a number $\lambda \in \mathbb{R}^*$ such that $\lambda D_1 \equiv D_2$.*

- *The* Neron-Severi space *is defined as the vector space*

$$N^1(X) := \mathrm{Div}_{\mathbb{R}}(X)/\equiv .$$

- *After exchanging divisors for 1-cycles, we define analogously numerical equivalence for 1-cycles: two 1-cycles C_1, C_2 are numerically equivalent, $C_1 \equiv C_2$, if for any Cartier divisor the intersection products are equal. We obtain the space $N_1(X) := Z_1(X)/\equiv$, where $Z_1(X)$ denotes the space of 1-cycles with real coefficients.*

Recall the following fact.

Fact 1.6. *The intersection product defines a perfect pairing*

$$N_1(X) \times N^1(X) \to \mathbb{R}.$$

Thus the spaces $N_1(X)_k$ and $N^1(X)_k$ are dual as $\mathbb{R}$-vector spaces. Their common dimension, called the Picard number *of X, is finite and denoted by $\rho(X)$.* □

Notation 1.7. We will use squared brackets, to denote the class of a divisor (resp. a curve) in $N^1(X)$ (resp. $N_1(X)$).

Definition 1.8 (Cones in the vector spaces $N^1(X)$ and $N_1(X)$)**.** *Let X be a normal projective variety. We define the following cones:*

- *The* pseudo-effective cone $\overline{\mathrm{NE}}^1(X) \subset N^1(X)$ *is the closure of the cone generated by classes of effective $\mathbb{R}$-Cartier divisors on X. An $\mathbb{R}$-divisor is called* pseudo-effective *if its class is contained in $\overline{\mathrm{NE}}^1(X)$.*
- *The* nef cone $\overline{\mathrm{NM}}^1(X) \subset N^1(X)$ *is the cone generated by nef divisors.*
- *The* Mori cone, *denoted by $\overline{\mathrm{NE}}_1(X) \subset N_1(X)$, is the closure of the cone generated by curves on X.*
- *The* moving cone $\overline{\mathrm{NM}}_1(X) \subset N_1(X)$ *is the closure of the cone generated by moving curves, i.e., by curves that are members of a dominating family of curves on X.*

We can now discuss some properties of these cones.

Remark 1.9 (Duality properties)*.* By definition, the nef cone and the Mori cone are dual. By Kleiman's ampleness criterion, see [Deb01, Theorem 1.27], an $\mathbb{R}$-Cartier divisor D is ample iff its class is contained in the interior of $\overline{\mathrm{NM}}^1(X)$. Consequently, an $\mathbb{R}$-Cartier divisor is big iff its class lies in the interior of $\overline{\mathrm{NE}}^1(X)$, see [Laz04a, Section 2.2].

By [BDPP04, Theorem 2.2], the cones $\overline{\mathrm{NM}}_1(X)$ and $\overline{\mathrm{NE}}^1(X)$ are dual, see also [Bar08, Section 1.1.1] and [Laz04b, Section 11.4.C].

Proposition 1.10. *Let X be a normal projective variety of dimension n. The cones defined in Definition 1.8 are closed, strict convex, i.e., they do not contain straight lines, and their interior (in the standard topology) is non-empty.*

Proof. By continuity of the intersection product, the nef cone is closed, and the other cones are closed by definition. Since ampleness is an open condition and X is projective, the cone $\overline{\mathrm{NM}}^1(X)$ has a non-empty interior, namely the ample cone. Consequently, the interior of the cone $\overline{\mathrm{NE}}^1(X)$ is non-empty as well.

By Lemma 1.4 and Remark 1.9, the cones $\overline{\mathrm{NM}}_1(X)$ and $\overline{\mathrm{NE}}_1(X)$ are strict convex. Note that it suffices to show that $\overline{\mathrm{NE}}^1(X)$ does not contain a line. Assume on the contrary that $\overline{\mathrm{NE}}^1(X)$ does contain a line L. As in the proof of Lemma 1.4, we can assume that $L \subset N^1(X)$ is a linear subspace. Then any moving class is trivial on L. In particular, there exists a class $0 \neq D \in \overline{\mathrm{NE}}^1(X)$ such that $D \cdot \gamma = 0$ for any $\gamma \in \overline{\mathrm{NM}}_1(X)$. Note that for any choice of ample $\mathbb{R}$-divisors $H_1, \ldots, H_{n-1}$, the intersection $H_1 \cdot H_2 \cdots H_{n-1}$ is a moving class, see [Neu10, Definition 3.2.1, Remark 3.2.2]. Write $D = H_1 - H_2$ as the difference of two ample $\mathbb{R}$-divisors, and let H be an ample integral divisor. Then we have

$$
\begin{aligned}
D \cdot H^{n-1} &= 0, && \text{and} \\
D^2 \cdot H^{n-2} &= D \cdot (H_1 - H_2) \cdot H^{n-2} = 0.
\end{aligned}
$$

Now the assertion is a consequence of the following proposition. ☐

Proposition 1.11 ([Kle66, Proposition 3, page 305])**.** *Let X be a normal projective variety of dimension at least two. Let H be an ample divisor and D an arbitrary $\mathbb{R}$-Cartier divisor. Assume that $H^{r-1} \cdot D = 0$ and $H^{r-2} \cdot D^2 \geq 0$ holds. Then D is numerically trivial.*

Proof. In [Kle66], Kleiman gives a proof for integral divisors, which immediately extends to $\mathbb{Q}$-divisors. We shortly explain why Kleiman's proof remains true if D is $\mathbb{R}$-Cartier.

The proof is by induction on the dimension, and follows from the Hodge Index Theorem [Har77, Chapter V, Theorem 1.9]. The Hodge Index Theorem in turn follows from standard computations of intersection numbers and from [Har77, Chapter V, Corollary 1.8]. The computations are still valid for $\mathbb{R}$-divisors, and it remains to show that [Har77, Chapter V, Corollary 1.8] still applies. We do this in following lemma. ☐

Lemma 1.12 ([Har77, Chapter V, Corollary 1.8])**.** *Let X be a smooth projective surface, H an ample divisor on X, and D an $\mathbb{R}$-divisor on X. If $D \cdot H > 0$ and $D^2 > 0$, then D is $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor.*

Proof. This lemma is proven in [Har77] if D is integral, and it remains obviously true for $\mathbb{Q}$-divisors.

Assume now that D is an arbitrary $\mathbb{R}$-divisor. Since the intersection product is continuous, there exists an effective $\mathbb{R}$-divisor E such that $D - E$ is a $\mathbb{Q}$-divisor, and

$$(D - E) \cdot H > 0 \quad \text{and} \quad (D - E)^2 > 0.$$

The statement for $\mathbb{Q}$-divisors implies that there exists an effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} (D - E)$, thus $D \sim_{\mathbb{R}} D' + E$, which is effective. $\square$

1.3 Singularities of pairs

When introducing the minimal model program, one might think that it is a good idea to consider, for simplicity, the smooth case at the beginning. For surfaces, this is a reasonable approach, but unfortunately, in higher dimensions a step of the minimal model program can lead out of the class of smooth varieties, see [Deb01, Chapter 6.18]. Consequently, in order to define the minimal model program, we must allow singularities.

On the other hand, at least the Cone Theorem, see Theorem 1.22, should hold. For that reason, the class of singular varieties, or even normal varieties is too large, since steps of a minimal model program do not even exist.

It turns out that there exist different types of singularities, called *terminal, canonical, klt, plt, dlt, lc*, which have good properties. These singularities are usually defined for *pairs* which consist of a normal variety X and an $\mathbb{R}$-divisor $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.

1.3.1 Definition of klt and dlt

The whole theory is well explained in [KM98, Chapter 2.3], thus we will only give the basic central definition. Although this definition is only given for $\mathbb{Q}$-divisors in [KM98], it can easily be extended to $\mathbb{R}$-divisors.

Definition 1.13 (Discrepancies, [KM98, Notation 2.26 and Definition 2.28])**.** *Let X be a normal variety and Δ an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $g : Y \to X$ be a birational morphism from a normal variety Y, and let E_i denote the irreducible exceptional divisors of g. Then there exist real numbers, which we denote $a(E_i, X, \Delta)$, such that*

$$K_Y + g_*^{-1}\Delta \equiv g^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta)E_i.$$

The number $a(E_i, X, \Delta)$ is called the discrepancy of E_i with respect to (X, Δ).

If $g : Y \to X$ is a birational morphism from a normal variety, we set $a(g) := \inf_i \{a(E_i, X, \Delta)\}$, and define the discrepancy *of (X, Δ) by*

$$\operatorname{discrep}(X, \Delta) := \inf\{a(E_i, X, \Delta) \,|\, g : Y \to X \text{ is birational, } Y \text{ is normal}\}.$$

As previously mentioned, we can now define the different types of singularities. Since we will only work with *klt* and *dlt* singularities, we only give the definitions of these types.

Definition 1.14 (klt, dlt, [KM98, Definitions 2.34 and 2.37])**.** *Let X be a normal projective variety and $\Delta \in \mathrm{WDiv}(X)_{\mathbb{R}}$ an $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We call (X, Δ)*

- *a* klt *pair, if $\lfloor \Delta \rfloor = 0$, and* $\operatorname{discrep}(X, \Delta) > -1$,
- *a* dlt *pair, if the coefficients of Δ lie in $[0, 1]$, and there is a closed subset $Z \subset X$ such that*
  1. *$X \setminus Z$ is smooth, and $\Delta|_{X \setminus Z}$ is snc on $X \setminus Z$.*
  2. *If $f : Y \to X$ is birational and $E \subset Y$ is an irreducible divisor such that $f(E) \subset Z$ then $a(E, X, \Delta) > -1$.*

We call a dlt (resp. klt) pair (X, Δ) projective *if the variety X is projective.*

Remark 1.15. The names given in Definition 1.14 are abbreviations for *Kawamata log terminal* and for *divisorially log terminal*, respectively.

The singularities defined above satisfy the following useful continuity properties, which we will frequently use:

Lemma 1.16 (Openness of klt and dlt, [KM98, Corollaries 2.35 and 2.39])**.** *Let (X, Δ) be a pair and Δ' be an effective $\mathbb{R}$-divisor.*

1. *If $(X, \Delta + \Delta')$ is klt (resp. dlt) then (X, Δ) is also klt (resp. dlt).*
2. *If (X, Δ) is klt then $(X, \Delta + \varepsilon\Delta')$ is also klt for $0 \le \varepsilon \ll 1$.*
3. *If (X, Δ) is dlt then $(X, \Delta + \varepsilon\Delta')$ is also dlt for $0 \le \varepsilon \ll 1$, assuming that* $\operatorname{Supp} \Delta' \subset \operatorname{Supp}(\Delta - \lfloor \Delta \rfloor)$. □

Remark 1.17. Again, our reference [KM98] states the above results only for $\mathbb{Q}$-divisors. Note that the proofs apply without change for $\mathbb{R}$-divisors.

Many results we need are only proven for klt pairs. To prove the Isotriviality Theorem, we have to extend these results to dlt pairs. The following section provides the technical background for this.

1.3.2 dlt is the limit of klt

The proof of the following Proposition 1.18 which is given in [KM98] for $\mathbb{Q}$-divisors does not directly apply to $\mathbb{R}$-divisors. For that reason and for lack of an adequate reference for $\mathbb{R}$-divisors, we provide short proofs of the results discussed in this section. A generalization of the following proposition for $\mathbb{R}$-divisors is then given in Proposition 1.21.

Proposition 1.18 ([KM98, Proposition 2.43]). *Assume that (X, Δ) is dlt (Δ a $\mathbb{Q}$-divisor) and X is quasi-projective with ample divisor H. Let Δ_1 be an effective $\mathbb{Q}$-divisor (not necessarily $\mathbb{Q}$-Cartier) such that $\Delta - \Delta_1$ is effective. Then there exists a rational number $c > 0$ and an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \Delta_1 + cH$ such that $(X, \Delta - \varepsilon\Delta_1 + \varepsilon D)$ is dlt for all rational numbers $0 < \varepsilon \ll 1$.*

If $\operatorname{Supp}\Delta_1 = \operatorname{Supp}\Delta$, then $(X, \Delta - \varepsilon\Delta_1 + \varepsilon D)$ is klt for all sufficiently small rational numbers $\varepsilon > 0$. □

Lemma 1.19 (See [Laz04b, Example 9.2.29]). *Let (X, Δ) be a projective klt pair and H an ample $\mathbb{R}$-divisor. Then H is $\mathbb{R}$-linearly equivalent to an effective divisor H' such that $(X, \Delta + H')$ is klt.*

Proof. We first consider an ample $\mathbb{Q}$-divisor H. Then for all sufficiently divisible $m \gg 0$, the divisor mH is a very ample integral Cartier divisor. Let $\tilde{H}$ be a general member of $|mH|$, and set $H' := \frac{1}{m}\tilde{H}$. Since m is chosen large, we have $\lfloor \Delta + H' \rfloor \leq 0$. Moreover, it follows from [KM98, Lemma 5.17] that the discrepancy of $(X, \Delta + H')$ is still greater than -1. This proves that $(X, \Delta + H')$ is klt.

Since any ample $\mathbb{R}$-divisor can be written as a positive linear combination of ample $\mathbb{Q}$-divisors, it suffices to prove the assertion for an ample $\mathbb{R}$-divisor of type λA, where $\lambda \in \mathbb{R}^+$ and A is an ample $\mathbb{Q}$-divisor. Choose a rational $l > \lambda$. As we have seen, there exists an ample $\mathbb{Q}$-divisor $A' \sim lA$ such that $(X, \Delta + A')$ is klt. Clearly, $\frac{\lambda}{l} < 1$ and $\frac{\lambda}{l}A' \sim \lambda A$. Therefore $\frac{\lambda}{l}A'$ has the required properties. □

Next, we use the continuity properties stated in Lemma 1.16 to show that the boundary divisor Δ can be replaced by a $\mathbb{Q}$-divisor which preserves the type of singularity.

Lemma 1.20 ($\mathbb{Q}$ is dense in $\mathbb{R}$). *Let (X, Δ) be a projective dlt (resp. klt) pair and $H \in \operatorname{Div}_{\mathbb{R}}(X)$ an ample $\mathbb{R}$-divisor. Then there exists an effective $\mathbb{Q}$-divisor Δ' such that (X, Δ') is dlt (resp. klt), and $H + \Delta - \Delta'$ is $\mathbb{R}$-ample.*

Proof. We give a proof for the case where (X, Δ) is dlt, the proof for the klt case is analogous.

To prove the existence of Δ', we first write Δ as a positive linear combination

$$\Delta = \sum_{i=1}^{n} r_i S_i,$$

where S_i are distinct prime Weil divisors and $r_i \in [0,1]$, for $i = 1, \dots, n$. Consider $K_X \in \mathrm{WDiv}(X)$ as a fixed divisor which represents the canonical class and set

$$Q := \left\{ K_X + \sum \lambda_i S_i \,|\, \lambda_i \in [0,1] \right\} \subset \mathrm{WDiv}_{\mathbb{R}}(X).$$

Note that Q is a rational polytope in $\mathrm{WDiv}_{\mathbb{R}}(X)$ and consequently, the intersection $B := Q \cap \mathrm{Div}_{\mathbb{R}}(X)$ is a rational polytope as well. Moreover, B is not empty because $K_X + \Delta \in B$. By Lemma 1.16 the property *dlt* is an open property on B. More precisely, there is an open neighborhood $U \subset B$ of $K_X + \Delta$ such that the pair (X, Γ) is dlt for any $K_X + \Gamma \in U$. Since ampleness is also an open property, we can assume that for any $K_X + \Gamma \in U$ the divisor $H + \Delta - \Gamma$ is ample.

Since B is a rational polytope, the set $B_{\mathbb{Q}} := Q \cap \mathrm{Div}_{\mathbb{Q}}(X)$ is dense in B. Therefore, there exists $K_X + \Delta' \in U$ with Δ' being a $\mathbb{Q}$-divisor. This finishes the proof. □

Proposition 1.21 (Generalization of Proposition 1.18 for $\mathbb{R}$-divisors)**.** *Let (X, Δ) be a projective dlt pair and H be an ample $\mathbb{R}$-divisor. Then for any $\varepsilon > 0$ there exists an effective $\mathbb{R}$-divisor $\Delta_\varepsilon \sim_{\mathbb{R}} \Delta + \varepsilon H$ such that the pair (X, Δ_ε) is klt.*

Proof. After rescaling of H we can assume without loss of generality that $\varepsilon = 1$. Moreover, because of Lemma 1.20 we can assume that Δ is a $\mathbb{Q}$-divisor. Since H is not necessarily a $\mathbb{Q}$-divisor, we write $H = H_1 + H_2$ such that H_1 is an ample $\mathbb{Q}$-divisor and H_2 is an ample $\mathbb{R}$-divisor.

There exists an $m \in \mathbb{N}$ such that mH_1 is integral and Cartier, thus we may apply Proposition 1.18 for $\Delta_1 = \Delta$ and mH_1. Accordingly there exists a rational number $c > 0$ and an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \Delta + cmH_1$ such that for any sufficiently small $\varepsilon' > 0$ the pair $(X, \Delta - \varepsilon'\Delta + \varepsilon' D)$ is klt. In particular, $\Delta + \varepsilon' cmH_1$ is $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor Δ_{H_1} such that (X, Δ_{H_1}) is klt. By Lemma 1.19, we can replace $\varepsilon' mcH_2$ by an $\mathbb{R}$-linear equivalent effective divisor H_3 such that $(X, \Delta_{H_1} + H_3)$ is klt. Note that

$$\Delta_{H_1} + H_3 \sim_{\mathbb{R}} \Delta + \varepsilon' mcH,$$

thus another application of Lemma 1.19 for $(1 - \varepsilon' mc)H$ yields that $\Delta + H$ is $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor Δ_H such that (X, Δ_H) is klt. This completes the proof. □

1.4 Cone Theorem and flips

The minimal model program consists of a sequence of birational maps. In order to show the existence of these maps, we need the following theorem.

Theorem 1.22 (Cone Theorem, [KM98, Theorem 3.7]). *Let (X, Δ) be a projective dlt pair. Then:*

1. *There are (countably many) rational curves $C_j \subset X$ such that*

$$\overline{\mathrm{NE}}_1(X) = \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0} + \sum_j \mathbb{R}^{\geq 0}[C_j].$$

2. *For any $\varepsilon > 0$ and ample $\mathbb{R}$-divisor H,*

$$\overline{\mathrm{NE}}_1(X) = \overline{\mathrm{NE}}_1(X)_{K_X+\Delta+\varepsilon H\geq 0} + \sum_{\textit{finite}} \mathbb{R}^{\geq 0}[C_j].$$

3. *Let $F \subset \overline{\mathrm{NE}}_1(X)$ be a $(K_X + \Delta)$-negative extremal face. Then there is a unique morphism $\mathrm{cont}_F : X \to Z$ to a projective variety such that $(\mathrm{cont}_F)_*\mathcal{O}_X = \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point by cont_F iff $[C] \in F$. cont_F is called the contraction of F.*

4. *Let F and $\mathrm{cont}_F : X \to Z$ be as in 3. Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Z on Z such that $L \cong \mathrm{cont}_F^* L_Z$.*

Proof. Our reference [KM98, Theorem 3.7] gives a proof if the pair is klt, and if all divisors are $\mathbb{Q}$-divisors. We will deduce the general case from this and from Proposition 1.21 and Lemma 1.20.

Note that (1) follows from (2). To show (2), let H be an arbitrary ample $\mathbb{R}$-divisor and $\varepsilon > 0$. After rescaling of H, we may assume that $\varepsilon = 1$. By Proposition 1.21 there exists an $\mathbb{R}$-divisor $\Delta_{\frac{1}{2}} \sim_{\mathbb{R}} \Delta + \frac{1}{2}H$ such that $(X, \Delta_{\frac{1}{2}})$ is klt. By Lemma 1.20 there exists a $\mathbb{Q}$-divisor Δ' such that (X, Δ') is klt and $H' := \frac{1}{2}H + \Delta_{\frac{1}{2}} - \Delta'$ is an ample $\mathbb{R}$-divisor. Observe that the following numerical equivalence holds:

$$K_X + \Delta + H \equiv K_X + \Delta' + H'.$$

Since H' is a positive linear combination of ample (integral) divisors, there exists an ample $\mathbb{Q}$-divisor $\tilde{H} \leq H'$ such that $H' - \tilde{H}$ is ample. Since (2) holds for $\mathbb{Q}$-divisors, there exist finitely many rational curves $C_1, \ldots, C_l$ such that

$$\overline{\mathrm{NE}}_1(X) = \overline{\mathrm{NE}}_1(X)_{K_X+\Delta'+\tilde{H}\geq 0} + \sum_{i=1}^{l} \mathbb{R}^{\geq 0}[C_i].$$

By Kleiman's criterion, each ample divisor intersects any element of $\overline{\mathrm{NE}}_1(X)$ positively. Consequently, the inclusion

$$\overline{\mathrm{NE}}_1(X)_{K_X+\Delta'+\tilde{H}\geq 0} \subset \overline{\mathrm{NE}}_1(X)_{K_X+\Delta+H\geq 0}$$

holds, which implies item (2) for $\mathbb{R}$-divisors and dlt pairs.

To show (3) and (4), let $F \subset \overline{\mathrm{NE}}_1(X)$ be a $(K_X+\Delta)$-negative extremal face. A similar argument as before shows that there exists a $\mathbb{Q}$-divisor Δ' such that (X, Δ') is klt and F is $(K_X+\Delta')$-negative. Then the existence of cont_F follows again from [KM98, Theorem 3.7]. Note that the properties of cont_F do not depend on the boundary divisor, which finishes the proof. □

The contractions given in the Cone Theorem will be classified next.

Fact 1.23 ([KM98, Proposition 2.5]). *Let (X, Δ) be a projective dlt pair and R a $(K_X+\Delta)$-negative extremal ray. By the Cone Theorem, there exists a contraction map $f : X \to Y$, which contracts a curve $C \subset X$ iff $R = \mathbb{R}^+[C]$. This map is one of the following type.*

- Divisorial contraction*: f contracts a divisor D, and outside D it is an isomorphism. In this case the pair $(Y, f_*\Delta)$ is dlt, and if X is $\mathbb{Q}$-factorial then Y is $\mathbb{Q}$-factorial.*
- Mori fibration *or* Mori fiber space*: The dimension of Y is less than the dimension of X. In this case the general fiber of f is dlt log Fano.*
- Small contraction*: f contracts a set of codimension greater than one. In this case the divisor $K_X+\Delta$ is not $\mathbb{Q}$-Cartier.* □

The obvious idea is to define the minimal model program as a sequence of maps which are contractions of extremal rays. Unfortunately, if (X, Δ) is a dlt pair, and $f : X \to Y$ is a small contraction, then the divisor $K_Y + f_*\Delta$ is not $\mathbb{R}$-Cartier. To overcome this problem, we must perform a flip.

Definition 1.24 (Flip, [KM98, Definition 3.33]). *Let (X, Δ) be a dlt pair. A* flipping contraction *is a proper birational morphism $f : X \to Y$ to a normal variety Y such that the exceptional set has codimension at least two in X and $-(K_X+\Delta)$ is f-ample.*

A normal variety X^+ together with a proper birational morphism

$$f^+ : X^+ \to Y$$

is called a $(K_X+\Delta)$-flip *of f if*

1. *$K_{X^+} + \Delta^+$ is $\mathbb{R}$-Cartier, where Δ^+ is the strict transform of Δ on X^+*
2. *$K_{X^+} + \Delta^+$ is f-ample, and*
3. *the exceptional set of f^+ has codimension at least two in X^+.*

A flip gives a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y, & \end{tikzcd}$$

where φ is an isomorphism in codimension one. By a slight abuse of terminology, we will also refer to the triangle above as a flip.

Finally, the previous results allow us to define the minimal model program.

Definition 1.25 (Minimal model program)**.** *Let (X, Δ) be a projective dlt (resp. klt) pair. Set $(X_0, \Delta_0) := (X, \Delta)$. A* minimal model program *is a possibly infinite sequence of birational maps*

$$X =: X_0 \overset{\varphi_1}{\dashrightarrow} X_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_n}{\dashrightarrow} X_n \overset{\varphi_{n+1}}{\dashrightarrow} \ldots$$

with the following properties:

1. *Define for each $i \in \mathbb{N}$ where φ_i is defined a divisor Δ_i on X_i recursively as*
$$\Delta_0 := \Delta \quad \textit{and} \quad \Delta_i := (\varphi_i)_* \Delta_{i-1}.$$
Then the pair (X_i, Δ_i) is a dlt (resp. klt pair).

2. *For each $i \in \mathbb{N}$ where φ_i is defined, there exists a $(K_{X_{i-1}} + \Delta_{i-1})$-negative extremal ray R_i of the cone $\overline{\mathrm{NE}}_1(X_{i-1})$ such that φ_i is either the divisorial contraction of R_i, or the contraction of R_i is a flipping contraction, and φ_i is the corresponding flip.*

3. *If the sequence is finite, then for the final outcome X_l one of the following properties holds:*
   (a) *Either the divisor $K_{X_l} + \Delta_l$ is nef, or*
   (b) *there exists a $(K_{X_l} + \Delta_l)$-negative extremal ray R_{l+1} such that its contraction $f : X_l \dashrightarrow Y$ is a Mori fibration.*

   In case (3a) we call (X_l, Δ_l) a minimal model*, and in case (3b) we call $X_l \to Y$ a* Mori fiber space.

A minimal model program is called a terminating minimal model program *if the sequence is finite.*

We say that the minimal model program is well-defined *or that* any minimal model program can be run *if for any finite $\tilde{I} = \{1, \ldots, l\}$ and any sequence of birational maps $(\varphi_i : X_{i-1} \dashrightarrow X_i)_{i \in \tilde{I}}$ that satisfies (1) and (2) the following holds. If R_{l+1} is a $(K_{X_l} + \Delta_l)$-negative extremal ray whose contraction f is small, then the flip of f exists.*

Remark 1.26. It is shown in [KM98, Corollary 3.44] that the properties *dlt* and *klt* are preserved under divisorial contractions and flips. If the minimal model program $(\varphi_i : X_{i-1} \dashrightarrow X_i)$ starts with a $\mathbb{Q}$-factorial variety X, then any X_i is $\mathbb{Q}$-factorial, see [KM98, Propositions 3.36 and 3.37]. Consequently, one can define a minimal model program inductively if flips are known to exist.

To see that a minimal model program terminates, we first remark that divisorial contractions decrease the Picard number by one, and flips leave the Picard number unchanged, see [KM98, Propositions 3.36 and 3.37] again. Therefore, a minimal model program terminates if it does not contain a sequence of flips of infinite length. It is indeed a serious problem to prove this in general, and we will see in Chapter 2 that termination can only be proved if the rays R_i are chosen properly.

The following proposition guarantees that minimal model programs can be run.

Proposition 1.27 ([BCHM10, Corollary 1.4.1]). *Let (X, Δ) be a projective dlt pair and let $\pi : X \to Z$ be a small $(K_X + \Delta)$-extremal contraction. Then the flip of π exists.* □

Remark 1.28. In [BCHM10] Proposition 1.27 is actually proven for klt pairs. The generalization to the dlt case is due to the fact that dlt pairs can be approximated by klt pairs, see Proposition 1.21 and Lemma 2.8 for a precise statement.

We will frequently use the following notation.

Notation 1.29. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair, and let

$$X =: X_0 \overset{\varphi_1}{\dashrightarrow} X_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_n}{\dashrightarrow} X_n \overset{\varphi_{n+1}}{\dashrightarrow} \dots$$

be a (possibly infinite) run of the minimal model program. Let $i \in \mathbb{N}$ such that the ith step φ_i exists.

1. Given an $\mathbb{R}$-divisor D on X, we set $D_0 := D$ and define an $\mathbb{R}$-divisor D_i on X_i recursively as
$$D_i := (\varphi_i)_* D_{i-1}.$$
2. We denote by $R_i \subset \overline{\mathrm{NE}}_1(X_{i-1})$ the $(K_{X_{i-1}} + \Delta_{i-1})$-negative extremal ray which is contracted or flipped by φ_i. If the minimal model program terminates with a Mori fiber space $X_m \to B$, we define R_{m+1} analogously.

1.5 The relative minimal model program

Given a morphism $f : X \to Z$ of varieties, one can define the *relative minimal model program*. This is a sequence of maps over Z, i.e.,

$$\begin{tikzcd} X_{i-1} \arrow[rr, dashed, "\varphi_i"] \arrow[dr, "f_{i-1}"'] & & X_i \arrow[dl, "f_i"] \\ & Z & \end{tikzcd}$$

which basically satisfies Definition 1.25. We refer to [KM98, Section 3.6] for a detailed treatment. We will use the relative minimal model program once in Chapter 4. In this particular case, we will not need any sophisticated results.

Chapter 2

The minimal model program with scaling

In this chapter we introduce the minimal model program with scaling and prove termination for the $\mathbb{Q}$-factorial dlt case. This generalizes a result of [BCHM10] from the klt to the dlt case. Although this generalization is probably well-known to experts, we will include a proof since the methods used will be very useful to prove Theorem 3.5.

2.1 Pushforward and pullback of curves

In the sequel we will sometimes have to take pushforward und pullback of numerical classes of 1-cycles. To define this, we use pullback and pushforward of classes of divisors and duality of the underlying vector spaces , see [Bar08, Chapter 3] and [Ara10, Chapter 4].

Definition 2.1 (Numerical pushforward and pullback of curves)**.** *Let $f : X \dashrightarrow Y$ be a birational map between $\mathbb{Q}$-factorial projective varieties which is surjective in codimension one. Then we define the* numerical pullback *and* numerical pushforward

$$f^* : N_1(Y) \to N_1(X) \quad \text{and} \quad f_* : N_1(X) \to N_1(Y)$$

as the dual maps of the pushforward and the pullback of divisors.

Remark 2.2. If a curve is contained in the domain of the map, then the pushforward of its class coincides with the class of its cycle-theoretic pushforward, see [Bar08, Corollary 3.12].

On the other hand it is difficult to see what the pullback or pushforward of a curve is if it is contained in the indeterminacy locus of the underlying map. There are examples where the pullback of a curve behaves rather counterintuitively, see [Ara10, Examples 4.2 and 4.3].

The definition above immediately implies the following identities.

Proposition 2.3 (Projection formulae). *Let $f : X \dashrightarrow Y$ be as in Definition 2.1.*

1. *If $\gamma \in N_1(X)$ and $[D] \in N^1(Y)$, then $f_*\gamma \cdot [D] = \gamma \cdot f^*[D]$.*
2. *If $\gamma \in N_1(Y)$ and $[D] \in N^1(X)$, then $f^*\gamma \cdot [D] = \gamma \cdot f_*[D]$.*

□

2.2 The minimal model program with scaling

The existence of terminating minimal model programs can be proved if we take a given divisor into account.

Definition 2.4 (Minimal model program with scaling). *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair, and let H be an ample $\mathbb{R}$-divisor such that $K_X + \Delta + H$ is nef. A* (terminating) minimal model program with scaling of H *is a (terminating) minimal model program*

$$X =: X_0 \overset{\varphi_1}{\dashrightarrow} X_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_n}{\dashrightarrow} X_n \overset{\varphi_{n+1}}{\dashrightarrow} \ldots$$

and a (finite) decreasing sequence of real numbers

$$s_0 \geq s_1 \geq \cdots \geq s_n \geq \cdots \geq 0,$$

such that for any i, where R_i is defined, the following holds.

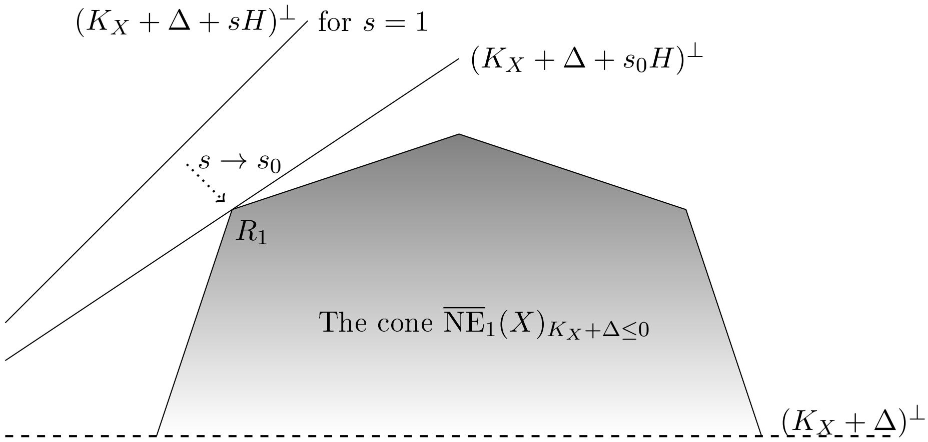
1. *The divisor $K_{X_{i-1}} + \Delta_{i-1} + s_{i-1}H_{i-1}$ is nef.*
2. *The ray R_i is contained in the hyperplane*
$$(K_{X_{i-1}} + \Delta_{i-1} + s_{i-1}H_{i-1})^{\perp} \subset N_1(X).$$
3. *If the minimal model program terminates with a Mori fiber space $X_m \to B$, then $R_{m+1} \subset (K_{X_m} + \Delta_m + s_m H_m)^{\perp}$.*

We will denote a minimal model program with scaling of H by the sequence of pairs $(\varphi_i, s_i)_i$.

Remark 2.5. An easy computation shows that the divisor $K_{X_i} + \Delta_i + s_{i-1}H_i$ is nef, see [Ara10, 3.8]. Properties (1) and (2) imply that s_i is uniquely determined by the equation

$$s_i = \inf\{s > 0 \,|\, K_{X_i} + \Delta_i + sH_i \text{ is nef}\}.$$

We can therefore view a step of the minimal model program with scaling as follows. The divisor $K_{X_i} + \Delta_i + s_{i-1}H_i$ is nef, and after scaling s down, the hyperplane $(K_{X_i}+\Delta_i+sH_i)^\perp$ approaches the Mori cone and determines the ray R_{i+1}. The first step is visualized in the following picture.



The first step of the minimal model program with scaling of H.

Remark 2.6. It is a priori not clear that minimal model programs with scaling exist generally, even if flips are known to exist. Given s_i as in Remark 2.5, we have to ensure the existence of an extremal ray $R \subset (K_{X_i}+\Delta_i+s_iH_i)^\perp$ that intersects $K_{X_i}+\Delta_i$ *negatively*. The statement that for dlt pairs such a ray indeed exists is given in [Bir10, Lemma 3.1]. Hence we can always run a minimal model program with scaling, if flips exist.

For the klt case, termination of the minimal model program with scaling is stated in the following Theorem, see [BCHM10, Corollary 1.3.3] and [Ara10, Theorem 3.9].

Theorem 2.7 (MMP with scaling for klt pairs)**.** *Let (X,Δ) be a $\mathbb{Q}$-factorial projective klt pair such that $K_X+\Delta$ is not pseudo-effective. Let H be an effective ample $\mathbb{R}$-divisor such that $K_X+\Delta+H$ is nef and klt. Then any minimal model program with scaling of H terminates with a Mori Fiber space.*

□

2.3 The minimal model program with scaling for dlt pairs

In Theorem 2.10 we will show that Theorem 2.7 still holds for dlt pairs. The proof uses that dlt pairs can be seen as the limit of klt pairs.

2.3.1 Termination of the minimal model program

The following lemma shows that a variation of the boundary divisor Δ does not affect flips.

Lemma 2.8 (Rigidity of flips)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair. Assume that R is a $(K_X + \Delta)$-negative extremal ray, and that the contraction f of R is small. Let D be an arbitrary $\mathbb{R}$-divisor on X such that R is $(K_X + D)$-negative. If the (X, Δ)-flip φ of f exists, then φ is also the (X, D)-flip of f.*

Proof. Assume that any flip

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

of f exists. We have to show that $K_{X^+} + \varphi_* D$ is f^+-ample. Let $C^+ \subset X^+$ be a curve which is contracted by f^+. Then it is shown in [Bar08, Lemma 4.13] that for the numerical pullback the following holds:

$$-\varphi^*[C^+] \in R.$$

Since $\varphi^* : N_1(X^+) \to N_1(X)$ is an isomorphism of vector spaces, the relative Picard number $\rho(X^+/Y)$ is one, and it suffices to show that $K_{X^+} + \varphi_* D$ intersects C^+ positively. This follows easily from the projection formula, thus φ is a flip for both (X, Δ) and (X, D). $\square$

Corollary 2.9. *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair. Then any minimal model program (with scaling) can be run for (X, Δ).*

Proof. Since flips exist for klt pairs, see [BCHM10, Corollary 1.4.1], Lemma 2.8 and Proposition 1.21 imply the existence of flips for dlt pairs. This implies the assertion for arbitrary minimal model programs. It remains to show that for each step of a minimal model program with scaling there exists an extremal ray which can be contracted. This is shown in [Bir10, Lemma 3.10]. $\square$

We are now able to generalize Theorem 2.7.

Theorem 2.10 (MMP with scaling for dlt pairs)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair, and H an ample $\mathbb{R}$-divisor such that $K_X + \Delta + H$ is nef. Assume that $K_X + \Delta$ is not pseudo-effective.*

1. *Set*
$$\sigma := \inf\{s > 0 \,|\, K_X + \Delta + sH \textit{ is pseudo-effective}\},$$
and let $0 \leq \varepsilon_1, \varepsilon_2 < \sigma$ be arbitrary real numbers. For $k \in \{1, 2\}$, let $\Delta^k := \Delta_{\varepsilon_k}$ be as in Proposition 1.21, if ε_k is positive, or set $\Delta^k := \Delta$, if $\varepsilon_k = 0$.

   If $(\varphi_i, s_i)_i$ is a minimal model program with scaling for the pair (X, Δ^1), then $(\varphi_i, s_i + (\varepsilon_1 - \varepsilon_2))_i$ is a minimal model program with scaling for the pair (X, Δ^2).

2. *Any minimal model program with scaling of H can be run for the pair (X, Δ) and terminates.*

Proof. It is shown in Corollary 2.9 that the minimal model program with scaling can be run for dlt pairs. Item (2) is then a consequence of (1) and Theorem 2.7.

To show (1), we first observe that for any i the numerical equivalence

$$K_{X_i} + \Delta_i^1 + s_i H_i \equiv K_{X_i} + \Delta_i^2 + (s_i + (\varepsilon_1 - \varepsilon_2))\, H_i$$

holds. In particular, the divisor $K_{X_i} + \Delta_i^2 + (s_i + (\varepsilon_1 - \varepsilon_2))\, H_i$ is nef and numerically trivial on R_{i+1}. Moreover, if R_{i+1} is $(K_{X_i} + \Delta_i^2)$-negative, then it follows from Lemma 2.8 that a flip of R_{i+1} does not depend on the numbers $\varepsilon_1, \varepsilon_2$. It therefore remains to show that for any i the following holds.

a) The number $s_i + (\varepsilon_1 - \varepsilon_2)$ is positive,

b) the ray R_{i+1} is $(K_{X_i} + \Delta_i^2)$-negative,

c) if the first sequence terminates, then so does the second one.

We first show that (a) implies (b), thus we assume that $s_i + (\varepsilon_1 - \varepsilon_2)$ is positive for some i. Since R_{i+1} is $(K_{X_i} + \Delta_i^1)$-negative and $(K_{X_i} + \Delta_i^1 + s_i H_i)$-trivial, R_{i+1} is H_i-positive. As we have seen before, the ray R_{i+1} is also $\big(K_{X_i} + \Delta_i^2 + (s_i + \varepsilon_1 - \varepsilon_2)\, H_i\big)$-trivial, and since $s_i + (\varepsilon_1 - \varepsilon_2)$ is positive, we conclude (b).

The next step is to show (a) by induction on i. For $i = 0$, it follows from Remark 2.5 that

$$s_0 = \inf\{s > 0 \,|\, K_X + \Delta_1 + sH \text{ is nef}\}.$$

In particular,

$$s_0 \geq \inf\{s > 0 \,|\, K_X + \Delta_1 + sH \text{ is pseudo-effective}\} = \sigma - \varepsilon_1.$$

Therefore, $s_0 + (\varepsilon_1 - \varepsilon_2) \geq \sigma - \varepsilon_2$, which is positive by assumption.

For the induction step we assume that $s_j + (\varepsilon_1 - \varepsilon_2)$ is positive for each $j \leq i$, and we aim to show that $s_{i+1} + (\varepsilon_1 - \varepsilon_2)$ is also positive. Assume this is not the case. This immediately implies $\varepsilon_2 > \varepsilon_1$, in particular (X, Δ^2) is klt. Moreover, the ray R_{i+1} is $(K_{X_i} + \Delta_i^2)$-negative, thus φ_{i+1} is a step of a (X, Δ^2)-minimal model program with scaling of H. We obtain the following nef $\mathbb{R}$-divisors on X_{i+1}.

$$\begin{aligned} &K_{X_{i+1}} + \Delta_{i+1}^2 + (s_i + (\varepsilon_1 - \varepsilon_2))\, H_{i+1} \quad \text{and} \\ &K_{X_{i+1}} + \Delta_{i+1}^2 + (s_{i+1} + (\varepsilon_1 - \varepsilon_2))\, H_{i+1} \end{aligned}$$

Convexity of the nef cone implies that also $K_{X_{i+1}} + \Delta_{i+1}^2$ is nef, thus a run of the minimal model program with scaling for the pair (X, Δ^2) terminates with a minimal model, a contradiction to Theorem 2.7.

It remains to show (c). We assume on the contrary that the first sequence terminates and the second one does not. This in particular implies that the first minimal model program terminates with a minimal model. Exchange ε_1 for ε_2, and we obtain a contradiction to (a). $\square$

Chapter 3

The moving cone of $\mathbb{Q}$-factorial dlt pairs

The goal of this chapter is to prove the Moving Cone Theorem 3.5. The proof is given in several steps. We first analyze an arbitrary Mori fiber space and specify the curves we want to pull back. More precisely, we construct the following subvariety of the Hilbert scheme.

Lemma 3.1. *Let $\lambda : X \dashrightarrow X'$ be a birational map between normal projective varieties which is surjective in codimension one. Let B be a variety with $\dim B < \dim X'$, and let $\pi : X' \to B$ a surjective morphism with connected fibers. Then there is an irreducible locally closed subvariety H of the Hilbert scheme of curves on X such that*

1. *any closed point of H corresponds to a moving curve that is contained in the open set where λ is an isomorphism,*
2. *any closed point of H corresponds to a curve C whose image $\lambda(C)$ lies in a fiber of π, and*
3. *if $Z \subset X$ has codimension greater than or equal to two, then the set*

$$H_Z := \{p \in H \,|\, p \text{ corresponds to a curve that avoids } Z\}$$

is non-empty and open in H.

Proof. Let $U \subset X$ denote the set where λ is an isomorphism onto its image $V := \lambda(U)$. We aim to find a dominating family of curves that is entirely contained in U.

To this end, we first remark that the inverse λ^{-1} does not contract any divisor, thus $\mathrm{codim}_{X'}(X' \setminus V) \geq 2$ holds. Therefore, if F is a general fiber of π, then $\mathrm{codim}_F(F \setminus V) \geq 2$, as well. Let k be the relative dimension of X' over B, and pick $k-1$ very ample divisors $H_1, \ldots, H_{k-1}$ on X'. If $D_1, \ldots, D_{k-1}$ are general members of the corresponding linear systems

$|H_1|, \ldots, |H_{k-1}|$ then the intersection $F \cap D_1 \cap \cdots \cap D_{k-1} \subset F$ is an irreducible smooth curve that avoids $X' \setminus V$. We conclude that there is an open subset $U \subset B \times |H_1| \times \cdots \times |H_{k-1}|$ such that for $(b, D_1, \ldots, D_{k-1}) \in U$ the intersection $\pi^{-1}(b) \cap D_1 \cap \cdots \cap D_{k-1}$ is a smooth curve. This defines a family of curves that are entirely contained in V. Moreover, if Z' is any subvariety of X' of codimension greater than or equal to two then the general member of this family avoids Z'.

Via λ we obtain the required family of curves on X which in turn defines the subset H of the Hilbert scheme. Moreover, if $Z \subset X$ has codimension greater than or equal to two, then $Z' := \lambda(U \cap Z) \subset X'$ has codimension greater than or equal to two as well. Thus a general point of H corresponds to a curve that avoids Z. $\square$

Corollary 3.2. *If a minimal model program leads to a Mori fiber space, then the numerical pullback of any curve on a fiber of the Mori fiber space is a moving class.*

Proof. Note that a minimal model program which leads to a Mori fiber space satisfies the condition of Lemma 3.1. Let $X' \to B$ denote the Mori fiber space, then the relative Picard number $\rho(X'/B)$ is one. Thus all classes of curves in fibers are numerically proportional in X', and Lemma 3.1 shows that these classes are moving. $\square$

The next step in the proof of Theorem 3.5 is the construction of a divisor suitable for running the minimal model program with scaling. This will be done in the following lemma, which is closely related to [Leh09, Lemma 4.3]. A similar statement is also given in [Ara10, Proof of Theorem 1.1].

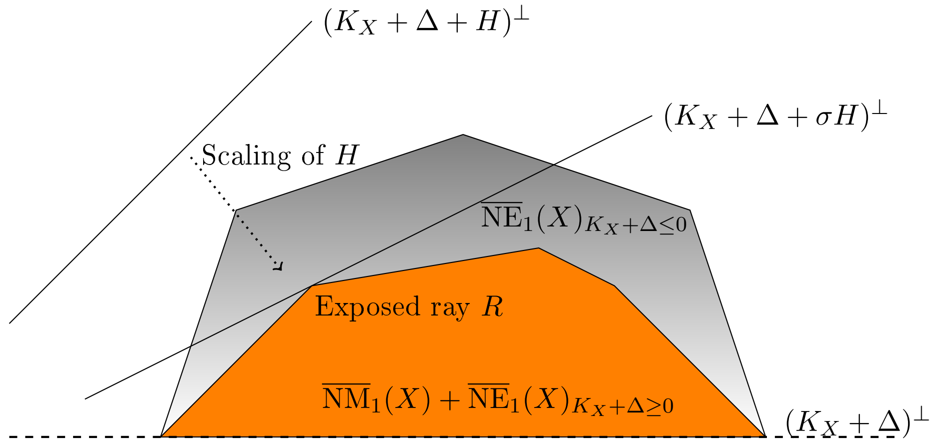
Lemma 3.3. *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair and let*

$$R \subset \overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}$$

be a $(K_X + \Delta)$-negative exposed ray. Then there is an $\mathbb{R}$-ample $\mathbb{R}$-divisor H such that for $\sigma := \inf\{s > 0 \,|\, K_X + \Delta + sH \in \overline{\mathrm{NE}}^1(X)\}$ the following holds.

1. *The divisor $K_X + \Delta + H$ is nef.*
2. *$(K_X + \Delta + \sigma H)^\perp \cap (\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}) = R$.*
3. *$(K_X + \Delta + sH)^\perp \cap (\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}) = 0$, if $s > \sigma$.*

Remark 3.4 (Picture)*.* The assertion of the previous lemma can be visualized in the following picture which shows the $(K_X+\Delta)$-negative part of the cones.



The picture suggests that the minimal model program with scaling of H terminates with the contraction of R.

Proof. We start with the construction of H. By definition of *exposed* there exists an $\mathbb{R}$-divisor D such that

$$R = D^\perp \cap (\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\ge 0}),$$

and D is non-negative on $\overline{\mathrm{NM}}_1(X)+\overline{\mathrm{NE}}_1(X)_{K_X+\Delta\ge 0}$. We claim that there is an $a>0$ such that $D-a(K_X+\Delta)$ is an ample $\mathbb{R}$-divisor. If $-(K_X+\Delta)$ is ample, we can take any sufficiently large a. Thus we may assume without loss of generality that $-(K_X+\Delta)$ is not ample. Since D and $K_X+\Delta$, considered as forms on $N_1(X)$, have no common zeros in $\overline{\mathrm{NE}}_1(X)\setminus\{0\}$, there exists a hyperplane $Z\subset N_1(X)$ such that

$$\left(D^\perp \cap (K_X+\Delta)^\perp\right) \subset Z \quad \text{and} \quad Z\cap \overline{\mathrm{NE}}_1(X) = \{0\}.$$

It follows from basic linear algebra that there exist $b, c\in\mathbb{R}$ such that $Z=(bD+c(K_X+\Delta))^\perp$, i.e., for any $x\in\overline{\mathrm{NE}}_1(X)\setminus\{0\}$ the inequality $(bD+c(K_X+\Delta))\cdot x\neq 0$ holds. This inequality still holds if we slightly vary b and c, thus we may assume that both b and c are not zero. We set $a:=-\frac{c}{b}$, and it remains to show that the resulting divisor is ample and that a is positive. Since $-(K_X+\Delta)$ is not ample, there exists $w\in\overline{\mathrm{NE}}_1(X)\setminus\{0\}$ intersecting $K_X+\Delta$ trivially. Thus we have $(D-a(K_X+\Delta))\cdot w = D\cdot w>0$, by the choice of D. Since the cone $\overline{\mathrm{NE}}_1(X)$ is connected, the divisor $D-a(K_X+\Delta)$ intersects any element of $\overline{\mathrm{NE}}_1(X)\setminus\{0\}$ positively, and Kleiman's ampleness criterion implies that the divisor is ample. To see that a is positive we consider the intersection product of $D-a(K_X+\Delta)$ with a generator z of R. Since this is positive, a is positive and the claim follows.

To finish the construction of H, let $l>0$ be a real number such that $K_X+\Delta+l(D-a(K_X+\Delta))$ is nef, and set

$$H := l(D-a(K_X+\Delta)).$$

It remains to show that H has the required properties. Property (1) follows immediately from the construction of H. To show Property (2), we first observe that D is numerically proportional to $K_X + \Delta + \frac{1}{al}H$. By [BDPP04, Theorem 2.2], the cones $\overline{\mathrm{NM}}_1(X)$ and $\overline{\mathrm{NE}}^1(X)$ are dual. Consequently, the divisor D is pseudo-effective, in particular $\sigma \leq \frac{1}{al}$. Moreover, $K_X + \Delta + sH$ intersects any generator of R negatively for any $s < \frac{1}{al}$. Therefore $\sigma = \frac{1}{al}$ and D is numerically proportional to $K_X + \Delta + \sigma H$, hence

$$\begin{aligned}(K_X + \Delta + \sigma H)^{\perp} &\cap (\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}) \\ &= D^{\perp} \cap (\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}) = R,\end{aligned}$$

as required.

To prove the last Property (3), recall that H is ample. This immediately implies that for any $s > 0$ and $\gamma \in \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}$ the intersection product $(K_X + \Delta + sH) \cdot \gamma$ is positive. Moreover, for any $s > \sigma$ the divisor $K_X + \Delta + sH = K_X + \Delta + \sigma H + (s - \sigma)H$ is big, thus it intersects any $\gamma \in \overline{\mathrm{NM}}_1(X)$ positively. □

With the previous lemmas at hand, we are now able to prove the following

Theorem 3.5 (Moving Cone Theorem)**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair, where Δ is an effective $\mathbb{R}$-divisor and X is projective. Let R be an exposed ray of the cone $\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}$ that intersects $(K_X + \Delta)$ negatively. Then there is an irreducible locally closed subset H_R of the Hilbert scheme of curves on X such that*

1. *each closed point of H_R corresponds to a curve that generates R,*

2. *for any closed subset $Z \subset X$ of $\operatorname{codim}_X(Z) \geq 2$, there is a non-empty open subset H_R^Z of H_R such that any curve that corresponds to a closed point of H_R^Z avoids Z,*

3. *there exists a run of the minimal model program with scaling that terminates with a Mori fiber Space*

$$\begin{tikzcd} X \arrow[r, dashed, "\lambda_R"] & X_R \arrow[d, "\pi_R"] \\ & B_R, \end{tikzcd}$$

such that any closed point of H_R corresponds to a curve that is contained in the open set $U \subset X$, where λ_R is an isomorphism of U onto its image. Moreover, the image of this curve via λ_R is contained in a fiber of π_R.

Proof. Let (X, Δ) and R be as in Lemma 3.3. We apply this lemma and obtain an $\mathbb{R}$-ample $\mathbb{R}$-divisor H and a positive number σ that satisfy properties (1), (2), (3). The existence of R implies that $K_X + \Delta$ is not pseudo-effective, see [BDPP04, Theorem 2.2]. By Theorem 2.10 we obtain a terminating minimal model program with scaling of H which we denote $(\varphi_i, s_i)_{i \in I}$. By Proposition 1.21, there exists for any $0 < \varepsilon < \sigma$ an $\mathbb{R}$-divisor $\Delta_\varepsilon \equiv \Delta + \varepsilon H$ such that (X, Δ_ε) is klt. It follows from Theorem 2.10 that the sequence $(\varphi_i, s_i - \varepsilon)_{i \in I}$ is a minimal model program with scaling of H for the pair (X, Δ_ε), and that both minimal model programs terminate with a Mori fiber space, say $\pi : X_l \to B$. Denote by λ the composition of all φ_i, $i \in I$, then we obtain the following diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\lambda"] & X_l \arrow[d, "\pi"] \\ & B. \end{tikzcd}$$

The family of curves constructed in Lemma 3.1 gives the required subset H_R of the Hilbert scheme. It remains to show that the class γ of a curve corresponding to a closed point of H_R generates R. Since γ is moving and because of Property (2) of Lemma 3.3, it suffices to prove that the equality

$$(K_X + \Delta + \sigma H) \cdot \gamma = 0$$

holds.

To this end, we consider the decreasing sequence of positive numbers

$$s_1 - \varepsilon \geq s_2 - \varepsilon \geq \cdots \geq s_l - \varepsilon \geq 0.$$

Since the inequality $s_l - \varepsilon \geq 0$ holds for all $\varepsilon \in [0, \sigma)$, we obtain $s_l \geq \sigma$. To show $s_l \leq \sigma$, we note that if C is any curve on a general fiber of π, then the class γ is numerically proportional to $\lambda^*([C])$. Therefore

$$\begin{aligned} 0 &= (K_{X_l} + \Delta_l + s_l \lambda_* H) \cdot C \\ &= (K_X + \Delta + s_l H) \cdot \gamma. \end{aligned}$$

Consequently, Property (3) of Lemma 3.3 implies $s_l = \sigma$. We now apply Property (2) of Lemma 3.3 again, which implies that R is generated by γ. $\square$

Chapter 4

$\mathbb{Q}$-factorializations of dlt pairs

4.1 $\mathbb{Q}$-factorialization

If (X, Δ) is a dlt pair where X is *not* $\mathbb{Q}$-factorial, then we cannot apply Theorem 3.5. To overcome this difficulty, we aim to replace X with a small, $\mathbb{Q}$-factorial modification.

Definition 4.1 ($\mathbb{Q}$-factorialization)**.** *Let X be a normal projective variety. A* $\mathbb{Q}$-factorialization of X *is a proper birational morphism $f : Y \to X$ where Y is a normal projective $\mathbb{Q}$-factorial variety and the exceptional set of f has codimension greater than or equal to two in Y.*

Example 4.2. *Let (Y, Δ) be a $\mathbb{Q}$-factorial dlt pair. Assume that there is a $(K_Y + \Delta)$-negative extremal ray R of the cone $\overline{\mathrm{NE}}_1(Y)$ whose associated contraction map $\mathrm{cont}_R : Y \to X$ is small. Then X is not $\mathbb{Q}$-factorial and $\mathrm{cont}_R : Y \to X$ is a $\mathbb{Q}$-factorialization of X.*

The existence of $\mathbb{Q}$-factorializations of dlt pairs is a result of [BCHM10].

Proposition 4.3 ([BCHM10, Corollary 1.4.3])**.** *Let (X, Δ) be a log canonical pair and let $f : W \to X$ be a log resolution. Suppose that there is a divisor Δ_0 such that $K_X + \Delta_0$ is klt. Let $\mathfrak{E}$ be any set of valuations of f-exceptional divisors which satisfies the following two properties:*

1. *$\mathfrak{E}$ contains only valuations of log discrepancy at most one, and*
2. *the centre of every valuation of log discrepancy one in $\mathfrak{E}$ does not contain any non-klt centres.*

Then we may find a proper birational morphism $\pi : Y \to X$, such that Y is $\mathbb{Q}$-factorial and the exceptional divisors of π correspond to the elements of $\mathfrak{E}$. □

We state the explicit result for dlt pairs in the following corollary. For klt pairs this is also explained in the discussion after the formulation of Corollary 1.4.3 in [BCHM10, p.9].

Corollary 4.4 (Existence of $\mathbb{Q}$-factorializations)**.** *Let (X, Δ) be a projective dlt (resp. klt) pair. Then a $\mathbb{Q}$-factorialization of X exists. Moreover, if $f : Y \to X$ is an arbitrary $\mathbb{Q}$-factorialization of X, and $\Delta_Y := f_*^{-1}\Delta$ is the strict transform of Δ, then the pair (Y, Δ_Y) is dlt (resp. klt).*

Proof. If (X, Δ) is dlt, then we may apply Proposition 1.21 and find a divisor Δ' such that (X, Δ') is klt. Therefore, the existence of a $\mathbb{Q}$-factorialization follows from Proposition 4.3, if we set $\mathfrak{E} = \emptyset$.

Now let $f : Y \to X$ be an arbitrary $\mathbb{Q}$-factorialization, and let Δ_Y be the strict transform of Δ. Note that f is small, thus the equalities

$$f^*(K_X + \Delta) = K_Y + \Delta_Y \quad \text{and} \quad f_*\Delta_Y = \Delta_X$$

hold. Moreover, the coefficients of Δ_Y are exactly the coefficients of Δ, hence $\lfloor \Delta \rfloor = 0$ iff $\lfloor \Delta_Y \rfloor = 0$. A straightforward calculation yields that the discrepancies of (X, Δ) and (Y, Δ_Y) are equal, which in turn implies that (Y, Δ_Y) is klt if (X, Δ) is klt; see also [KM98, Lemma 2.30].

To show that the property *dlt* is preserved, recall its definition, [KM98, Definition 2.37]. According to this, it remains to prove that the strict transform of an snc divisor on the smooth locus U of X is an snc divisor on $f^{-1}(U) \subset Y$. We even claim that $f|_{f^{-1}(U)}$ is an isomorphism. Indeed, if $x \in U$ is a point where the inverse map f^{-1} is not regular, then [Sha94, Chapter II.4, Theorem 2] immediately implies that f contracts a divisor. This contradicts the assumption that f does not contract divisors. □

Notation 4.5. Given a dlt pair (X, Δ) and a $\mathbb{Q}$-factorialization $f : Y \to X$, we will denote by Δ_Y the strict transform of Δ as defined in Corollary 4.4.

Remark 4.6. In fact, $\mathbb{Q}$-factorializations of a given variety are generally not unique. As we will see in Section 4.3, any log flop of a $\mathbb{Q}$-factorialization yields a new $\mathbb{Q}$-factorialization.

4.2 $\mathbb{Q}$-factorializations of log Fano varieties

We consider dlt pairs (X, Δ) with $-(K_X + \Delta)$ ample. Unfortunately, if $f : Y \to X$ is a $\mathbb{Q}$-factorialization, then the divisor $-(K_Y + \Delta_Y)$ is generally not ample, unless f is the identity. Nevertheless, the following lemmas hold.

Lemma 4.7. *Let (X, Δ) be a projective dlt pair with $-(K_X + \Delta)$ ample, and let $f : Y \to X$ be a $\mathbb{Q}$-factorialization of X. Then the divisor $-(K_Y + \Delta_Y)$ is big and nef.*

Proof. Since $-(K_X + \Delta)$ is ample, it is in particular big and nef. The pullback of a big and nef $\mathbb{R}$-Cartier divisor via a birational morphism is again big and nef. □

Lemma 4.8. *Let (X, Δ) be a $\mathbb{Q}$-factorial projective klt pair, and assume that $-(K_X+\Delta)$ is big and nef. Then the cones $\overline{\mathrm{NM}}_1(X)$ and $\overline{\mathrm{NE}}_1(X)$ are rational polyhedrons. Moreover, for any divisor D any minimal model program for the pair (X, D) can be run and terminates.*

Proof. Recall from [BDPP04, Theorem 2.2] that a divisor is big if and only if it intersects any $\gamma \in \overline{\mathrm{NM}}_1(X)\setminus\{0\}$ positively. Hence, the cones $\overline{\mathrm{NM}}_1(X)\setminus\{0\}$ and $\overline{\mathrm{NE}}_1(X)_{K_X+\Delta=0} \setminus \{0\}$ are disjoint, and by convexity there exists an $\mathbb{R}$-divisor B that separates these cones, i.e.,

$$\begin{aligned} \overline{\mathrm{NM}}_1(X) \setminus \{0\} &\subset N_1(X)_{B>0}, \quad \text{and} \\ \overline{\mathrm{NE}}_1(X)_{K_X+\Delta=0} \setminus \{0\} &\subset N_1(X)_{B<0}. \end{aligned}$$

In particular, the divisor B is big.

We claim that for sufficiently small $\varepsilon > 0$ the pair $(X, \Delta + \varepsilon B)$ is still klt and the divisor $-(K_X + \Delta + \varepsilon B)$ is ample. To prove the claim we first note that for any sufficiently small $\varepsilon > 0$ the pair $(X, \Delta + \varepsilon B)$ is klt, see Lemma 1.16. To show that $-(K_X + \Delta + \varepsilon B)$ is ample for $0 < \varepsilon \ll 1$, we use Kleiman's ampleness criterion. According to this, we must show that the intersection product with any class $\gamma \in \overline{\mathrm{NE}}_1(X) \setminus \{0\}$ is positive. This is obviously true for $\gamma \in \overline{\mathrm{NE}}_1(X)_{B<0} \setminus \{0\}$, thus it remains to show that the intersection product with any class $\gamma \in \overline{\mathrm{NE}}_1(X)_{B\geq 0} \setminus \{0\}$ is positive. Let $H \subset N_1(X)_{\mathbb{R}} \setminus \{0\}$ be an affine hyperplane such that its intersection with the Mori cone is a *cross section*, i.e.,

$$\emptyset \neq \overline{\mathrm{NE}}_1(X)|_H := H \cap \overline{\mathrm{NE}}_1(X)$$

is compact, and

$$\overline{\mathrm{NE}}_1(X) = \mathbb{R}^{\geq 0} \cdot \overline{\mathrm{NE}}_1(X)|_H.$$

It suffices to show that $-(K_X+\Delta+\varepsilon B)$ intersects any class $\gamma \in \overline{\mathrm{NE}}_1(X)|_{H,B\geq 0}$ positively. Since $\overline{\mathrm{NE}}_1(X)|_{H,B\geq 0}$ is compact, the continuous function

$$\begin{aligned} \overline{\mathrm{NE}}_1(X)|_{H,B\geq 0} &\to \mathbb{R} \\ \gamma &\mapsto -(K_X + \Delta + \varepsilon B) \cdot \gamma \end{aligned}$$

has a global minimum $m_\varepsilon \in \mathbb{R}$. This minimum depends continuously on ε and is positive for $\varepsilon = 0$. Consequently, the claim follows.

The Cone Theorem implies that $\overline{\mathrm{NE}}_1(X)$ is a rational polyhedron, and the assertion for $\overline{\mathrm{NM}}_1(X)$ is proved in [BCHM10, Corollary 1.3.5]. To show that for any divisor D the minimal model program terminates, we apply [BCHM10, Corollary 1.3.2] to $(X, \Delta + \varepsilon B)$. According to this, the variety X is a Mori dream space (see [HK00, Definition 1.10] for the definition), and it follows from [HK00, Proposition 1.11] that the minimal model program can be run for any divisor and terminates. ∎

Corollary 4.9. *Let (X, Δ) be a projective dlt pair with $-(K_X + \Delta)$ ample, and let $f : Y \to X$ be any $\mathbb{Q}$-factorialization of X. Then the cones $\overline{\mathrm{NE}}_1(Y)$ and $\overline{\mathrm{NM}}_1(Y)$ are rational polyhedrons and for any divisor the minimal model program can be run and terminates.*

Proof. By Lemma 4.8 it suffices to show that there is a divisor Δ' on Y such that (Y, Δ') is klt and $-(K_Y + \Delta')$ is big and nef. In order to prove the existence of Δ' we first pick an ample divisor H on X. It follows from Proposition 1.21 that for any $\varepsilon > 0$ the divisor $\Delta + \varepsilon H$ is $\mathbb{R}$-linearly equivalent to a divisor Δ_ε such that (X, Δ_ε) is klt. Moreover, if ε is sufficiently small then $-(K_X + \Delta_\varepsilon)$ is still ample. By Corollary 4.4 the pair $\big(Y, f_*^{-1}(\Delta_\varepsilon)\big)$ is klt, and Lemma 4.7 implies that $-(K_Y + \Delta')$ is big and nef. ☐

4.3 Log flops of $\mathbb{Q}$-factorializations

One main step in the proof of the Isotriviality Theorem 5.1 is to find a certain exposed moving ray which intersects a given pseudo-effective divisor D non-trivially. This is not a big problem if the pair (X, Δ) is $\mathbb{Q}$-factorial and log Fano. However, if we drop the assumption that X is $\mathbb{Q}$-factorial, then we have to switch over to a $\mathbb{Q}$-factorialization $f : Y \to X$ which is generally not log Fano, as we have seen. Indeed, it could happen in this situation that the set of exposed moving rays is entirely contained in the hyperplane $(f_*^{-1}D)^\perp$ in $N_1(Y)$.

To prove the Isotriviality Theorem 5.1 in the non-$\mathbb{Q}$-factorial case we have to find the *right* $\mathbb{Q}$-factorialization. We will see that a certain class of birational maps gives us new $\mathbb{Q}$-factorializations. These *log flops* are strongly connected to flips.

Definition 4.10 (Log flops, see [Mat02, Conjecture 11.3.3])**.** *Let (X, Δ) be a dlt pair. A* flopping contraction *is a proper birational morphism $f : X \to Y$ to a normal variety Y such that the exceptional set has codimension at least two in X and $K_X + \Delta$ is numerically f-trivial.*

*Assume that there exists an $\mathbb{R}$-Cartier divisor D on X such that the divisor $-(K_X + \Delta + D)$ is f-ample, and the $(K_X + \Delta + D)$-flip of f exists. Then this flip is also called the D-*log flop of f *or* log flop *for short.*

Remark 4.11. If $\Delta = 0$, a log flop is a *flop*, see [KM98, Definition 6.10].

Lemma 4.12 (Existence of log flops on $\mathbb{Q}$-factorializations)**.** *Let (X, Δ) be a log Fano dlt pair with $\mathbb{Q}$-factorialization (Y, Δ_Y). Let D be an arbitrary $\mathbb{R}$-divisor on Y, and let $F \subset \overline{\mathrm{NE}}_1(Y)_{K_Y + \Delta_Y = 0}$ be an extremal face that is contained in $D < 0$. Then*

1. *the contraction $g : Y \to Z$ of F exists and factorizes the $\mathbb{Q}$-factorialization map $f : Y \to X$, and*

2. the D-log flop of F exists and is another $\mathbb{Q}$-factorialization of X.

Proof. By Corollary 4.9 the minimal model program for the pair (Y, D) is well-defined, in particular the contraction $g : Y \to Z$ of F exists. To prove that g is small, we note that the map $f : Y \to X$ is the contraction of the extremal face $G := \overline{\mathrm{NE}}_1(Y) \cap (K_Y + \Delta_Y)^\perp$. Indeed, it is easy to see that a curve C is contracted by f iff it intersects $K_Y + \Delta_Y$ trivially. Since this is a small contraction and $F \subset G$ is a subface, any curve that is contracted by g is also contracted by f. Therefore, the exceptional set of g has codimension at least two, hence g is a small contraction. It remains to show that g factorizes f. We have already seen that f contracts each fiber of g. Thus the assertion follows immediately from [Deb01, Lemma 1.15(b)]. This implies (1).

Item (2) is an immediate consequence of Corollary 4.9, and is visualized in the following commuting diagram.

$$\begin{tikzcd}
Y \arrow[rr, dashed, "D\text{−log flop}"] \arrow[dr, "g"] \arrow[ddr, "f"'] & & Y^+ \arrow[dl, "g^+"'] \arrow[ddl, "f^+"] \\
& Z \arrow[d] & \\
& X &
\end{tikzcd}$$

The map f^+ is the new $\mathbb{Q}$-factorialization which is obtained by the D-log flop. $\square$

We finally come to the main result of this section. Roughly speaking, the following proposition asserts that for any effective Weil-divisor D on X, there exists a $\mathbb{Q}$-factorialization $f : Y \to X$ such that $(f_*^{-1} D)^\perp$ is in a sufficiently general position relative to the moving cone $\overline{\mathrm{NM}}_1(Y)$.

Proposition 4.13. *Let (X, Δ) be a projective dlt pair with $-(K_X + \Delta)$ ample, and let $D \neq 0$ be an effective $\mathbb{R}$-Weil-divisor on X. Then there exists a $\mathbb{Q}$-factorialization (Y, Δ_Y) such that the cone $\overline{\mathrm{NE}}_1(Y)_{K_Y+\Delta_Y=0} + \overline{\mathrm{NM}}_1(Y)$ has a $(K_Y + \Delta_Y)$-negative exposed ray which is not contained in $D_Y^\perp$, where D_Y is the strict transform of D.*

The proof of Proposition 4.13 is quite long, and will be given in the following two Sections 4.3.1 and 4.3.2.

4.3.1 Preparation for the proof of Proposition 4.13

The proof of the proposition consists of the following steps:

1. Use log flops to construct the $\mathbb{Q}$-factorialization, and
2. prove that the $\mathbb{Q}$-factorialization satisfies Proposition 4.13.

Since the second part includes some tedious but not very challenging computations, we divide these computations into the following two lemmas. The first lemma provides a criterion to decide whether a given ray in a cone is extremal, and can be formulated in terms of convex geometry, the second one analyzes the image of exposed moving rays via flips.

Lemma 4.14 (Criterion of extremeness)**.** *Let V be a finite dimensional real vector space, and let $\mathcal{C}^1, \mathcal{C}^2 \subset V$ be two closed, convex cones. Let $\alpha \in V^\vee$ be a linear form and $R \subset \mathcal{C}^1$ a ray such that the following conditions hold.*

- $R = \mathcal{C}^1_{\alpha=0}$, *and* $\mathcal{C}^1 \subset \{\alpha \geq 0\}$,
- $\mathcal{C}^2 \subset \{\alpha \geq 0\}$, *and*
- $R \not\subset \mathcal{C}^2$ *and* $(-R) \not\subset \mathcal{C}^2$.

Then R is an extremal ray of $\mathcal{C}^1 + \mathcal{C}^2$.

Proof. Observe that the set $\mathcal{D} := (\mathcal{C}^1 + \mathcal{C}^2)_{\alpha=0}$ is an extremal face of $\mathcal{C}^1 + \mathcal{C}^2$. Therefore, the face $\mathcal{D}$ decomposes into

$$\mathcal{D} = \mathcal{C}^1_{\alpha=0} + \mathcal{C}^2_{\alpha=0} = R + \mathcal{C}^2_{\alpha=0}.$$

Since $R \not\subset \mathcal{C}_2$ and $(-R) \not\subset \mathcal{C}^2$, it follows that R is an extremal ray of $\mathcal{D}$. To finish the proof, recall that being extremal is a transitive property, i.e., since R is extremal in $\mathcal{D}$ and $\mathcal{D}$ is extremal in $\mathcal{C}^1 + \mathcal{C}^2$, the ray R is also extremal in $\mathcal{C}^1 + \mathcal{C}^2$, as required. $\square$

Remark 4.15. The ray $R \subset \mathcal{C}^1 + \mathcal{C}^2$ is not necessarily exposed.

Lemma 4.16 (Flips of exposed rays)**.** *Let X, Y be $\mathbb{Q}$-factorial normal projective varieties, and let $\varphi : X \dashrightarrow Y$ be a birational map which is an isomorphism in codimension one. Let $F \subset \overline{\mathrm{NM}}_1(X)$ be an exposed face, cut out by a pseudo-effective $\mathbb{R}$-divisor D. Then the image $\varphi_*(F)$ of F via the numerical pushforward of curves is an exposed face of $\overline{\mathrm{NM}}_1(Y)$ which is cut out by $\varphi_*(D)$.*

Proof. The assumptions imply that the vector spaces $N^1(X)_\mathbb{R}$ and $N^1(Y)_\mathbb{R}$ are isomorphic via the pullback and pushforward of divisors. Moreover, the restriction of the pushforward map to $\overline{\mathrm{NE}}^1(X)$ gives a bijection between the pseudo-effective cones $\overline{\mathrm{NE}}^1(X)$ and $\overline{\mathrm{NE}}^1(Y)$. By duality, the numerical pushforward and pullback of curve classes yields an isomorphism between $N_1(X)_\mathbb{R}$ and $N_1(Y)_\mathbb{R}$, and by [BDPP04, Theorem 2.2], a bijection between $\overline{\mathrm{NM}}_1(X)$ and $\overline{\mathrm{NM}}_1(Y)$, in particular $\varphi_*(F) \subset \overline{\mathrm{NM}}_1(Y)$. Since the divisor D is pseudo-effective, its pushforward $\varphi_*(D)$ is pseudo-effective as well.

It remains to prove that the equality $\varphi_*(D)^\perp \cap \overline{\mathrm{NM}}_1(Y) = \varphi_*(F)$ holds. This follows easily from the projection formula and the fact that pushforward and pullback are mutually inverse bijections. These computations are straightforward, thus we omit them. $\square$

Remark 4.17. The lemma is also true for extremal faces, but becomes false if the map is not an isomorphism in codimension one, e.g., if φ is a divisorial contraction.

4.3.2 Proof of Proposition 4.13

We start with an arbitrary $\mathbb{Q}$-factorialization $f_0 : Y_0 \to X$. Set $\Delta_0 := \Delta_{Y_0}$, and let $D_0 := (f_0^{-1})_* D$ be the strict transform of the effective Weil divisor D on X. Let R_0 be a $(K_{Y_0} + \Delta_{Y_0})$-negative extremal ray of the moving cone $\overline{\mathrm{NM}}_1(Y_0)$ which is not contained in $D_0^{\perp}$. By Corollary 4.9, the cone $\overline{\mathrm{NM}}_1(Y_0)$ is polyhedral, therefore R_0 is exposed and there is a pseudo-effective $\mathbb{R}$-divisor D_{R_0} such that

$$R_0 = \overline{\mathrm{NM}}_1(Y_0)_{D_{R_0}=0}.$$

Because of Corollary 4.9 we can run the relative minimal model program for the pair $(Y_0, \Delta_0 + D_{R_0})$ over X. Observe that this minimal model program only involves log flops and yields by Lemma 4.12 a sequence of $\mathbb{Q}$-factorializations of X. Because of Corollary 4.9 we eventually obtain a minimal model over X which is expressed in the following commutative diagram

$$\begin{tikzcd} Y_0 \arrow[rr, dashed, "\varphi"] \arrow[dr, "f_0"'] & & Y \arrow[dl, "f"] \\ & X, & \end{tikzcd}$$

moreover the divisor $K_Y + \varphi_*(D_{R_0}) + \varphi_*(\Delta_0)$ is f-nef.

This finishes the construction of the $\mathbb{Q}$-factorialization, and it remains to show that Y has the required properties. To this end, we first observe that Y_0, Y, and φ satisfy the conditions of Lemma 4.16, hence the ray $R := \varphi_*(R_0)$ is an exposed ray of $\overline{\mathrm{NM}}_1(Y)$, cut out by $D_R := \varphi_*(D_{R_0})$. Moreover, since $K_Y + \varphi_*(\Delta_0) + D_R$ is f-nef and any $K_Y + \varphi_*(\Delta_0)$-trivial curve is contracted by f, we obtain the inclusion

$$\overline{\mathrm{NE}}_1(Y)_{K_Y+\varphi_*(\Delta_0)=0} \subset \{D_R \geq 0\}.$$

By Lemma 4.7, the divisor $K_Y + \varphi_*(\Delta_0)$ is big, thus

$$\overline{\mathrm{NE}}_1(Y)_{K_Y+\varphi_*(\Delta_0)=0} \cap \overline{\mathrm{NM}}_1(Y) = 0,$$

and Lemma 4.14 applies. Altogether, the ray R is an extremal ray of $\overline{\mathrm{NE}}_1(Y)_{K_Y+\varphi_*(\Delta_0)=0} + \overline{\mathrm{NM}}_1(Y)$. Since this cone is polyhedral by Corollary 4.9, the ray R is even an exposed ray.

To finish the proof, we have to show that R is not contained in the hyperplane $D_Y^{\perp}$, where D_Y is the strict transform of D. Since φ is an isomorphism in codimension one, the divisor D_Y is also given by the pushforward of D_0 via φ. The projection formula immediately implies that D_Y intersects any non-zero class $\gamma \in R$ positively, and the proof is finished. $\square$

Chapter 5

Families over log Fano varieties

In this section we will prove the Isotriviality Theorem.

Theorem 5.1 (Isotriviality Theorem)**.** *Let (X, Δ) be a dlt pair where Δ is an effective $\mathbb{R}$-divisor, where $-(K_X + \Delta)$ is $\mathbb{R}$-ample, and X is projective. Let $T \subset X$ be a subvariety of $\operatorname{codim}_X(T) \geq 2$ such that $X \setminus (T \cup \operatorname{Supp}\lfloor\Delta\rfloor)$ is smooth. Then any smooth family of canonically polarized varieties over $X \setminus (T \cup \operatorname{Supp}\lfloor\Delta\rfloor)$ is isotrivial.*

The proof of the Isotriviality Theorem is by induction over the dimension of X. As a part of the induction we prove Theorem 5.2, which is stated below. Assuming that Theorem 5.2 holds in dimension n, we first show that the family is necessarily isotrivial on certain moving curves, namely the curves we constructed in Theorem 3.5. Next we show that for any proper algebraic subset Z of X there exists a moving curve that is not contained in Z and intersects Z properly. On this curve the family is isotrivial. This finally finishes the proof of the Isotriviality Theorem 5.1 for n-dimensional varieties.

Assuming that Theorem 5.1 holds in dimension n we will prove Theorem 5.2 in the $(n + 1)$-dimensional case. This finally finishes the proof of both theorems in arbitrary dimension.

5.1 A result of Kebekus and Kovács

Given a smooth projective family of canonically polarized varieties, it is proved in [KK10, Theorem 1.2] that any run of the minimal model program for the base terminates with a Kodaira or Mori fiber space that factors the moduli map birationally, if the dimension of the base is less than or equal to three. A proof for surfaces can be found in [KK08]. Since we discuss log Fano varieties, we will focus on the case of negative Kodaira-Iitaka dimension. As part of the induction we show that this result holds in arbitrary dimension.

Theorem 5.2 (Moduli and the minimal model program, [KK10, Theorem 1.2])**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair of negative Kodaira-Iitaka-dimension. Let $T \subset X$ be a subvariety of* $\operatorname{codim}_X(T) \geq 2$ *such that* $X \setminus (T \cup \operatorname{Supp}\lfloor\Delta\rfloor)$ *is smooth, and let* $\mu : X \setminus (T \cup \operatorname{Supp}(\lfloor\Delta\rfloor)) \to \mathfrak{M}$ *be a map to the coarse moduli space of canonically polarized manifolds which is induced by a smooth projective family over* $X \setminus (T \cup \operatorname{Supp}\lfloor\Delta\rfloor)$*.*

Then any terminating minimal model program $\lambda : X \dashrightarrow X'$ leads to a Mori fiber space $\pi : X' \to B$ which factors the moduli map μ via $\pi \circ \lambda$ birationally. In other words, there exists a rational map $\nu : B \dashrightarrow \mathfrak{M}$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, dashed, "\lambda"] \arrow[d, dashed, "\mu"'] & X' \arrow[d, "\pi"] \\ \mathfrak{M} & B. \arrow[l, dashed, "\nu"'] \end{tikzcd}$$

5.2 Proof of Theorems 5.1 and 5.2

5.2.1 General strategy and setup

The proof of the Theorems 5.1 and 5.2 is by induction on the dimension. For arbitrary x the notation *Theorem* x_n stands for "Theorem x in dimension at most n". The proof is given in the following three steps.

Step 1: The case where the Picard number of X is one

In this case, both theorems assert that a smooth family of canonically polarized varieties is isotrivial over a logarithmic log Fano dlt pair with Picard number one. A proof of this case is given in [KK10] if $\dim X \leq 3$. It can be generalized to arbitrary dimension, since the Bogomolov-Sommese vanishing for $\mathbb{Q}$-factorial lc pairs holds in arbitrary dimensions, see [GKKP10]. Note that this case implies both theorems if X is a curve.

Step 2: Theorem 5.2$_n$ implies Theorem 5.1$_n$

Assuming Theorem 5.2$_n$, it follows from Proposition 4.13 and Theorem 3.5 that the family is isotrivial on "sufficiently many" moving curves. This implies Theorem 5.1$_n$.

Step 3: Theorem 5.1$_n$ implies Theorem 5.2$_{n+1}$

Finally, we can apply Theorem 5.1$_n$ to the general fiber of a Mori fiber space, which in turn implies Theorem 5.2$_{n+1}$.

5.2.2 The case where the Picard number of X is one

To show that Theorem 5.2 holds if the Picard number is one, we have to use certain invertible sheaves $\mathcal{A} \subset \operatorname{Sym}^n \Omega^1_X(\log \Delta)$ which were introduced by Viehweg and Zuo in [VZ02]. These *Viehweg-Zuo sheaves* are also discussed in [KK10, Chapter 5].

Theorem 5.3 ([KK10, Theorem 6.1])**.** *Let (Z, Δ) be a log canonical logarithmic pair where Z is projective $\mathbb{Q}$-factorial. Assume that there exists a Viehweg-Zuo sheaf $\mathcal{A}$ of positive Kodaira-Iitaka dimension, and that the divisor $-(K_Z + \Delta)$ is nef. Then the Picard number of Z is greater than one.*

Proof. After replacing the old version of the Bogomolov-Sommese Vanishing Theorem [KK10, Theorem 3.5] with the new one [GKKP10, Theorem 7.2], the proof given in [KK10, Theorem 6.1] applies verbatim for arbitrary dimension. □

Lemma 5.4 (Picard number one)**.** *Let (X, Δ) and μ be as in Theorem 5.2 and assume that the Picard number of X is one. Then μ is constant.*

Proof. Assume that μ is not constant. Since the Picard number is one, the $\mathbb{R}$-divisor Δ is nef, in particular the pair $(X, \lfloor\Delta\rfloor)$ is dlt log Fano. Thus we can assume without loss of generality that Δ is reduced.

Let $\pi : \tilde{X} \to X$ be a log resolution of (X, Δ) such that $\pi^{-1}(T)$ is contained in the π-exceptional divisor $E \in \operatorname{Div}(\tilde{X})$. Set $\tilde{\Delta} := E + \pi_*^{-1}(\Delta)$, and note that $\tilde{\Delta}$ is snc and $\pi_*\tilde{\Delta} = \Delta$. Use π to obtain a family of positive variation over $\tilde{X} \setminus \operatorname{Supp} \tilde{\Delta}$. It follows from [VZ02, Theorem 1.4] that there exists a Viehweg-Zuo sheaf $\tilde{\mathcal{A}} \subset \operatorname{Sym}^n \Omega^1_{\tilde{X}}(\log \tilde{\Delta})$ with $\kappa(\tilde{\mathcal{A}}) > 0$. Apply [KK10, Lemma 5.2] to obtain a Viehweg-Zuo sheaf $\mathcal{A} \subset \operatorname{Sym}^n \Omega^1_X(\log \Delta)$ with $\kappa(\mathcal{A}) \geq \kappa(\tilde{\mathcal{A}}) > 0$. By Theorem 5.3 the Picard number of X is greater than one, which is a contradiction. □

Since curves always have Picard number one, we obtain the start of the induction.

Corollary 5.5 (Start of induction, [Kov00, 0.2])**.** *Theorem 5.2 and Theorem 5.1 hold in dimension one.* □

5.2.3 Theorem 5.2_n implies Theorem 5.1_n

We first use Theorem 5.2_n to show that a smooth family of canonically polarized varieties is isotrivial on certain moving curves.

Proposition 5.6. *Assume Theorem 5.2_n. Let (X, Δ), T and μ be as in Theorem 5.2_n. Let R be a $(K_X + \Delta)$-negative exposed ray of the cone $\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}$. Let H_R be the associated subset of the Hilbert scheme as in Theorem 3.5. Then there exists a non-empty open subset $H_{R,\mu}$*

of H_R such that any curve $C \subset X$ that corresponds to a closed point of $H_{R,\mu}$ satisfies the following properties.

1. *The curve C is not contained in $T \cup \operatorname{Supp}\lfloor\Delta\rfloor$,*
2. *the moduli map μ is constant on $C \cap (X \setminus (T \cup \operatorname{Supp}\lfloor\Delta\rfloor))$,*
3. *for any closed subset $Z \subset X$ of $\operatorname{codim}_X(Z) \geq 2$, there is a non-empty open subset $H^Z_{R,\mu}$ of H_R such that any curve that corresponds to a closed point of $H^Z_{R,\mu}$ avoids Z.*

Proof. We apply the Moving Cone Theorem 3.5 and obtain an associated minimal model program $\lambda : X \dashrightarrow X'$ and a Mori fibration $\pi : X' \to B$ such that any curve that corresponds to a point of H_R is contained in the locus where λ is well-defined and is mapped to a fiber of π. Theorem 5.2_n gives a commutative diagram of rational maps

$$\begin{tikzcd} X \arrow[r, dashed, "\lambda"] \arrow[d, dashed, "\mu"'] & X' \arrow[d, "\pi"] \\ \mathfrak{M} & B, \arrow[l, dashed] \end{tikzcd}$$

which becomes a diagram of morphisms on appropriate non-empty open sets. More precisely, let $V \subset B$ be the domain of $B \dashrightarrow \mathfrak{M}$ and let $U' \subset X$ be the intersection of the domains of μ and λ. Then, if we set $U := \lambda|_{U'}^{-1}(\pi^{-1}(V))$, we obtain the following commutative diagram of morphisms

$$\begin{tikzcd} U \arrow[r, "\lambda"] \arrow[d, "\mu"'] & \pi^{-1}(V) \arrow[d, "\pi"] \\ \mathfrak{M} & V. \arrow[l] \end{tikzcd}$$

Let A be a very ample divisor on X in general position. Then the intersection $S := (\operatorname{Supp} A \cap (X \setminus U)) \subset X$ is a subvariety of $\operatorname{codim}_X(S) \geq 2$. Property (2) of Theorem 3.5 implies that there is an open subset H^S_R of H_R such that any closed point of H^S_R corresponds to a curve that avoids S. We set $H_{R,\mu} := H^S_R$, and it remains to show that $H_{R,\mu}$ has the required properties (1), (2) and (3). Let $C \subset X$ be curve that corresponds to a closed point of $H_{R,\mu}$.

Since A is chosen to be ample, C intersects A positively in a point $p \in \operatorname{Supp} A$. By definition, $p \notin X \setminus U$, which implies (1).

Since C is not entirely contained in $X \setminus U$, the image $\pi \circ \lambda(C \cap U)$ is a point of V, thus the family is isotrivial on C. This implies (2).

To prove the last property (3), recall that H_R is irreducible and that $H_{R,\mu} \subset H_R$ is open. For $Z \subset X$ of $\operatorname{codim}_X(Z) \geq 2$ let H^Z_R be as in Property (2) of Theorem 3.5. We set $H^Z_{R,\mu} := H_{R,\mu} \cap H^Z_R$ which is non-empty and open in $H_{R,\mu}$. This implies (3). $\square$

Lemma 5.7. *Theorem 5.2$_n$ implies Theorem 5.1$_n$.*

Proof. Let (X, Δ) and T be as in Theorem 5.1, and $\dim X = n$. Let $\mathfrak{X} \to X \setminus (T \cup \operatorname{Supp}\lfloor\Delta\rfloor)$ be a smooth projective family of canonically polarized manifolds. As before, we denote by $\mu : X \dashrightarrow \mathfrak{M}$ the induced moduli map to the coarse moduli space of canonically polarized manifolds. To prove that μ is constant we argue by contradiction and assume that this is not the case. Since $\mathfrak{M}$ is quasi-projective, see [Vie95, Theorem 1.11], we may choose a general hyperplane section H on $\mathfrak{M}$. This is a divisor which intersects the image of μ properly, hence we can take its strict transform via μ, denoted by $D_X \in \operatorname{WDiv}(X)$. This is an effective Weil divisor to which we apply Proposition 4.13. Accordingly, we obtain a $\mathbb{Q}$-factorialization $f : Y \to X$ with boundary divisor $\Delta_Y := f_*^{-1}\Delta$ and a $(K_Y + \Delta_Y)$-negative exposed ray R of the cone $\overline{\mathrm{NM}}_1(Y) + \overline{\mathrm{NE}}_1(Y)_{K_Y+\Delta_Y \geq 0}$ which is not contained in the hyperplane $(f_*^{-1}(D_X))^{\perp}$ defined by the strict transform $D_Y := f_*^{-1}(D_X)$. Observe that the family over $X \setminus (T \cup \operatorname{Supp}\lfloor\Delta\rfloor)$ can be pulled back along f to a family over $Y \setminus (f^{-1}(T) \cup \operatorname{Supp}\lfloor\Delta_Y\rfloor)$, and the induced moduli map is given by $\mu_Y := \mu \circ f$. Since f is small, the set $f^{-1}(T)$ has codimension greater than or equal to two, thus the conditions of Proposition 5.6 are still satisfied.

Consequently, we obtain a subset H_{R,μ_Y} of the Hilbert scheme such that μ_Y is constant on any curve C in H_{R,μ_Y}. Denote by $S \subset \operatorname{Supp} D_Y$ the set of points where the moduli map μ_Y is not defined. Since $\operatorname{codim}_Y S \geq 2$ and because of Property (3) of Proposition 5.6, there is an open subset H^S_{R,μ_Y} of H_{R,μ_Y} such that the curves that correspond to this subset avoid S. Moreover, if A is a very ample divisor in general position on Y, then we can assume, after shrinking H_{R,μ_Y} if necessary, that any such curve avoids $(\operatorname{Supp} A) \cap (\operatorname{Supp} D_Y)$. In particular, any curve that corresponds to a closed point of H_{R,μ_Y} is not entirely contained in $\operatorname{Supp} D_Y$.

Let C be an arbitrary curve that corresponds to a closed point of H^S_{R,μ_Y}. Due to Proposition 5.6, the image of C is a point $p \in \mathfrak{M}$. Since C intersects D_Y outside S, this point p is an element of the hyperplane section H which in turn implies that C is contained in D_Y. This finally contradicts the choice of C. $\square$

Remark 5.8. Note that the assumption that (X, Δ) is log Fano is only needed to apply Proposition 4.13. More precisely, the proof of Theorem 5.1 still works if we assume that Proposition 4.13 holds for the pair (X, Δ), instead of assuming that (X, Δ) log Fano. We will see in Chapter 6 that this has an interesting consequence.

5.2.4 Theorem 5.1$_n$ implies Theorem 5.2$_{n+1}$, end of proof

To finish the proof, we show the following

Lemma 5.9. *Theorem 5.1$_n$ implies Theorem 5.2$_{n+1}$.*

Proof. Let $\lambda : X \dashrightarrow X'$ be a minimal model program which leads to a Mori fiber space $\pi : X' \to B$. Set $\Delta' := \lambda_* \Delta$, and let T' be the union of the indeterminacy locus of λ^{-1} and the closure of the image of T. Note that $\operatorname{codim}_{X'} T' \geq 2$ holds. We use λ^{-1} to pull the family back to a family $f' : Y' \to X' \setminus (\operatorname{Supp}\lfloor \Delta' \rfloor \cup T')$. Then we have to show that the family is isotrival on a general fiber of π.

If the Picard number $\rho(X')$ of X' is one, then (X', Δ') is in particular log Fano. In this case Lemma 5.4 implies the assertion.

Otherwise, if $\rho(X') > 1$, then $\dim B \geq 1$. Let F be a general fiber of π, then $(F, \Delta'|_F)$ is dlt log Fano. Moreover, $\operatorname{codim}_F(F \cap T') \geq 2$, and $\lfloor \Delta'|_F \rfloor = \lfloor \Delta' \rfloor|_F$. Since $\dim F \leq n$, Theorem 5.1$_n$ implies that the family restricted to F is isotrivial, which finishes the proof. □

Chapter 6

A corollary of Theorem 5.1

We are now able to discuss some properties of the cone

$$\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}.$$

First we recall some well-known facts.

Fact 6.1 ([Leh09, Theorem 1.1] and [BCHM10, Corollary 1.35])**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial projective dlt pair, then the following holds.*

- *If $-(K_X + \Delta)$ is ample, then $\overline{\mathrm{NM}}_1(X)$ is a rational polyhedron.*
- *More generally, there are countably many rays $(R_i)_{i\in\mathbb{N}} \subset \overline{\mathrm{NM}}_1(X)$ such that*

  $$\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0} = \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0} + \sum_i R_i.$$

  These rays are locally discrete away from hyperplanes that support both $\overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}$ and $\overline{\mathrm{NM}}_1(X)$.

If (X, Δ) is a pair that admits a family of positive variation we can apply our proof of Theorem 5.1 to obtain another result. Remark 5.8 implies that Proposition 4.13 cannot hold for (X, Δ). This in turn implies the following observation.

Observation 6.2. If (X, Δ) is a dlt pair that admits a non-isotrivial family, then Proposition 4.13 does not hold for (X, Δ). In particular, if X is $\mathbb{Q}$-factorial, then there is a hyperplane $H \subset N_1(X)$ such that any $(K_X + \Delta)$-negative exposed ray of $\overline{\mathrm{NM}}_1(X) + \overline{\mathrm{NE}}_1(X)_{K_X+\Delta\geq 0}$ is contained in H.

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