Time-inhomogeneous Lévy processes in interest rate and credit risk models

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik und Physik
der Albert-Ludwigs-Universität Freiburg im Breisgau

vorgelegt von

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Juli 2005

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Datum der Promotion: 11. Oktober 2005

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Abstract

In this thesis, we present interest rate models and a credit risk model, all driven by time-inhomogeneous Lévy processes, i.e. stochastic processes whose increments are independent but in general not stationary.

In the interest rate part, we discuss a Heath–Jarrow–Morton forward rate model (the *Lévy term structure model*), a model for forward bond prices (the *Lévy forward price model*) and a Libor model (the *Lévy Libor model*) which generalizes the Libor market model. In all of these models, explicit valuation formulae are established for the most liquid interest rate derivatives, namely caps, floors, and swaptions. The formulae can numerically be evaluated fast and thus allow to calibrate the models to market data. In the Lévy term structure model, we also price floating range notes. Their payoffs are path-dependent.

In the credit risk part, the Lévy Libor model (and therewith, as a special case, the Libor market model) is extended to defaultable forward Libor rates. We present a rigorous construction of the model and price some of the most heavily traded credit derivatives, namely credit default swaps, total rate of return swaps, credit spread options and credit default swaptions.

Acknowledgements

I am indebted to many people for contributing to my development in mathematics and finance or to my personal development or to both. Any list would definitely be incomplete, so let me just name those that, in my opinion, have the closest link to this thesis:

First and foremost, I would like to thank my academic teacher and supervisor Prof. Dr. Ernst Eberlein. He introduced me to the theory of Lévy processes as well as to mathematical finance and gave me valuable advice and support. I would also like to thank all my colleagues from the Department of Mathematical Stochastics at the University of Freiburg, and especially Jan Bergenthum and Antonis Papapantoleon, for discussing mathematical and non-mathematical ideas with me. Further thanks go to Monika Hattenbach, Antonis Papapantoleon, and Albrecht Zimmermann for valuable comments on the manuscript.

Very special thanks go to my family and to my friends for their support. In particular, I would like to thank Marie-Luise Kluge, Wieland Kluge, Andreas Kluge, and Vera Henrike Meyer-Borgstädt.

Financial support by the Deutsche Forschungsgemeinschaft (DFG) is gratefully acknowledged. I also acknowledge the permission from Blackwell Publishing to use the content of Eberlein and Kluge (2005) in this thesis.

Contents

Chapter 1

Introduction

A trader in a bank asked for an offer by a client, who wants to purchase a financial product, faces two questions:

- What price shall he offer the product for?
- Assuming the client accepts the offer, how can the trader hedge himself?

The answers to these questions are easy if there is a liquid market for this product. The trader can use the market price, add a margin and offer it to the client. Suppose his offer is accepted, then he buys the product at the market, sells it to the client and keeps the margin.

In case there is no liquid market for the product, the questions are much harder to answer. For a specific financial product, namely a call (or put) option on a stock price, these questions have been addressed in a famous article by Black and Scholes (1973). Under the assumption of existence of a liquid market for the underlying stock and a risk-free asset, they duplicate the option by a permanently re-balanced portfolio consisting of stocks and a certain amount of the risk-free asset. In this way, they derive a unique price for the option and a perfect hedging strategy at the same time.

Unfortunately, the price as well as the hedging strategy depend heavily on the way Black and Scholes model the stock price. They model it – first suggested by Samuelson (1965) – as a geometric Brownian motion. Since their article has been published, a large variety of different approaches for modelling stock prices has been proposed by several authors. These approaches usually lead to different option prices and hedging strategies. Often, they do not even produce unique prices for options or perfect hedging strategies. Similar observations can be made for other derivatives, as e.g. interest rate, foreign exchange or credit derivatives: a reasonable price for a derivative will usually depend on the way the underlying is modelled, a fact that imposes the next question: How should it be modelled?

Of course, there is no objective answer to this question since every model has its advantages and drawbacks. Two desirable model features are generality and tractability. However, these two properties do not, usually, come together. Very general models, e.g. models driven by general semimartingales, are often not tractable. On the other hand, models driven by Brownian motions usually can be handled nicely but cannot explain statistical behaviour of real-life financial data (e.g. fat-tailed log return distributions or volatility smiles).

Lévy processes are excellent tools for modelling price processes of financial securities. The resulting models are very flexible and allow for tractable pricing formulae for derivatives in many cases. In this thesis, we go a step further towards generality and consider various models driven by time-inhomogeneous (also called non-homogeneous) Lévy processes. They are usually as easy to handle as models driven by (homogeneous) Lévy processes but allow for additional flexibility. Our main focus lies on the construction of the models as well as on the derivation of pricing formulae for options and other derivatives. Hedging strategies are not addressed.

Among the derivatives that are traded over-the-counter (OTC) interest rate products form the largest part. According to the Bank for International Settlements (BIS), the notional amount of outstanding OTC interest rate derivative contracts added up to 187 trillions of US dollars in December 2004, compared to 248 trillions for all OTC derivative contracts.[1] We devote chapters 2 and 3 to modelling of interest rates and to pricing of interest rate derivatives. In chapter 2, a Heath–Jarrow–Morton forward rate model introduced by Eberlein and Raible (1999), the *Lévy term structure model*, is presented. Explicit valuation formulae for caps, floors, swaptions and for a derivative with a path-dependent payoff, namely a floating range note, are derived. All formulae can numerically be evaluated fast, a fact that allows us to calibrate the model to market data. The main results of chapter 2 can also be found in Eberlein and Kluge (2004, 2005). Two models for effective rates due to Eberlein and Özkan (2005), the *Lévy forward price model* and the *Lévy Libor model*, are presented in chapter 3. We show that the first one can be seen as a special case of the Lévy term structure model. The second approach generalizes the Libor market model to non-homogeneous Lévy processes. Eberlein and Özkan (2005) deduce approximate pricing solutions for caps and floors but do not consider swaption pricing. We suggest an alternative approximation to price caps and floors and compare the results. Moreover, a valuation formula for swaptions (also based on an approximation) is derived. Symmetry results relating the prices of caps and floors in all of these interest rate models are established in Eberlein, Kluge, and Papapantoleon (2005). However, they are not part of this thesis.

A market that is growing rapidly in size is the market for credit derivatives. "According to the data presented in the *Triennial Central Bank Survey of Foreign Exchange and Derivatives Market Activity*, the growth of credit related derivatives in the three years ending June 2004 amounted to 568 %, against 121 % for all OTC products".[2] In chapter 4 we develop a credit risk model that extends the Lévy Libor model (and therewith the Libor market model). Pricing formulae for some of the most popular credit derivatives, as e.g. credit default swaps, total rate of return swaps, credit spread options and credit default swaptions, are also deduced.

[1] Source: BIS Quarterly Review, June 2005, p. A99.
[2] Source: BIS Quarterly Review, June 2005, p. 50.

In the remaining sections of this chapter, we give very brief introductions to interest rate and credit risk modelling. Moreover, non-homogeneous Lévy processes are defined and some of their properties are established.

1.1 Interest rate models

In this section, we introduce notations and give definitions for some very common financial products and mathematical concepts related to interest rate theory which are used in the subsequent chapters. Moreover, a brief survey of the most important types of interest rate models is presented. However, this section cannot, and is not supposed to, serve as an introduction to interest rate theory. There is a large amount of literature discussing the subject in detail. To name just a few references, let us mention the books of Musiela and Rutkowski (1998), Brigo and Mercurio (2001), and Björk (2004).

A *zero coupon bond* is a financial security that pays an amount of one currency unit to its owner at maturity of the contract. The price at time t of a zero coupon bond with maturity T ($t \leq T$) is denoted by $B(t,T)$ in what follows. Obviously, we have $B(T,T) = 1$ for any maturity date T. Since there are no intermediate payments, one could expect the value of a zero coupon bond to be always less than or equal to one. However, this property does not hold in models that admit negative interest rates. To put it differently, not all models allow to carry cash at no costs.

Instantaneous (continuously compounded) forward rates are mathematical concepts rather than observable rates in the markets. However, they are very convenient for modelling purposes. By $f(t,T)$ we denote the forward rate as seen at time t for borrowing or lending money over an infinitesimal time period starting at T. Formally, instantaneous forward rates are defined by $f(t,T) := -\frac{\partial}{\partial T} \log B(t,T)$ (assuming the derivative exists). Zero coupon bond prices can be recovered from forward rates via $B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$.

The instantaneous interest rate that prevails at time t for immediate lending or borrowing over an infinitesimal time interval is called *spot rate* or *short rate* and denoted by r_t. Clearly, we have the relationship $r_t = f(t,t)$. An amount of one unit of cash at time 0 that is continuously reinvested at the short rate yields $B_t := \exp \int_0^t r_s\,\mathrm{d}s$ at time t. B is usually referred to as *savings account*, *money market account* or *discount factor*.

Libor is an abbreviation for *London Inter-Bank Offer Rate* and refers to the interest rate that is paid between banks. In contrast to instantaneous rates, Libor rates are *effective* or *simply compounded*, i.e. interest accrues according to a discrete grid. Although banks can default, many interest rate models neglect this risk and assume that Libor rates are default-free. Under this assumption, the Libor rate for a period of length δ starting at T can be related to zero coupon bond prices via $L(T,T) := \frac{1}{\delta}\left(\frac{B(T,T)}{B(T,T+\delta)} - 1\right)$. The *forward Libor rate* $L(t,T)$ is the Libor rate at T as it is seen by the market at time t. More precisely, $L(t,T) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$.

A *(plain vanilla) interest rate swap* is an agreement between two parties to exchange fixed against floating interest rate payments. More precisely, one party agrees to pay a fixed interest rate on a notional principal in return for a floating interest rate (usually the Libor) on the same notional and for the same period of time. The fixed rate that makes the initial value of this contract equal to zero is called *swap rate*.

A *cap* (resp. *floor*) consists of a series of call (resp. put) options on subsequent Libor rates. These single options are called *caplets* (resp. *floorlets*). A caplet that is settled in arrears with a notional of 1, maturity T and a strike rate of K on the Libor rate $L(T,T)$ pays off $\delta(L(T,T)-K)^+$ at $T+\delta$. The payoff of the respective floorlet equals $\delta(K-L(T,T))^+$. Note that a caplet can be seen as a put option on a zero coupon bond, since a payoff of $\delta(L(T,T)-K)^+$ at $T+\delta$ equals a payoff of $B(T,T+\delta)\delta(L(T,T)-K)^+$ at T and

$$\begin{aligned} B(T,T+\delta)\delta(L(T,T)-K)^+ &= (1-(1+\delta K)B(T,T+\delta))^+ \\ &= (1+\delta K)((1+\delta K)^{-1}-B(T,T+\delta))^+. \end{aligned}$$

Similarly, a floorlet can be regarded as a call option on a zero coupon bond. Caps and floors are commonly used as insurances against rising or falling interest rates.

A *swaption* is an option on a forward swap, i.e. on an interest rate swap which starts in the future. At maturity of the option, its holder has the right to enter into the swap at a pre-specified fixed rate. There are payer and receiver swaptions giving their owners the right to enter into the swap as fixed rate payer or receiver respectively. The holder of a payer (receiver) swaption will exercise the option if the swap rate at option maturity is higher (lower) than the strike rate of the swaption.

The most classical approaches to modelling fixed income markets are short rate models. They exogenously specify the dynamics of the short rate r. Derivatives in these models are not only caps, floors, and swaptions but also zero coupon bonds. In other words, initial zero coupon bond prices are an *output of* and not an *input to* the model. Brigo and Mercurio (2001) give a good overview of various models, also commenting on their particular advantages and drawbacks. Short rate models describe the evolution of the whole fixed income market by one explanatory variable (the short rate) only, a feature that is often criticized. Another common handicap of all (time-homogeneous) short rate models is their inability to exactly reproduce a given initial term structure, i.e. the bond prices $B(0,\cdot)$ that are observed in the market (see e.g. Björk (2004)). Nevertheless, short rate models are still widely used.

The idea to exogenously specify the evolution of the whole term structure of interest rates was pioneered by Heath, Jarrow, and Morton (1992). Subject to modelling in a *Heath–Jarrow–Morton* (henceforth HJM) framework are either zero coupon bond prices or instantaneous forward rates. Initial bond prices enter as a model input, i.e. any given initial term structure is perfectly reproduced. There are various HJM-type models differing mainly in the specification of the process that drives the forward rates or bond prices. At the high end as far as generality is concerned let us mention the semimartingale approach of Björk, Di Masi, Kabanov, and Runggaldier (1997). These authors use a finite number of Wiener processes plus an integer-valued random measure as drivers.

In a series of papers by Sandmann, Sondermann, and Miltersen (1995), Miltersen, Sandmann, and Sondermann (1997), Brace, Gatarek, and Musiela (1997), Jamshidian (1997), and Musiela and Rutkowski (1997) the *forward Libor model* and the *forward swap model* were developed. Subject to modelling in these so-called *market models* are the dynamics of forward Libor or swap rates. Among practitioners, the models are very popular since they reproduce well-established market formulae for caps/floors and swaptions respectively. More precisely, the forward Libor model can be calibrated perfectly to (at-the-money) quotes of caps and floors whereas the forward swap model is able to reproduce market prices of swaptions exactly. For an extensive survey of the market models we refer to Brigo and Mercurio (2001, Section 6).

1.2 Modelling credit risk

Most credit risk models may be classified into two categories: structural models and reduced form models. We comment very briefly on the two approaches and refer to Schönbucher (2000) and Bielecki and Rutkowski (2002) for more details.

In *structural models* or *firm's value models* the value process of a firm's assets is exogenously specified. Default is defined as the first time at which the value process hits or falls below a default triggering barrier. The barrier is a second exogenously specified process that is usually related to the debt of the firm. An advantage of this approach is the direct link between default and the firm's capital structure. The structural approach is well-suited whenever products that depend on more than one security issued by a firm (e.g. a callable bond which can be converted into shares when called) have to be priced. A question that arises naturally in this setting is wherefrom to observe the value process? Another drawback is the fact that defaultable bonds are no fundamentals but an output of the model. Moreover, in structural models that are driven by continuous processes the time of default is a predictable stopping time. As a consequence, credit spreads tend to zero as the time to maturity of a debt tends to zero. This contradicts what can be observed in real markets. From the large amount of papers that have contributed to the structural approach let us mention a few: Merton (1974), Black and Cox (1976), Geske (1977), Longstaff and Schwartz (1995), and Zhou (1997).

Reduced form models, also called *intensity based models* or *hazard rate models*, specify default by a totally inaccessible stopping time. In many cases, exogenous specifications of *hazard rates* or *stochastic intensities* are used to construct the time of default. Some authors link the hazard rate to the value of the firm's assets or stocks, as e.g. Madan and Unal (1998, 2000). The resulting models are often referred to as *hybrid models* since they combine elements from the structural and the intensity based approach. However, most reduced form models do not rely on the value of the firm's assets at all. Instead, they often extend default-free interest rate models. The *Lévy Libor model with default risk* which we present in chapter 4 is one example of the intensity based approach. From the long list of other examples, let us mention a very small sample: Artzner and Delbaen (1992), Jarrow and Turnbull (1995), Lando (1997), and Duffie and Singleton (1999).

1.3 Non-homogeneous Lévy processes

In financial mathematics Lévy processes are very popular since they allow for models that are much more flexible than models driven by Brownian motions. At the same time, these models are often still tractable. Important special cases of Lévy processes include Brownian motions, Poisson and compound Poisson processes as well as Lévy processes generated by variance gamma (introduced by Madan and Seneta (1987, 1990)), CGMY (Carr, Geman, Madan, and Yor (2002)), normal inverse Gaussian or generalized hyperbolic distributions (Barndorff-Nielsen (1977, 1998) and Eberlein (2001)). We are not going to review the definition and properties of a Lévy process. Instead, we refer to existing literature. There are many books examining these processes in detail, e.g. Bertoin (1996), Sato (1999), and Applebaum (2004). Schoutens (2003) discusses applications of Lévy processes to mathematical finance.

Non-homogeneous Lévy processes are more general than (homogeneous) Lévy processes. In contrast to their homogeneous counterparts, non-homogeneous Lévy processes do not, generally, possess stationary increments. As we will see in the subsequent chapters, relaxation of the stationarity assumption provides us with additional flexibility in the models. Fortunately, this flexibility does not come at a high pice, i.e. the models usually do not become more complicated. A book that examines non-homogeneous Lévy processes in some detail is Cont and Tankov (2003).

We assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ to be given, i.e. a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. By a filtration we mean an increasing and right continuous family of sub-σ-fields of $\mathcal{F}$ (compare Jacod and Shiryaev (2003, Definition I.1.2)). Note that, unless explicitly stated, we do not assume completeness of the stochastic basis, i.e. we do not require that the *usual conditions* hold. Since all models that are considered in what follows have a finite time horizon T^*, we work with the following definition for a non-homogenous Lévy process:

Definition 1.1 *An adapted stochastic process* $L = (L_t)_{0\leq t\leq T^*}$ *with values in* $\mathbb{R}^d$ *is a* non-homogeneous Lévy process*, sometimes also called* time-inhomogeneous Lévy process *or process with* independent increments and absolutely continuous characteristics*, henceforth abbreviated by* PIIAC*, if the following conditions hold:*

1. *L has independent increments,*

   i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ $\qquad (0 \leq s < t \leq T^*)$.

2. *For every $t \in [0, T^*]$, the law of L_t is characterized by the characteristic*

function

$$\mathbb{E}\left[e^{\mathrm{i}\langle u,L_t\rangle}\right] = \exp\int_0^t \Bigg(\mathrm{i}\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle \tag{1.1}$$
$$+ \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u, x\rangle \mathbb{1}_{\{|x|\le 1\}}) F_s(\mathrm{d}x) \Bigg)\, \mathrm{d}s.$$

Here, $b_s \in \mathbb{R}^d$, c_s is a symmetric nonnegative-definite $d\times d$ matrix, and F_s is a measure on $\mathbb{R}^d$ that integrates $(|x|^2 \wedge 1)$ and satisfies $F_s(\{0\}) = 0$. The Euclidian scalar product on $\mathbb{R}^d$ is denoted by $\langle\cdot,\cdot\rangle$, the respective norm by $|\cdot|$. It is assumed that

$$\int_0^{T^*} \Bigg(|b_s| + ||c_s|| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(\mathrm{d}x) \Bigg)\, \mathrm{d}s < \infty, \tag{1.2}$$

where $||\cdot||$ denotes any norm on the set of $d \times d$ matrices.

We call $(b, c, F) := (b_s, c_s, F_s)_{0\le s\le T^*}$ the characteristics *of L.*

Remark: It is also possible to define a non-homogeneous Lévy process L on a probability space without assuming a filtration to be given in advance. Instead, the probability space can then be equipped with the filtration generated by L.

Note that definition 1.1 can also be used to define a homogeneous Lévy process if b_s, c_s, and F_s are assumed not to depend on s. Let us point out some properties of a non-homogeneous Lévy process L:

Lemma 1.2 *Fix $t \in [0, T^*]$. The distribution of L_t is infinitely divisible with Lévy–Khintchine triplet (b, c, F), where*

$$b := \int_0^t b_s\,\mathrm{d}s, \quad c := \int_0^t c_s\,\mathrm{d}s, \quad F(\mathrm{d}x) := \int_0^t F_s(\mathrm{d}x)\,\mathrm{d}s.$$

(The integrals are to be understood componentwise.)

Proof: Clearly, $b \in \mathbb{R}^d$ and c is a symmetric nonnegative-definite $d\times d$ matrix. A monotone convergence argument yields that F is a measure on the Borel sets of $\mathbb{R}^d$ and we get

$$\int_{\mathbb{R}^d} f(x)F(\mathrm{d}x) = \int_0^t \int_{\mathbb{R}^d} f(x)F_s(\mathrm{d}x)\,\mathrm{d}s \tag{1.3}$$

for any integrable function f. Thus, $\int (|x|^2 \wedge 1) F(\mathrm{d}x) < \infty$ by (1.2) and $F(\{0\}) = 0$. The claim now follows from (1.1) and the Lévy–Khintchine formula. $\square$

Lemma 1.3 *L is an additive process in law, i.e. a stochastically continuous process with independent increments and $L_0 = 0$ a.s.*

PROOF: The last property follows directly from the characteristic function of L_0. To verify stochastic continuity let us have a look at the characteristic function of $L_t - L_v$ for $v < t$. By the independence of the increments

$$\begin{aligned} \mathbb{E}\left[e^{\mathrm{i}\langle u, L_t - L_v\rangle}\right] &= \frac{\mathbb{E}\left[e^{\mathrm{i}\langle u, L_t\rangle}\right]}{\mathbb{E}\left[e^{\mathrm{i}\langle u, L_v\rangle}\right]} \\ &= \exp \int_v^t \Bigg(\mathrm{i}\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle \\ &\qquad + \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle u, x\rangle} - 1 - \mathrm{i}\langle u, x\rangle \mathbb{1}_{\{|x|\le 1\}}) F_s(\mathrm{d}x) \Bigg) \mathrm{d}s. \end{aligned}$$

As v approaches t, the characteristic function of $L_t - L_v$ converges pointwise to 1. Thus, $L_t - L_v$ converges to 0 in distribution and, since that is a constant, also stochastically. □

Every additive process in law has a modification that is càdlàg, which means that all paths are right-continuous and admit left-hand limits (see e.g. Sato (1999, Theorem 11.5)). We will always work with this modification of L. Although not every càdlàg adapted process with independent increments is a semimartingale (Jacod and Shiryaev (2003, Chapter II, §4c) give a counterexample), non-homogeneous Lévy processes are semimartingales:

Lemma 1.4 *The process L is a semimartingale with respect to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$.*

PROOF: This property can be established by looking at the characteristic function of L_t. More precisely, (1.2) implies that for every $u \in \mathbb{R}^d$ the function $t \mapsto f(u)_t$ where

$$\begin{aligned} f(u)_t &:= \log \mathbb{E}\left[e^{\mathrm{i}\langle u, L_t\rangle}\right] \\ &= \int_0^t \Bigg(\mathrm{i}\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle \\ &\qquad + \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle u, x\rangle} - 1 - \mathrm{i}\langle u, x\rangle \mathbb{1}_{\{|x|\le 1\}}) F_s(\mathrm{d}x) \Bigg) \mathrm{d}s \end{aligned}$$

has finite variation over finite intervals. Hence, $t \mapsto \exp f(u)_t$ has finite variation over finite intervals and Jacod and Shiryaev (2003, Chapter II, Theorem 4.14) yields that L is a semimartingale. □

We describe L by its *semimartingale characteristics* in the sense of Jacod and Shiryaev (2003, Chapter II, Definition 2.6):

Lemma 1.5 *The semimartingale characteristics of L associated with the truncation function $h(x) := 1\!\!1_{\{|x|\leq 1\}}$ are given by*

$$B_t = \int_0^t b_s \,\mathrm{d}s, \quad C_t = \int_0^t c_s \,\mathrm{d}s, \quad \nu([0,t]\times A) = \int_0^t \int_A F_s(\mathrm{d}x)\ \mathrm{d}s \quad (A \in \mathcal{B}(\mathbb{R}^d)).$$

PROOF: We use Jacod and Shiryaev (2003, Chapter II, Corollary 2.48) and look at

$$A(u)_t := \mathrm{i}\langle u, B_t\rangle - \frac{1}{2}\langle u, C_t u\rangle + \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u, x\rangle 1\!\!1_{\{|x|\leq 1\}})\nu([0,t]\times \mathrm{d}x)$$

for $u \in \mathbb{R}^d$. The function $t \mapsto A(u)_t$ is continuous and has finite variation over finite intervals. Moreover, $A(u)_t$ equals the characteristic exponent of L_t. Thus,

$$\mathcal{E}[A(u)] = \exp A(u) = \mathbb{E}\left[e^{\mathrm{i}\langle u,L.\rangle}\right],$$

where $\mathcal{E}$ denotes the stochastic exponential. By independence of the increments of L

$$\begin{aligned}\mathbb{E}[e^{\mathrm{i}\langle u,L_t\rangle}|\mathcal{F}_s] &= \mathbb{E}[e^{\mathrm{i}\langle u,L_t-L_s\rangle}e^{\mathrm{i}\langle u,L_s\rangle}|\mathcal{F}_s]\\ &= \frac{\mathbb{E}[e^{\mathrm{i}\langle u,L_t\rangle}]}{\mathbb{E}[e^{\mathrm{i}\langle u,L_s\rangle}]}e^{\mathrm{i}\langle u,L_s\rangle}.\end{aligned}$$

Hence, $e^{\mathrm{i}\langle u,L.\rangle}/\mathcal{E}[A(u)]$ is a martingale and the cited corollary yields that B, C and ν are indeed the characteristics of L. □

These characteristics allow us to write L in its *canonical representation* (see Jacod and Shiryaev (2003, II.2.34))

$$L_t = \int_0^t b_s \,\mathrm{d}s + L_t^c + \int_0^t \int_{\mathbb{R}^d} x 1\!\!1_{\{|x|\leq 1\}}(\mu-\nu)(\mathrm{d}s,\mathrm{d}x) + \sum_{s\leq t} \Delta L_s 1\!\!1_{\{|\Delta L_s|>1\}}. \quad (1.4)$$

Here, L^c denotes the continuous martingale part of L and μ is the random measure associated with the jumps of L. From the characteristic C we can conclude that $L_t^c = \int_0^t \sqrt{c_s}\,\mathrm{d}W_s$, where W is a standard d-dimensional Brownian motion and $\sqrt{c_s}$ is a measurable version of the square root of c_s.

In many applications we will require the existence of exponential moments and put the following integrability condition on the measures F_s:

Assumption ($\mathbb{EM}$). *There are constants $M, \varepsilon > 0$ such that for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$*

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\langle u, x\rangle F_s(\mathrm{d}x)\,\mathrm{d}s < \infty. \quad (1.5)$$

Without loss of generality, $\int_{\{|x|>1\}} \exp\langle u,x\rangle F_s(\mathrm{d}x)$ *is assumed to be finite for all* s*.*

An equivalent condition to (1.5) in terms of the existence of exponential moments of L is given by the following lemma:

Lemma 1.6 *Assumption* $(\mathbb{EM})$ *holds if and only if there are constants* $M, \varepsilon > 0$ *such that* $\mathbb{E}[\exp\langle u, L_t\rangle] < \infty$ *for all* $t \in [0, T^*]$ *and* $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$*.*

PROOF: Assume that (1.5) holds and fix $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$ and $t \in [0, T^*]$. Let $\widetilde{L}$ be a Lévy process with $\widetilde{L}_1 \sim L_t$. Then, its generating triplet (b, c, F) is given by lemma 1.2. By (1.3) we have

$$\int_{\{|x|>1\}} \exp\langle u,x\rangle F(\mathrm{d}x) = \int_0^t \int_{\{|x|>1\}} \exp\langle u,x\rangle F_s(\mathrm{d}x)\,\mathrm{d}s < \infty.$$

Sato (1999, Theorem 25.3) yields that $\mathbb{E}[\exp\langle u, \widetilde{L}_1\rangle] < \infty$. Since $\widetilde{L}_1 \sim L_t$, we also get $\mathbb{E}[\exp\langle u, L_t\rangle] < \infty$.

Conversely, let us assume that for $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$ we have $\mathbb{E}[\exp\langle u, L_{T^*}\rangle] < \infty$. Let $\widetilde{L}$ be a Lévy process with $\widetilde{L}_1 \sim L_{T^*}$ and (b, c, F) its generating triplet. Then $\mathbb{E}[\exp\langle u, \widetilde{L}_1\rangle] = \mathbb{E}[\exp\langle u, L_{T^*}\rangle] < \infty$. Again (1.3) and Sato (1999, Theorem 25.3) imply

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\langle u,x\rangle F_s(\mathrm{d}x)\,\mathrm{d}s = \int_{\{|x|>1\}} \exp\langle u,x\rangle F(\mathrm{d}x) < \infty. \qquad \square$$

From the preceding lemma we can conclude that under assumption $(\mathbb{EM})$ the expected value of L_t is finite. Hence, the characteristic function of L_t can be written as

$$\mathbb{E}\left[e^{\mathrm{i}\langle u, L_t\rangle}\right] = \exp \int_0^t \Bigg(\mathrm{i}\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u,x\rangle) F_s(\mathrm{d}x) \Bigg)\,\mathrm{d}s. \tag{1.6}$$

Of course, the b_s in this representation differ from those in equation (1.1) since we changed the truncation function. In fact, it follows from Jacod and Shiryaev (2003, II.2.30) that they differ by $\int_{\mathbb{R}^d} x \mathbb{1}_{\{|x|>1\}} F_s(\mathrm{d}x)$. Henceforth, whenever we work under assumption $(\mathbb{EM})$, we will use the characteristics that correspond to equation (1.6). Also, in this setting L is not only a semimartingale but a special semimartingale:

Lemma 1.7 L *is a special semimartingale.*

PROOF: We use Jacod and Shiryaev (2003, II.2.29) and show that $(|x|^2 \wedge |x|) * \nu$ is an adapted process with locally integrable variation. Since $(|x|^2 \wedge |x|) * \nu$ is increasing and deterministic, we only need to show finiteness of

$$\begin{aligned}(|x|^2 \wedge |x|) * \nu_{T^*} &= \int_0^{T^*} \int_{\mathbb{R}^d} (|x|^2 \wedge |x|) F_s(\mathrm{d}x)\,\mathrm{d}s \\ &= \int_0^{T^*} \int_{\{|x|\le 1\}} |x|^2 F_s(\mathrm{d}x)\,\mathrm{d}s + \int_0^{T^*} \int_{\{|x|>1\}} |x| F_s(\mathrm{d}x)\,\mathrm{d}s.\end{aligned}$$

The finiteness of the first term is guaranteed by (1.2), while (1.5) implies that the second summand is finite. □

Consequently, the canonical representation of L simplifies to

$$L_t = \int_0^t b_s\,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu)(\mathrm{d}s, \mathrm{d}x). \tag{1.7}$$

In one of the models that follow we will need an assumption which is slightly stronger than assumption ($\mathbb{EM}$) from a mathematical point of view. In applications, both assumptions are practically equal and not very restrictive:

Assumption ($\mathbb{SUP}$). *It holds that*

$$\sup_{0\le s\le T^*} \left(|b_s| + ||c_s|| + \int_{\mathbb{R}^d} (|x|^2 \wedge |x|) F_s(\mathrm{d}x) \right) < \infty \tag{1.8}$$

(where $||\cdot||$ denotes any norm on the set of $d \times d$ matrices) and there are constants M, $\varepsilon > 0$ such that for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$

$$\sup_{0\le s\le T^*} \left(\int_{\{|x|>1\}} \exp\langle u, x\rangle F_s(\mathrm{d}x) \right) < \infty. \tag{1.9}$$

In the remaining part of this section we assume that ($\mathbb{EM}$) is in force and present a proposition that proves to be very useful for the derivation of drift conditions in term structure models as well as for option pricing. To simplify notation, let us denote by θ_s the cumulant associated with the infinitely divisible distribution characterized by the Lévy–Khintchine triplet (b_s, c_s, F_s), i.e. for $z \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$ where M is the constant from assumption ($\mathbb{EM}$) we have

$$\theta_s(z) := \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} (e^{\langle z,x\rangle} - 1 - \langle z, x\rangle) F_s(\mathrm{d}x). \tag{1.10}$$

According to Sato (1999, Theorem 25.17) we can extend θ_s to complex numbers $z \in \mathbb{C}^d$ with $\Re(z_j) \in [-(1+\varepsilon)M, (1+\varepsilon)M]$ for $j \in \{1, \ldots, d\}$ and write the characteristic function of L_t as

$$\mathbb{E}\left[e^{\mathrm{i}\langle u, L_t\rangle}\right] = \exp \int_0^t \theta_s(\mathrm{i}u)\,\mathrm{d}s \tag{1.11}$$

Note that $\mathrm{i}u := (\mathrm{i}u_j)_{1\le j\le d}$ and the scalar product on $\mathbb{R}^d$ is extended to complex numbers, that is $\langle w, z\rangle := \sum_{j=1}^d w_j z_j$ for $w, z \in \mathbb{C}^d$. Hence, $\langle \cdot, \cdot\rangle$ is not the Hermitian scalar product. If L is a (homogeneous) Lévy process, i.e. the increments of L are stationary, b_s, c_s, and F_s and thus also θ_s do not depend on s. In this case we write θ for short. θ then equals the cumulant (also called log moment generating function) of L_1.

The characteristic function of L_t can also be extended to a strip in the complex plane, as the following lemma shows:

Lemma 1.8 *Fix $t \in [0, T^*]$. For $z \in \mathbb{C}^d$ with $\Re(z) \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$ we have $\mathbb{E}[|e^{\langle z, L_t\rangle}|] < \infty$ and*

$$\mathbb{E}[e^{\langle z, L_t\rangle}] = \exp \int_0^t \theta_s(z)\,\mathrm{d}s. \tag{1.12}$$

Proof: We get $\mathbb{E}[|e^{\langle z, L_t\rangle}|] = \mathbb{E}[e^{\langle \Re(z), L_t\rangle}] < \infty$ from lemma 1.6. Let $\widetilde{L}$ be a Lévy process with $\widetilde{L}_1 \sim L_t$. Clearly, $\mathbb{E}[e^{\langle \Re(z), \widetilde{L}_1\rangle}] = \mathbb{E}[e^{\langle \Re(z), L_t\rangle}] < \infty$. The Lévy–Khintchine triplet (b, c, F) of $\widetilde{L}$ is given by lemma 1.2 and Sato (1999, Theorem 25.17) yields that

$$\Psi(z) := \langle z, b\rangle + \frac{1}{2}\langle z, cz\rangle + \int_{\mathbb{R}^d} (e^{\langle z, x\rangle} - 1 - \langle z, x\rangle) F(\mathrm{d}x)$$

is definable and $\mathbb{E}[e^{\langle z, \widetilde{L}_1\rangle}] = e^{\Psi(z)}$. By using (1.3) we see that $\Psi(z)$ equals $\int_0^t \theta_s(z)\,\mathrm{d}s$ and therefore $\mathbb{E}[e^{\langle z, L_t\rangle}] = \exp \int_0^t \theta_s(z)\,\mathrm{d}s$. □

The following proposition will frequently be used for option pricing:

Proposition 1.9 *Suppose that $f : \mathbb{R}_+ \to \mathbb{C}^d$ is a continuous function such that $|\Re(f^i(x))| \le M$ for all $i \in \{1, \ldots, d\}$ and $x \in \mathbb{R}_+$, then*

$$\mathbb{E}\left[\exp\left(\int_t^T f(s)\,\mathrm{d}L_s\right)\right] = \exp \int_t^T \theta_s(f(s))\,\mathrm{d}s.$$

(The integrals are to be understood componentwise for real and imaginary part.)

Proof: This proof uses the idea of the proof of lemma 3.1 in Eberlein and Raible (1999). By independence of the increments of L it is sufficient to consider the case $t = 0$. Since f is continuous and deterministic, it is locally bounded (take $T_n := n$ as localizing sequence) and we have

$$\int_0^T f(s)\,\mathrm{d}L_s = \sum_{i=1}^d \int_0^T f^i(s)\,\mathrm{d}L_s^i.$$

For any partition $0 = t_0 < t_1 < \ldots < t_{N+1} = T$

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\sum_{k=0}^N \langle f(t_k), L_{t_{k+1}} - L_{t_k}\rangle\right)\right] &= \prod_{k=0}^N \mathbb{E}\left[\exp\langle f(t_k), L_{t_{k+1}} - L_{t_k}\rangle\right] \\
&= \prod_{k=0}^N \frac{\mathbb{E}\left[\exp\langle f(t_k), L_{t_{k+1}}\rangle\right]}{\mathbb{E}\left[\exp\langle f(t_k), L_{t_k}\rangle\right]} \\
&= \exp\left(\sum_{k=0}^N \int_{t_k}^{t_{k+1}} \theta_s(f(t_k))\,\mathrm{d}s\right).
\end{aligned}$$

We used the independence of the increments of L for the first two equalities and lemma 1.8 for the third. Now let the mesh of the partition go to zero. The right-hand side converges to $\exp \int_0^T \theta_s(f(s))\,\mathrm{d}s$.

Let us have a look at the left-hand side. According to Jacod and Shiryaev (2003, Proposition I.4.44)

$$\sum_{k=0}^N f^i(t_k)(L^i_{t_{k+1}} - L^i_{t_k}) \longrightarrow \int_0^T f^i(s)\,\mathrm{d}L^i_s \qquad \text{in measure for each i.}$$

Continuous transformations preserve convergence in measure. Consequently,

$$\begin{aligned}
\exp\left(\sum_{k=0}^N \langle f(t_k), L_{t_{k+1}} - L_{t_k}\rangle\right) &= \exp\left(\sum_{i=1}^d \sum_{k=0}^N f^i(t_k)(L^i_{t_{k+1}} - L^i_{t_k})\right) \\
&\longrightarrow \exp\left(\sum_{i=1}^d \int_0^T f^i(s)\,\mathrm{d}L^i_s\right) \\
&= \exp \int_0^T f(s)\,\mathrm{d}L_s \qquad \text{in measure.}
\end{aligned}$$

If we can show that the approximating sequence is uniformly integrable, convergence in measure will imply convergence in L^1 and the claim is proved. To show uniform integrability, we use Dellacherie and Meyer (1978, Theorem

II.22) and check that the sequence is bounded in $L^{1+\varepsilon}$:

$$\begin{aligned}
\mathbb{E}&\left[\left|\exp\left(\sum_{k=0}^{N}\langle f(t_k), L_{t_{k+1}} - L_{t_k}\rangle\right)\right|^{1+\varepsilon}\right] \\
&\qquad = \mathbb{E}\left[\exp\left(\sum_{k=0}^{N}\langle (1+\varepsilon)\Re(f(t_k)), L_{t_{k+1}} - L_{t_k}\rangle\right)\right] \\
&\qquad = \exp\left(\sum_{k=0}^{N}\int_{t_k}^{t_{k+1}} \theta_s((1+\varepsilon)\Re(f(t_k)))\,\mathrm{d}s\right),
\end{aligned}$$

where the last equality follows as in the chain of equations above. The term on the right-hand side converges in $\mathbb{R}$ if the mesh of the partition goes to zero. Hence, we can find an upper bound for this term independent of N. $\square$

Chapter 2

The Lévy term structure model

In designing a model for fixed income markets that is interesting for both, the academic world as well as the financial industry, one has to have two aspects in mind: the model should allow for analytical expressions at least for the most important interest rate-sensitive instruments such as bonds, swaps, caps, floors and swaptions. At the same time, it should be possible to calibrate it fast and accurately to market data. In particular, models should be able to reproduce a given term structure and prices of the most liquid interest rate derivatives, namely caps, floors and swaptions, with a sufficient degree of accuracy. We try to fulfill both needs by presenting a generalization of the Lévy term structure model introduced in Eberlein and Raible (1999) to non-homogeneous Lévy processes (see also Eberlein, Jacod, and Raible (2005)). Within this framework, we derive explicit formulae for the prices of caps, floors and swaptions. These formulae can numerically be evaluated fast and allow for a calibration of the model to market data. Moreover, we provide a valuation formula for a derivative with a path-dependent payoff, namely a floating range note.

Among the variety of different interest rate models, the most popular approach is probably the *Libor market model*. Its popularity results from the fact that it is consistent with the market practice of pricing caps and floors. In other words, the model allows for a perfect calibration to cap and floor quotes. Unfortunately, the model prices fit only the at-the-money quotes well. Away-from-the-money there may be substantial misvaluations. Our goal is not only to reproduce the market prices of at-the-money caps. We intend to get an accurate calibration to cap prices across different strike rates and across all maturities with a reasonable number of parameters.

It is well known that exponential Lévy models for stock prices allow for an excellent calibration to implied volatility patterns for single maturities and also for a certain range of maturities, but fail to reproduce option prices with the same accuracy over the full range of different maturities. We made a similar observation in the Lévy term structure model. It performs very well when calibrating prices of caps with different strikes for a certain range of maturities. However, results become worse when the calibration is done across strikes and across the full range of maturities. This is due to the restrictive assumption of stationary increments. We drop this assumption and allow for a non-homogeneous Lévy process as driving process.

The outline of the chapter is as follows. Section 2.1 presents the details of the model. Some mathematical tools that are needed for derivative pricing in the subsequent sections are established in section 2.2. In section 2.3 we discuss the pricing of caps and floors. The main techniques used are *change-of-numeraire* and Laplace transformation methods. Analytical formulae that can numerically be evaluated fast are derived. The same tools together with an idea of Jamshidian (1989) can be applied to price swaptions under an additional assumption on the volatility structure. This is the content of section 2.4. Change-of-numeraire and Laplace transformation techniques can also be employed in a path dependent context. Concretely, we use a non-standard numeraire plus Laplace transformation methods in section 2.5 to determine the value of floating range notes. As a necessary tool and nice side result, digital options are priced. In section 2.6 we give an example of a model calibration to real market prices of caps as well as to swaption prices. Results for driving homogeneous Lévy processes are compared to those that are obtained when a non-homogeneous Lévy process is used. Section 2.7 concludes.

2.1 Presentation of the model

Let us briefly recall the HJM framework for modelling the term structure of interest rates. Suppose that $T^* > 0$ is a fixed time horizon and assume that for every $T \in [0, T^*]$ there is a zero coupon bond maturing at T traded on the market. Subject to modelling are either zero coupon bond prices or instantaneous, continuously compounded forward rates. Since forward rates can be deduced from bond prices and vice versa (see section 1.1), the term structure can be modelled by specifying either of them. Here, forward rates are specified and zero coupon bond prices are deduced.

The model is driven by a d-dimensional non-homogeneous Lévy process L with characteristics (b, c, F) on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with the filtration $(\mathcal{F}_s)_{0 \le s \le T^*}$ which is generated by L. More precisely, we assume $\mathcal{F} = \mathcal{F}_{T^*}$ and $(\mathcal{F}_s)_{0 \le s \le T^*}$ is the smallest right continuous filtration to which L is adapted. The dynamics of the instantaneous forward rates for $T \in [0, T^*]$ are postulated to be given by

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,\mathrm{d}s - \int_0^t \sigma(s,T)\,\mathrm{d}L_s \qquad (0 \le t \le T). \tag{2.1}$$

The initial values $f(0,T)$ are deterministic, and bounded and measurable in T. Moreover, α and σ are stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively defined on $\Omega \times [0,T^*] \times [0,T^*]$ that satisfy the following conditions:

1. $(\omega, s, T) \mapsto \alpha(\omega, s, T)$ and $(\omega, s, T) \mapsto \sigma(\omega, s, T)$ are measurable with respect to $\mathcal{P} \otimes \mathcal{B}([0, T^*])$.

2. For $s > T$ we have $\alpha(\omega, s, T) = 0$ and $\sigma(\omega, s, T) = 0$.

3. $\sup_{s,T \le T^*}(|\alpha(\omega, s, T)| + |\sigma(\omega, s, T)|) < \infty$.

These conditions ensure that we can find a "joint version" of all $f(t,T)$ such that $(\omega, t, T) \mapsto f(t,T)(\omega)1\!\!1_{\{t \le T\}}$ is $\mathcal{O} \otimes \mathcal{B}([0,T^*])$-measurable (see Eberlein, Jacod, and Raible (2005)). Here, $\mathcal{P}$ and $\mathcal{O}$ denote the predictable and the optional σ-field on $\Omega \times [0, T^*]$.

From the forward rates we can deduce explicit expressions for zero coupon bond prices and the risk free savings account:

Lemma 2.1 *The bond price $B(t,T)$ is given by*

$$B(t,T) = B(0,T) \exp\left(\int_0^t (r(s) - A(s,T))\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s \right), \tag{2.2}$$

where

$$A(s,T) := \int_{s \wedge T}^{T} \alpha(s,u)\,\mathrm{d}u \quad \text{and} \quad \Sigma(s,T) := \int_{s \wedge T}^{T} \sigma(s,u)\,\mathrm{d}u. \tag{2.3}$$

Proof: The claim can be proved in the same way as proposition 3.1 in Özkan (2002). Although the statement there is for a Lévy process and special semimartingale L, the proof neither uses the stationarity of the increments nor the fact that L is a special semimartingale. Thus, it also applies in the present setting. □

Setting $T = t$ in lemma 2.1, the risk free savings account $B_t := \exp \int_0^t r(s)\,\mathrm{d}s$ can be written as

$$B_t = \frac{1}{B(0,t)} \exp\left(\int_0^t A(s,t)\,\mathrm{d}s - \int_0^t \Sigma(s,t)\,\mathrm{d}L_s \right). \tag{2.4}$$

This leads to the following representation for the bond price which will be useful later:

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(-\int_0^t A(s,t,T)\,\mathrm{d}s + \int_0^t \Sigma(s,t,T)\,\mathrm{d}L_s \right), \tag{2.5}$$

where we used the abbreviations

$$A(s,t,T) := A(s,T) - A(s,t)$$

and

$$\Sigma(s,t,T) := \Sigma(s,T) - \Sigma(s,t). \tag{2.6}$$

In the remaining part of this chapter, we consider only deterministic volatility structures. More precisely, we require that the driving process L satisfies assumption $(\mathbb{EM})$ (see chapter 1) as well as the following condition:

Assumption ($\mathbb{DET}$). *The volatility structure σ is deterministic and bounded. For $0 \leq s, T \leq T^*$ we have*

$$0 \leq \Sigma^i(s,T) \leq M \qquad (i \in \{1, \dots, d\}) \tag{2.7}$$

where Σ is given by (2.3) and M is the constant from assumption ($\mathbb{EM}$). Note that $\Sigma(s,s) = 0$.

Our goal in the remaining part of this section is to derive a condition on the drift term that ensures the martingale property for discounted bond prices. In other words, we are looking for a condition on the drift term that makes the model work directly under a martingale measure. This will allow us to price integrable contingent claims by taking the $\mathbb{P}$-expectation of their discounted payoffs.

Let us use proposition 1.9 with $f(s) := \Sigma(s,T)$ for a fixed $T \in [0, T^*]$. The proposition yields

$$\mathbb{E}\left[\exp\left(\int_0^t \Sigma(s,T)\,\mathrm{d}L_s\right)\right] = \exp\int_0^t \theta_s(\Sigma(s,T))\,\mathrm{d}s.$$

If we set $A(s,T) := \theta_s(\Sigma(s,T))$ and $X := \int_0^\bullet \Sigma(s,T)\,\mathrm{d}L_s$, this reads as

$$\mathbb{E}[\exp X_t] = \exp\int_0^t A(s,T)\,\mathrm{d}s.$$

Lemma 2.1 implies the following expression for the discounted bond price:

$$Z(t,T) := \frac{1}{B_t}B(t,T) = B(0,T)\frac{\exp X_t}{\mathbb{E}[\exp X_t]}.$$

X is a process with independent increments. Therefore

$$\mathbb{E}[\exp X_t | \mathcal{F}_s] = \mathbb{E}[\exp(X_t - X_s)| \mathcal{F}_s]\exp X_s = \frac{\mathbb{E}[\exp X_t]}{\mathbb{E}[\exp X_s]}\exp X_s.$$

Hence, $Z(\cdot, T)$ is a martingale. Summing up, we get the following proposition:

Proposition 2.2 *Let*

$$A(s,T) := \theta_s(\Sigma(s,T)), \tag{2.8}$$

then for all $T \in [0, T^*]$ the discounted bond price process $Z(t,T) := \frac{1}{B_t}B(t,T)$ is a martingale.

Remark: The drift condition from proposition 2.2 ensures that $\mathbb{P}$ is a risk-neutral measure. If the dimension of the driving process L is $d = 1$, $\mathbb{P}$ is the unique martingale measure. For $d \geq 2$ this property does not hold in general. A discussion on the uniqueness of martingale measures in this model framework can be found in Eberlein, Jacod, and Raible (2005). If there is more than one martingale measure the problem of which one to choose arises. In this case, we assume that $\mathbb{P}$ is the risk-neutral measure chosen by the market and price integrable contingent claims by taking the $\mathbb{P}$-expectation of the discounted payoffs.

Note that in the special case of a driving standard Brownian motion, i.e. $\theta_s(z) = \frac{\langle z,z\rangle}{2}$ for $z \in \mathbb{C}^d$, equation (2.8) is the well-known Heath–Jarrow–Morton drift condition for the multifactor Gaussian HJM model. In all sections that follow, the drift condition from proposition 2.2 is assumed to be in force. Expression (2.5) for the bond price can then be expressed as

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\bigg(\int_0^t \Big(\theta_s(\Sigma(s,t)) - \theta_s(\Sigma(s,T)) \Big)\,\mathrm{d}s + \int_0^t \Sigma(s,t,T)\,\mathrm{d}L_s \bigg). \tag{2.9}$$

2.2 Tools for derivative valuation

The aim of this section is to present the main mathematical tools that are needed for derivative pricing in the subsequent sections. One method that will be applied is the *change-of-numeraire* technique developed by Geman, El Karoui, and Rochet (1995). It will prevent us from having to evaluate joint probability laws and, therefore, save us time in the computation of derivative prices. Standard and non-standard numeraires will be used, i.e. we employ forward martingale measures as well as a measure that we call adjusted forward measure. The other pillar on which the derivation of pricing formulae for derivatives will rest is an integral transform method. Integral transform methods are very useful whenever the characteristic function or bilateral Laplace transform of the underlying is known in closed form. They go back to Carr and Madan (1999) who use Fourier transforms and to Raible (2000) whose approach is based on bilateral Laplace transforms. In the context of deriving hedging strategies similar methods have been used by Hubalek and Krawczyk (1998). The idea to use characteristic functions for option pricing has already been applied in Heston (1993).

Remember that the *forward martingale measure* for the settlement day T, denoted by $\mathbb{P}_T$, is defined by the Radon–Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}} := \frac{1}{B_T B(0,T)}. \tag{2.10}$$

Usually, this measure is defined on $(\Omega, \mathcal{F}_T)$ only, but we can and do define it on $(\Omega, \mathcal{F}_{T^*})$. $\mathbb{P}$ and $\mathbb{P}_T$ are equivalent and from (2.4) we get the explicit expression

$$\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}} = \exp\bigg(-\int_0^T A(s,T)\,\mathrm{d}s + \int_0^T \Sigma(s,T)\,\mathrm{d}L_s \bigg). \tag{2.11}$$

Restricted to the σ-field $\mathcal{F}_t$ for $t \leq T$ this becomes

$$\begin{aligned} \frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} &= \mathbb{E}_{\mathbb{P}}\left[\frac{1}{B_T B(0,T)}\Big|\mathcal{F}_t\right] = \frac{B(t,T)}{B_t B(0,T)} \\ &= \exp\left(-\int_0^t A(s,T)\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s\right). \end{aligned} \tag{2.12}$$

Let us derive some properties of L under $\mathbb{P}_T$. The next proposition provides the semimartingale characteristics of L under the forward measure. Since we assume ($\mathbb{EM}$), L is a special semimartingale with respect to $\mathbb{P}$. However, it might not be a $\mathbb{P}_T$-special semimartingale (at least it is not clear at this point that it actually is). Hence, we work with a general truncation function h:

Proposition 2.3 *The semimartingale characteristics of L with respect to $\mathbb{P}_T$ associated with the truncation function h are given by*

$$\begin{aligned} (B_s^T)^i(h) &= B_s^i(h) + \int_0^s c_u^i \Sigma(u,T)\,\mathrm{d}u + \int_0^s \int_{\mathbb{R}^d} h^i(x)(e^{\langle \Sigma(u,T),x\rangle} - 1)\nu(\mathrm{d}u,\mathrm{d}x), \\ C_s^T &= C_s, \\ \nu^T(\mathrm{d}s,\mathrm{d}x) &= e^{\langle \Sigma(s,T),x\rangle}\nu(\mathrm{d}s,\mathrm{d}x), \end{aligned}$$

where c_u^i denotes the i-th row of the matrix c_u.

Proof: To derive the semimartingale characteristics we use Girsanov's Theorem for semimartingales as presented in Jacod and Shiryaev (2003, III.3.24). That is, we look for two predictable processes β and Y describing the change of measure. Denote by Z the density process as given in (2.12). Then, combining proposition 2.2, equation (1.10), and the canonical decomposition of L, we get

$$\begin{aligned} Z_s &= \exp\Bigg(-\int_0^s \langle \Sigma(u,T), b_u\rangle\,\mathrm{d}u - \frac{1}{2}\int_0^s \langle \Sigma(u,T), c_u\Sigma(u,T)\rangle\,\mathrm{d}u \\ &\qquad - \int_0^s\int_{\mathbb{R}^d}\left(e^{\langle \Sigma(u,T),x\rangle} - 1 - \langle \Sigma(u,T),x\rangle\right)\nu(\mathrm{d}u,\mathrm{d}x) \\ &\qquad + \int_0^s \langle \Sigma(u,T), b_u\rangle\,\mathrm{d}u + \int_0^s \sqrt{c_u}\Sigma(u,T)\,\mathrm{d}W_u \\ &\qquad + \int_0^s\int_{\mathbb{R}^d} \langle \Sigma(u,T),x\rangle(\mu^L - \nu)(\mathrm{d}u,\mathrm{d}x)\Bigg) \\ &= \mathcal{E}_s\Bigg(\int_0^\bullet \sqrt{c_u}\beta(u)\,\mathrm{d}W_u \\ &\qquad + \int_0^\bullet\int_{\mathbb{R}^d}(Y(u,x)-1)\,(\mu^L-\nu)(\mathrm{d}u,\mathrm{d}x)\Bigg) \end{aligned}$$

where $\beta(u) := \Sigma(u,T)$ and $Y(u,x) := \exp\langle\Sigma(u,T),x\rangle$ are the candidates for Girsanov's Theorem. The last equality follows from Kallsen and Shiryaev (2002, Lemma 2.6). The following lemma shows that these candidates meet the prerequisites of the theorem (compare also Shiryaev (1999, Chapter VII, Section 3g, Theorem 1)). We obtain the characteristics as given above. □

Lemma 2.4 *Define β, Y, and Z as above. Then*

$$\langle Z^c, L^{i,c}\rangle_t = \int_0^t Z_{u-}c_u^i\beta(u)\,\mathrm{d}u$$

and Y is a nonnegative version of $M^{\mathbb{P}}_{\mu^L}\left(\frac{Z}{Z_-}\mathbb{1}_{\{Z_->0\}}\middle|\widetilde{\mathcal{P}}\right)$.

Remark: Here, $\langle\cdot,\cdot\rangle$. denotes the angle bracket relative to $\mathbb{P}$ and $M^{\mathbb{P}}_{\mu^L}$ is a positive measure on $(\Omega\times[0,T^*]\times\mathbb{R}^d, \mathcal{F}\otimes\mathcal{B}([0,T^*])\otimes\mathcal{B}(\mathbb{R}^d))$ defined by $M^{\mathbb{P}}_{\mu^L}(W) := \mathbb{E}[W * \mu_{T^*}]$ for a measurable nonnegative function W. $M^{\mathbb{P}}_{\mu^L}(\cdot|\widetilde{\mathcal{P}})$ denotes the "conditional expectation" relative to $M^{\mathbb{P}}_{\mu^L}$ with respect to the σ-field $\widetilde{\mathcal{P}} := \mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$. For more details we refer to Jacod and Shiryaev (2003, Section III.3c).

Proof: To show the first claim, define a process N by

$$N_s := \int_0^s \sqrt{c_u}\beta(u)\,\mathrm{d}W_u + \int_0^s\int_{\mathbb{R}^d}(Y(u,x)-1)(\mu^L-\nu)(\mathrm{d}u,\mathrm{d}x).$$

N is well defined since $(Y-1)^2 * \nu_{T^*} < \infty$ and thus, by Jacod and Shiryaev (2003, Theorem II.1.33), $(Y-1)\in G_{\mathrm{loc}}$. Moreover, $Z=\mathcal{E}(N)$ or, written differently, $Z_s = 1+\int_0^s Z_{u-}\,\mathrm{d}N_u$. Thus, for $1\le i\le d$

$$\begin{aligned}
\langle Z^c, L^{i,c}\rangle_t &= \left\langle \int_0^{\bullet} Z_{s-}\,\mathrm{d}N_s^c, \left(\int_0^{\bullet}\sqrt{c_s}\,\mathrm{d}W_s\right)^i\right\rangle_t \\
&= \int_0^t Z_{u-}\,\mathrm{d}\left\langle N^c, \left(\int_0^{\bullet}\sqrt{c_s}\,\mathrm{d}W_s\right)^i\right\rangle_u \\
&= \int_0^t Z_{u-}c_u^i\beta(u)\,\mathrm{d}u.
\end{aligned}$$

To prove the second claim, we have to show that for any $\widetilde{\mathcal{P}}$-measurable nonnegative U the equation $M^{\mathbb{P}}_{\mu^L}(YU) = M^{\mathbb{P}}_{\mu^L}(\frac{Z}{Z_-}\mathbb{1}_{\{Z_->0\}}U)$ holds. Since

$(\frac{Z}{Z_-}1\!\!1_{\{Z_->0\}})_s = \exp\langle\Sigma(s,T),\Delta L_s\rangle$, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T^*}\int_{\mathbb{R}^d} Y(s,x)U(s,x)\mu^L(\mathrm{d}s,\mathrm{d}x)\right]\\
&\qquad = \mathbb{E}\left[\sum_{0\le s\le T^*} e^{\langle\Sigma(s,T),\Delta L_s\rangle}U(s,\Delta L_s)1\!\!1_{\{\Delta L_s\neq 0\}}\right]\\
&\qquad = \mathbb{E}\left[\int_0^{T^*}\int_{\mathbb{R}^d}\frac{Z_s}{Z_{s-}}1\!\!1_{\{Z_{s-}>0\}}U(s,x)\mu^L(\mathrm{d}s,\mathrm{d}x)\right]
\end{aligned}$$

and the assertion is proved. □

Lemma 2.5 *L is a special semimartingale and a non-homogeneous Lévy process with respect to $\mathbb{P}_T$.*

PROOF: To verify that L is a special semimartingale, note that

$$\begin{aligned}
(|x|^2\wedge|x|)*\nu^T_{T^*} &= \int_0^{T^*}\int_{\mathbb{R}^d}(|x|^2\wedge|x|)e^{\langle\Sigma(s,T),x\rangle}F_s(\mathrm{d}x)\,\mathrm{d}s\\
&\le C_1\int_0^{T^*}\int_{\{|x|\le 1\}}|x|^2F_s(\mathrm{d}x)\,\mathrm{d}s\\
&\quad+\int_0^{T^*}\int_{\{|x|>1\}}|x|e^{\langle\Sigma(s,T),x\rangle}F_s(\mathrm{d}x)\,\mathrm{d}s
\end{aligned}$$

for some constant C_1 and use the arguments of the proof of lemma 1.7.

L is a process with independent increments since there is a deterministic version of its semimartingale characteristics given in proposition 2.3 (see Jacod and Shiryaev (2003, II.4.15)). The same theorem (or alternatively proposition 1.9) can be used to calculate the characteristic function of L under $\mathbb{P}_T$:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_T}\left[e^{\mathrm{i}\langle u,L_s\rangle}\right] = \exp\Bigg(&\mathrm{i}\langle u,B^T_s(h)\rangle-\frac{1}{2}\langle u,C_su\rangle\\
&+\int_0^s\int_{\mathbb{R}^d}(e^{\mathrm{i}\langle u,x\rangle}-1-\mathrm{i}\langle u,h(x)\rangle)\nu^T(\mathrm{d}t,\mathrm{d}x)\Bigg).
\end{aligned}$$

We can write this characteristic function in the same form as in (1.1) with

$$\begin{aligned}
b^T_s &= b_s + c_s\Sigma(s,T)+\int_{\mathbb{R}^d}\left(e^{\langle\Sigma(s,T),x\rangle}-1\right)x1\!\!1_{\{|x|\le 1\}}\,F_s(\mathrm{d}x),\\
c^T_s &= c_s,\\
F^T_s(\mathrm{d}x) &= e^{\langle\Sigma(s,T),x\rangle}F_s(\mathrm{d}x).
\end{aligned}$$

It can easily be checked that b^T, c^T, and F^T satisfy an equation analogous to (1.2). Consequently, L is a non-homogeneous Lévy process with respect to $\mathbb{P}_T$. Of course, path properties are preserved under the equivalent change of measure. □

Another measure of which we will make use in the context of pricing floating range notes is the following:
For $T' < T$ we define the *adjusted forward measure* $\mathbb{P}_{T',T}$ on $(\Omega, \mathcal{F}_{T^*})$ via

$$\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}_T} := \frac{F(T',T',T)}{F(0,T',T)} = \frac{B(0,T)}{B(0,T')B(T',T)}, \tag{2.13}$$

where $F(\cdot,T',T) := \frac{B(\cdot,T')}{B(\cdot,T)}$ denotes the forward price process. Restricting this density to $\mathcal{F}_t$ for $t \leq T'$ we get

$$\left.\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}_T}\right|_{\mathcal{F}_t} = \frac{F(t,T',T)}{F(0,T',T)} = \frac{B(0,T)B(t,T')}{B(0,T')B(t,T)} \tag{2.14}$$

since $(F(t,T',T))_{0\leq t\leq T'}$ is a $\mathbb{P}_T$-martingale. Thus we have

$$\left.\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{B(0,T)B(t,T')}{B(0,T')B(t,T)}\frac{B(t,T)}{B_t B(0,T)} = \frac{B(t,T')}{B_t B(0,T')},$$

i.e. the forward measure $\mathbb{P}_{T'}$ and the adjusted forward measure $\mathbb{P}_{T',T}$ are equal once restricted to $(\Omega, \mathcal{F}_t)$ for $t \leq T'$. However, on $(\Omega, \mathcal{F}_t)$ for $t > T'$ they are usually different. Choose for example $T' < t < T$, then in general

$$\begin{aligned}
\left.\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} &= \frac{B(t,T)}{B(T',T)B_t B(0,T')} \\
&\overset{(2.2)}{=} \frac{1}{B_{T'}B(0,T')}\exp\left(-\int_{T'}^{t} A(s,T)\,\mathrm{d}s + \int_{T'}^{t} \Sigma(s,T)\,\mathrm{d}L_s\right) \\
&\neq \frac{1}{B_{T'}B(0,T')} \\
&= \left.\frac{\mathrm{d}\mathbb{P}_{T'}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}.
\end{aligned}$$

Using (2.2) and (2.4) we can write the density process Z' of $\mathbb{P}_{T',T}$ with respect to $\mathbb{P}$ as

$$Z'_t = \left.\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t A_{T',T}(s)\,\mathrm{d}s + \int_0^t \Sigma_{T',T}(s)\,\mathrm{d}L_s\right), \tag{2.15}$$

where

$$A_{T',T}(s) := A(s,T')\mathbb{1}_{\{s\leq T'\}} + A(s,T)\mathbb{1}_{\{s>T'\}} \tag{2.16}$$

and

$$\Sigma_{T',T}(s) := \Sigma(s,T')\mathbb{1}_{\{s\leq T'\}} + \Sigma(s,T)\mathbb{1}_{\{s>T'\}}. \tag{2.17}$$

Proceeding as in the proof of proposition 2.3, the two predictable processes in Girsanov's Theorem can be identified as

$$\beta(u) = \Sigma_{T',T}(u) \qquad \text{and} \qquad Y(u,x) = \exp\langle \Sigma_{T',T}(u), x\rangle.$$

Similar arguments as in the proof of lemma 2.5 lead to the conclusion that L is also a non-homogeneous Lévy process and a special semimartingale with respect to the adjusted forward measure $\mathbb{P}_{T',T}$.

The integral transform method we are going to apply traces back to Raible (2000). The key idea is to express derivative prices as convolutions and perform Laplace transformations followed by inverse Laplace transformations. This technique exploits the fact that the Laplace transform of a convolution equals the product of the Laplace transforms of the convolution factors. It is especially useful whenever these Laplace transforms of the convolution factors can easily be calculated or are even known in closed form. We cite the following theorems from Raible (2000, Theorems B.2 and B.3) since they will be used very frequently in the subsequent sections and chapters:

Theorem 2.6 *Let F_1 and F_2 be measurable complex-valued functions on the real line. Let $z \in \mathbb{C}$ and $R := \Re z$. If*

$$\int_{\mathbb{R}} e^{-Rx}|F_1(x)|\,\mathrm{d}x < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} e^{-Rx}|F_2(x)|\,\mathrm{d}x < \infty$$

and if $x \mapsto e^{-Rx}|F_1(x)|$ is bounded, then the convolution $F(x) := F_1 * F_2(x)$ exists and is continuous for all $x \in \mathbb{R}$, and we have

$$\int_{\mathbb{R}} e^{-Rx}|F(x)|\,\mathrm{d}x < \infty$$

and

$$\int_{\mathbb{R}} e^{-zx}F(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^{-zx}F_1(x)\,\mathrm{d}x \int_{\mathbb{R}} e^{-zx}F_2(x)\,\mathrm{d}x.$$

Theorem 2.7 *Let F be a measurable complex-valued function on the real line. Let $R \in \mathbb{R}$ be such that*

$$f(z) = \int_{\mathbb{R}} e^{-zx}F(x)\,\mathrm{d}x \qquad (z \in \mathbb{C},\ \Re z = R),$$

with the integral converging absolutely for $z = R$. Let $x \in \mathbb{R}$ be such that the integral

$$\int_{R-i\infty}^{R+i\infty} e^{zx}f(z)\,\mathrm{d}z$$

exists as a Cauchy principal value. Assume that F is continuous at the point x. Then

$$F(x) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} e^{zx}f(z)\,\mathrm{d}z,$$

where the integral is to be understood as the Cauchy principal value if the integrand is not absolutely integrable.

2.3 Valuation of caps and floors

Our goal in this section is to derive explicit formulae for the prices of caps and floors that can numerically be evaluated fast. This is crucial in order to be able to calibrate the model to market prices of caps and floors in reasonable time.

Remember that a cap (resp. floor) is a series of call (resp. put) options on subsequent variable rates. These single options are called *caplets* (resp. *floorlets*). Each caplet is equivalent to a put option on a zero coupon bond, each floorlet can be considered as a call option (see section 1.1 or James and Webber (2000, 3.1.5)). Thus, if we derive suitable formulae for calls and puts on zero coupon bonds, we immediately have formulae for caps and floors.

As described in a previous section, the discounted bond price process $Z(\cdot, T)$ is a martingale with respect to the measure $\mathbb{P}$ and the given filtration for each $T \in [0, T^*]$. Consequently, we can price an integrable contingent claim by taking the conditional expectation of the discounted payoff. The time-s value of a call with strike K and maturity t on a bond which matures at T is then given by

$$C_s(t, T, K) := \mathbb{E}\left[\frac{1}{B_t}(B(t,T) - K)^+ \,|\mathcal{F}_s\right] \qquad (s \leq t).$$

To determine today's value of the call we need to evaluate $\mathbb{E}\left[B_t^{-1}(B(t,T) - K)^+\right]$. A straightforward approach is to derive the joint (conditional) distribution of the random variables B_t and $B(t, T)$. Although this can easily be done analytically (compare Eberlein and Raible (1999)), the numerical evaluation of the resulting expression is extremely time consuming. Instead, we use the *change-of-numeraire* technique to circumvent the calculation of the joint probability law, i.e. we switch from the spot martingale measure $\mathbb{P}$ to the *forward martingale measure* for the settlement day t, denoted by $\mathbb{P}_t$. Equation (2.10) for the density and expression (2.9) for the bond price lead to the following price for the call:

$$\begin{aligned} C_0(t, T, K) &= B(0,t)\mathbb{E}_{\mathbb{P}_t}[(B(t,T) - K)^+] \\ &= B(0,t)\mathbb{E}_{\mathbb{P}_t}[(D\exp(X) - K)^+], \end{aligned}$$

where

$$D := \frac{B(0,T)}{B(0,t)} \exp\left(\int_0^t \Big(\theta_s(\Sigma(s,t)) - \theta_s(\Sigma(s,T))\Big)\,\mathrm{d}s\right)$$

is deterministic and

$$X := \int_0^t \Sigma(s, t, T)\,\mathrm{d}L_s$$

is $\mathcal{F}_t$-measurable. To calculate the option price we only need to know the distribution of X under the measure $\mathbb{P}_t$, which we denote by $\mathbb{P}_t^X$. Suppose that this distribution possesses a Lebesgue-density φ in $\mathbb{R}$, then

$$C_0(t,T,K) = B(0,t) \int_{\mathbb{R}} (De^x - K)^+ \varphi(x)\,\mathrm{d}x. \tag{2.18}$$

Before deriving a formula for the option price, let us shortly discuss the assumption that $\mathbb{P}_t^X$ possesses a Lebesgue-density. This distribution possesses a Lebesgue-density if and only if it is absolutely continuous (with respect to the Lebesgue-measure on $\mathbb{R}$). Since $\mathbb{P}$ and $\mathbb{P}_t$ are equivalent, this is the case if and only if the distribution of X with respect to $\mathbb{P}$, denoted by $\mathbb{P}^X$, is absolutely continuous. Whether or not $\mathbb{P}^X$ is absolutely continuous cannot be answered in general. The answer depends on the choice of the volatility structure and the driving process, as the following examples show:

1. Let $\Sigma(s,t) = \Sigma(s,T)$ for $s \in [0,t]$ (i.e. $\Sigma(s,t,T) = 0$), then $X = 0$ and $\mathbb{P}^X$ cannot be absolutely continuous.

2. Choose the Ho–Lee volatility structure, i.e. $\Sigma(s,T) = \widehat{\sigma}(T-s)$, and let L be a Poisson process, then $X = \widehat{\sigma}(T-t)L_t$, whose distribution is not absolutely continuous since the distribution of L_t is not.

The following proposition gives sufficient conditions for absolute continuity:

Proposition 2.8 *Assume that $\Sigma(s,t,T) \neq 0$ for $s \in [0,t]$. Then each of the following conditions implies that $\mathbb{P}^X$ is absolutely continuous with respect to the Lebesgue-measure λ:*

1. *There is a Borel set $S \subset [0,t]$ with $\lambda(S) > 0$ such that c_s is positive definite for $s \in S$.*

2. *Denote by Φ_s the characteristic function associated with the Lévy–Khintchine triplet (b_s, c_s, F_s), i.e. for $u \in \mathbb{R}^d$*

$$\begin{aligned}\Phi_s(u) &= \exp\left(\mathrm{i}\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u,x\rangle) F_s(\mathrm{d}x) \right)\\ &= \exp(\theta_s(\mathrm{i}u)).\end{aligned}$$

   Then

$$|\Phi_s(u)| \leq C \exp\left(-\gamma |u|^\eta\right) \qquad (s \in [0,t])$$

   for real constants C, $\gamma > 0$, $\eta > 0$ that do not depend on s.

Proof: We show that Φ^X, i.e. the characteristic function of X, is integrable. Using proposition 1.9 we get

$$\Phi^X(u) = \exp \int_0^t \theta_s(\mathrm{i}u\Sigma(s,t,T))\,\mathrm{d}s \qquad (u \in \mathbb{R}). \tag{2.19}$$

Let us first suppose that condition 1 is satisfied and define L^1 and L^2 by

$$\begin{aligned} L_t^1 &:= \int_0^t b_s \,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s, \\ L_t^2 &:= \int_0^t \int_{\mathbb{R}^d} x(\mu-\nu)(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

Both processes are PIIACs and $L = L^1 + L^2$. By (2.19) and using proposition 1.9 on L^1 and L^2, we have

$$\Phi^X(u) = \Phi^{X^1}(u)\Phi^{X^2}(u),$$

where $\Phi^{X^j}(u) := \mathbb{E}\left[\exp\left(\mathrm{i}u\int_0^t \Sigma(s,t,T)\,\mathrm{d}L_s^j\right)\right]$ for $j \in \{1,2\}$. Since both factors are bounded above by 1, it is enough to show that one of them is integrable. But

$$\begin{aligned} \left|\Phi^{X^1}(u)\right| &= \exp\Big(-\frac{1}{2}u^2\int_0^t \langle\Sigma(s,t,T), c_s\Sigma(s,t,T)\rangle\,\mathrm{d}s\Big) \\ &=: \exp\Big(-\frac{1}{2}u^2 C\Big), \end{aligned}$$

where $C > 0$ due to the positive definiteness of c_s for $s \in S$ and the fact that $\lambda(S) > 0$. Hence Φ^{X^1} is integrable.

Now suppose condition 2 is satisfied. Then by (2.19)

$$\begin{aligned} |\Phi^X(u)| &= \exp\int_0^t \Re\Big(\theta_s(\mathrm{i}u\Sigma(s,t,T))\Big)\,\mathrm{d}s \\ &= \exp\int_0^t \log|\Phi_s(u\Sigma(s,t,T))|\,\mathrm{d}s \\ &\leq \exp\int_0^t \log\Big(C\exp(-\gamma|u\Sigma(s,t,T)|^\eta)\Big)\,\mathrm{d}s \\ &= C^t\exp\Big(-\gamma|u|^\eta\int_0^t|\Sigma(s,t,T)|^\eta\,\mathrm{d}s\Big) \\ &=: C^t\exp\left(-\widetilde{\gamma}|u|^\eta\right), \end{aligned}$$

where $\widetilde{\gamma} > 0$ since $\Sigma(s,t,T) \neq 0$ for $s \in [0,t]$. Consequently, Φ^X is integrable. □

Let us come back to option pricing and denote by M_t^X the moment generating function of the random variable X with respect to the measure $\mathbb{P}_t$. The next theorem gives an analytic expression for the price of the call:

Theorem 2.9 *Suppose that the distribution of X possesses a Lebesgue-density. Choose an $R < -1$ such that $M_t^X(-R) < \infty$. Then we have*

$$C_0(t,T,K) = \frac{1}{2\pi} K B(0,t) e^{R\xi} \int_{-\infty}^{\infty} e^{\mathrm{i}u\xi} \frac{1}{(R+\mathrm{i}u)(R+1+\mathrm{i}u)} M_t^X(-R-\mathrm{i}u)\,\mathrm{d}u \tag{2.20}$$

with

$$\xi := \log \frac{B(0,t)}{B(0,T)} - \int_0^t \Big(\theta_s(\Sigma(s,t)) - \theta_s(\Sigma(s,T))\Big)\,\mathrm{d}s + \log K.$$

Before proving the theorem let us point out that it is always possible to find an R which satisfies the prerequisites of the theorem. This is part of the following lemma which also gives an explicit expression for $M_t^X(-R-\mathrm{i}u)$:

Lemma 2.10 *Choose M and ε in assumption ($\mathbb{EM}$) such that $\Sigma(s,T) \leq M'$ componentwise for an $M' < M$ and for all $s,T \in [0,T^*]$. Then, for each $R \in [-1 - \frac{M-M'}{M'}, -1)$ we have $M_t^X(-R) < \infty$. Moreover, for $z \in \mathbb{C}$ with $\Re z = -R$*

$$M_t^X(z) = \exp \int_0^t \Big(\theta_s(z\Sigma(s,T) + (1-z)\Sigma(s,t)) - \theta_s(\Sigma(s,t))\Big)\,\mathrm{d}s. \tag{2.21}$$

Proof: Fix $R \in [-1 - \frac{M-M'}{M'}, -1)$. For $z \in \mathbb{C}$ with $\Re z = -R$ we have

$$\begin{aligned} \big|\Re\big(z\Sigma^i(s,T) + (1-z)\Sigma^i(s,t)\big)\big| &= \big|\Re\big(\Sigma^i(s,T) + (1-z)(\Sigma^i(s,t) - \Sigma^i(s,T))\big)\big| \\ &\leq \Sigma^i(s,T) + |1+R||\Sigma^i(s,t) - \Sigma^i(s,T)| \\ &\leq M' + \frac{M-M'}{M'}M' = M. \end{aligned}$$

Hence,

$$\begin{aligned} M_t^X(z) &= \mathbb{E}_{\mathbb{P}_t}\left[\exp\left(z\int_0^t (\Sigma(s,T) - \Sigma(s,t))\,\mathrm{d}L_s\right)\right] \\ &\overset{(2.11)}{=} \exp\left(-\int_0^t A(s,t)\,\mathrm{d}s\right) \\ &\qquad \times \mathbb{E}_{\mathbb{P}}\left[\exp\left(\int_0^t (z\Sigma(s,T) + (1-z)\Sigma(s,t))\,\mathrm{d}L_s\right)\right] \\ &= \exp\int_0^t \Big(\theta_s(z\Sigma(s,T) + (1-z)\Sigma(s,t)) - \theta_s(\Sigma(s,t))\Big)\,\mathrm{d}s, \end{aligned}$$

where the last equality follows from (2.8) and proposition 1.9. In particular, $M_t^X(-R)$ is finite. □

PROOF OF THEOREM 2.9: The arguments are similar to the proof of theorem 3.2 in Raible (2000).

Using representation (2.18) for the option price and defining $\xi := -\log D + \log K$ and $v(x) := (e^{-x} - 1)^+$ yields

$$\begin{aligned} C_0(t,T,K) &= B(0,t) \int_{\mathbb{R}} (De^x - K)^+ \varphi(x)\,\mathrm{d}x \\ &= KB(0,t) \int_{\mathbb{R}} (DK^{-1}e^x - 1)^+ \varphi(x)\,\mathrm{d}x \\ &= KB(0,t) \int_{\mathbb{R}} v(\xi - x)\varphi(x)\,\mathrm{d}x \\ &= KB(0,t)(v * \varphi)(\xi) =: V(\xi). \end{aligned}$$

We apply theorem 2.6 to the functions $F_1(x) := v(x)$ and $F_2(x) := \varphi(x)$, that is we express the bilateral Laplace transform of their convolution as the product of the bilateral Laplace transforms of the convolution factors. The prerequisites of the theorem are satisfied since $x \mapsto e^{-Rx}v(x)$ is bounded,

$$\int_{\mathbb{R}} e^{-Rx}|v(x)|\,\mathrm{d}x = \frac{1}{R(R+1)} < \infty,$$

and

$$\int_{\mathbb{R}} e^{-Rx}|\varphi(x)|\,\mathrm{d}x = \int_{\mathbb{R}} e^{-Rx}\varphi(x)\,\mathrm{d}x = M_t^X(-R) < \infty$$

by assumption. The cited theorem yields

$$L[V](R + \mathrm{i}u) = B(0,t)KL[v](R + \mathrm{i}u)L[\varphi](R + \mathrm{i}u) \qquad (u \in \mathbb{R})$$

where $L[V]$ denotes the bilateral Laplace transform of V (analogously for v and φ). It also yields that $\xi \mapsto V(\xi)$ is continuous and $\int_{\mathbb{R}} e^{-R\xi}V(\xi)\,\mathrm{d}\xi$ is absolutely convergent. Therefore, we may apply theorem 2.7 and get

$$\begin{aligned} V(\xi) &= \frac{1}{2\pi i} \lim_{Y\to\infty} \int_{R-\mathrm{i}Y}^{R+\mathrm{i}Y} e^{z\xi} L[V](z)\,\mathrm{d}z \\ &= \frac{1}{2\pi} \lim_{Y\to\infty} \int_{-Y}^{Y} e^{(R+\mathrm{i}u)\xi} L[V](R+\mathrm{i}u)\,\mathrm{d}u \\ &= \frac{1}{2\pi} B(0,t)Ke^{R\xi} \lim_{Y\to\infty} \int_{-Y}^{Y} e^{\mathrm{i}u\xi} L[v](R+\mathrm{i}u)L[\varphi](R+\mathrm{i}u)\,\mathrm{d}u, \end{aligned}$$

if this limit exists. We have

$$L[\varphi](R + \mathrm{i}u) = M_t^X(-R - \mathrm{i}u)$$

and, according to Raible (2000, p. 66),

$$L[v](R+\mathrm{i}u) = \frac{1}{(R+\mathrm{i}u)(R+1+\mathrm{i}u)}.$$

The above limit exists (moreover, the integral converges absolutely) due to the fact that $|e^{\mathrm{i}u\xi}| = 1$, $|M_t^X(-R-\mathrm{i}u)| \leq M_t^X(-R) < \infty$ independent of u and $\int_{\mathbb{R}} \left| \frac{1}{(R+\mathrm{i}u)(R+1+\mathrm{i}u)} \right| \mathrm{d}u < \infty$. This proves our assertion. □

In a similar manner we can derive the price $P_0(t,T,K)$ for a put with strike K and maturity t on a bond which matures at T:

Corollary 2.11 *Suppose the distribution of X possesses a Lebesgue-density. Choose an $R > 0$ such that $M_t^X(-R) < \infty$. Then we have*

$$P_0(t,T,K) = \frac{1}{2\pi} K B(0,t) e^{R\xi} \int_{-\infty}^{\infty} e^{\mathrm{i}u\xi} \frac{1}{(R+\mathrm{i}u)(R+1+\mathrm{i}u)} M_t^X(-R-\mathrm{i}u)\,\mathrm{d}u \tag{2.22}$$

with

$$\xi := \log \frac{B(0,t)}{B(0,T)} - \int_0^t \Big(\theta_s(\Sigma(s,t)) - \theta_s(\Sigma(s,T))\Big)\,\mathrm{d}s + \log K,$$

Remark: Note that the formulae for the call and the corresponding put coincide. The difference is in the permitted values for R.

2.4 Swaption pricing

In this section, an explicit formula for pricing swaptions is derived under an additional assumption on the volatility structure. The numerical evaluation of the pricing formula can be done fast. Once again, this is crucial for calibration purposes.

Remember that a swaption is an option on a forward swap, i.e. on a swap which starts in the future. At maturity of the option the holder has the right to enter into the swap at a pre-specified fixed rate. There are payer and receiver swaptions giving their owners the right to enter into the swap as fixed rate payer or receiver respectively. We interpret the swaption as a right to exchange a coupon bond having the fixed rate of the swap as its coupon against a floater, whose value is always equal to 1. Thus, a receiver (resp. payer) swaption can be seen as a call (resp. put) on a coupon bond with an exercise price of 1 (compare Musiela and Rutkowski (1998, Section 16.2.3)).

We price options on coupon bonds in our model framework with the following restriction on the volatility structure:

Assumption ($\mathbb{VOL}$). *For all $T \in [0,T^*]$ we have $\sigma(\cdot,T) \not\equiv (0,\ldots,0)$ and*

$$\sigma(s,T) = \sigma_2(T)\sigma_1(s) \qquad (0 \leq s \leq T)$$

where $\sigma_1 : [0, T^*] \to \mathbb{R}^d$ *and* $\sigma_2 : [0, T^*] \to \mathbb{R}_+$ *are continuously differentiable.*

REMARKS:

1. Under assumption ($\mathbb{VOL}$), the short rate process r is Markovian. This can be proved using exactly the same arguments as in Eberlein and Raible (1999, Theorems 4.2 and 4.3) even though L does not possess stationary increments.

2. In the one-dimensional case ($d = 1$) and for a homogeneous Lévy process L which is not identically zero, the converse is also true. That is, a Markovian short rate implies that the volatility structure factorizes as above. This has been proved by Eberlein and Raible (1999) under an additional assumption on L and by Küchler and Naumann (2003) in the general case.

3. In this setting, $\mathbb{P}$ is the unique martingale measure. This is an immediate consequence of Eberlein, Jacod, and Raible (2005, Theorem 6.4) since the dimension d_t of the vector space $E_t := span(\Sigma(t.T) : T \in [0, T^*])$ satisfies $d_t \leq 1$ for all t and we can thus reduce the dimensionality of the model to one (e.g. by using the PIIAC $\int_0^\bullet \sigma_1(s)\,\mathrm{d}L_s$ as driving motion).

Example 2.12 *The following volatility structures satisfy the condition above:*

1. $\sigma(s, T) = \widehat{\sigma}$ *(Ho–Lee volatility structure),*
2. $\sigma(s, T) = \widehat{\sigma} e^{-a(T-s)}$ *(Vasiček volatility structure),*
3. $\sigma(s, T) = \widehat{\sigma}\,\frac{1+\gamma T}{1+\gamma s}\,e^{-a(T-s)}$ *(Moraleda–Vorst volatility structure)*

for real constants $\widehat{\sigma}$, $\gamma > 0$ *and* $a \neq 0$.

To price options on coupon bonds we use an idea of Jamshidian (1989) as well as change of numeraire and Laplace transformation techniques. Denote by $B_C(t, T_1, \ldots, T_n)$ the time t price of a coupon bond with maturity T_n paying to its owner an amount of $C_1, \ldots, C_n$ at the dates $T_1, \ldots, T_n$. Then, for $0 \leq t < T_1$

$$B_C(t, T_1, \ldots, T_n) = C_1 B(t, T_1) + C_2 B(t, T_2) + \ldots + C_n B(t, T_n).$$

The time-0 price of a call with strike price 1 and maturity t on that bond is obtained by taking the expectation of the discounted payoff, i.e.

$$\begin{aligned}
C_0 &:= C_0(t, T_1, \ldots, T_n, C_1, \ldots, C_n) \\
&:= \mathbb{E}\left[\frac{1}{B_t}\Big(\sum_{i=1}^{n} C_i B(t, T_i) - 1\Big)^+\right] \\
&= B(0, t)\mathbb{E}_{\mathbb{P}_t}\left[\Big(\sum_{i=1}^{n} C_i B(t, T_i) - 1\Big)^+\right] \\
&= B(0, t)\mathbb{E}_{\mathbb{P}_t}\left[\Big(\sum_{i=1}^{n} D_i \exp\Big(\int_0^t \Sigma(s, t, T_i)\,\mathrm{d}L_s\Big) - 1\Big)^+\right].
\end{aligned}$$

For the last equation we used (2.9) and defined the constants

$$D_i := \frac{B(0,T_i)}{B(0,t)} C_i \exp\left(\int_0^t \Big(\theta_s(\Sigma(s,t)) - \theta_s(\Sigma(s,T_i)) \Big) \,\mathrm{d}s \right).$$

Since for $0 \leq s \leq t \leq T \leq T^*$

$$\Sigma(s,t,T) = \int_t^T \sigma(s,u)\,\mathrm{d}u = \int_t^T \sigma_2(u)\,\mathrm{d}u\,\sigma_1(s),$$

we get

$$\Sigma(s,t,T_i) = \frac{\int_t^{T_i} \sigma_2(u)\,\mathrm{d}u}{\int_t^{T_n} \sigma_2(u)\,\mathrm{d}u} \Sigma(s,t,T_n).$$

Hence,

$$C_0 = B(0,t)\mathbb{E}_{\mathbb{P}_t}\left[\left(\sum_{i=1}^n D_i e^{B_i X} - 1 \right)^+ \right], \tag{2.23}$$

where

$$0 < B_i := \frac{\int_t^{T_i} \sigma_2(u)\,\mathrm{d}u}{\int_t^{T_n} \sigma_2(u)\,\mathrm{d}u} \leq 1 \qquad (i \in \{1,\ldots,n\})$$

and

$$X := \int_0^t \Sigma(s,t,T_n)\,\mathrm{d}L_s.$$

We only need to know the distribution of X with respect to the measure $\mathbb{P}_t$ to calculate the option price. Suppose that this distribution $\mathbb{P}_t^X$ possesses a Lebesgue-density in $\mathbb{R}$, then

$$C_0 = B(0,t) \int_{\mathbb{R}} \left(\sum_{i=1}^n D_i e^{B_i x} - 1 \right)^+ \varphi(x)\,\mathrm{d}x, \tag{2.24}$$

where $\varphi := \frac{\mathrm{d}\mathbb{P}_t^X}{\mathrm{d}\lambda\!\!\lambda}$.

We proceed as in the previous section by performing a Laplace transformation followed by an inverse Laplace transformation. As before, denote by M_t^X the moment generating function of the random variable X with respect to the measure $\mathbb{P}_t$.

Theorem 2.13 *Suppose the distribution of X possesses a Lebesgue-density. Choose an $R < -1$ such that $M_t^X(-R) < \infty$ and let Z be the unique zero of the strictly increasing and continuous function*

$$g(x) := \sum_{i=1}^n D_i e^{B_i x} - 1.$$

Then we have

$$C_0 = \frac{1}{2\pi} B(0,t) \lim_{Y\to\infty} \int_{-Y}^{Y} L[v](R+\mathrm{i}u) M_t^X(-R-\mathrm{i}u)\,\mathrm{d}u, \tag{2.25}$$

where $L[v]$ denotes the bilateral Laplace transform of $v : \mathbb{R} \to \mathbb{R}$ defined by $v(x) := (g(-x))^+$. Moreover,

$$L[v](R+\mathrm{i}u) = e^{(R+\mathrm{i}u)Z} \left(\sum_{i=1}^{n} \left(D_i e^{B_i Z} \frac{-1}{B_i + R + \mathrm{i}u} \right) + \frac{1}{R+\mathrm{i}u} \right) \qquad (u \in \mathbb{R})$$

and for $z \in \mathbb{C}$ with $\Re z = -R$

$$M_t^X(z) = \exp \int_0^t \Big(\theta_s(z\Sigma(s,T_n) + (1-z)\Sigma(s,t)) - \theta_s(\Sigma(s,t)) \Big)\,\mathrm{d}s.$$

PROOF: Since g is a strictly increasing and continuous function taking negative as well as positive values, it has a unique zero. Consequently, there is a real number Z such that

$$(g(x))^+ = \mathbb{1}_{[Z,\infty)}(x) g(x).$$

Define $v(x) := (g(-x))^+$, then (2.24) implies

$$C_0 = B(0,t)(v * \varphi)(0).$$

First, we calculate the bilateral Laplace transform of v:

$$\begin{aligned} L[v](z) &:= \int_{\mathbb{R}} e^{-zx} \left(\sum_{i=1}^{n} D_i e^{-B_i x} - 1 \right)^+ \mathrm{d}x \\ &= \int_{-\infty}^{-Z} e^{-zx} \left(\sum_{i=1}^{n} D_i e^{-B_i x} - 1 \right) \mathrm{d}x \\ &= \sum_{i=1}^{n} \left(D_i \int_{-\infty}^{-Z} e^{-zx} e^{-B_i x}\,\mathrm{d}x \right) - \int_{-\infty}^{-Z} e^{-zx}\,\mathrm{d}x. \end{aligned}$$

For any constant $0 \le C \le 1$ and any $z \in \mathbb{C}$ with $\Re z < -C$

$$\begin{aligned} \int_{-\infty}^{-Z} e^{-(C+z)x}\,\mathrm{d}x &= \int_{-\infty}^{0} e^{-(C+z)(x-Z)}\,\mathrm{d}x \\ &= e^{(C+z)Z} \int_0^1 t^{-(C+z)} \frac{1}{t}\,\mathrm{d}t \\ &= e^{(C+z)Z} B(-C-z, 1) \\ &= e^{(C+z)Z} \frac{\Gamma(-C-z)\Gamma(1)}{\Gamma(-C-z+1)} \\ &= e^{(C+z)Z} \frac{-1}{C+z}. \end{aligned}$$

$B(\cdot,\cdot)$ and $\Gamma(\cdot)$ denote the Euler Beta and Gamma function respectively. For some comments on the chain of equalities we refer to Raible (2000, p. 66) where the bilateral Laplace transform of a similar function is derived. It can be concluded that the Laplace transform of v exists for $z \in \mathbb{C}$ with $\Re z < -1$ and that it is given by

$$L[v](z) = e^{zZ}\left(\sum_{i=1}^{n}\left(D_i e^{B_i Z}\frac{-1}{B_i + z}\right) + \frac{1}{z}\right).$$

Again, we apply theorem 2.6 to the functions $F_1(x) := v(x)$ and $F_2(x) := \varphi(x)$ and, proceeding similar to the proof of theorem 2.9, we obtain

$$C_0 = \frac{1}{2\pi}B(0,t)\lim_{Y\to\infty}\int_{-Y}^{Y} L[v](R+\mathrm{i}u)M_t^X(-R-\mathrm{i}u)\,\mathrm{d}u,$$

if this limit exists. The next lemma shows that it exists, although the integral does not converge absolutely as it does in formulae (2.20) and (2.22). The expression for M_t^X can be derived as in lemma 2.10. □

Lemma 2.14 *For any $C < 0$ and under the assumptions of theorem 2.13 the limit*

$$\lim_{Y\to\infty}\int_{-Y}^{Y} e^{\mathrm{i}uZ}\frac{1}{C+\mathrm{i}u}M_t^X(-R-\mathrm{i}u)\,\mathrm{d}u$$

exists.

Proof: As before, let φ denote the Lebesgue-density of $\mathbb{P}_t^X$. For $Y \in \mathbb{R}_+$ Fubini's theorem yields

$$\begin{aligned}
I(Y) &:= \int_{-Y}^{Y} e^{\mathrm{i}uZ}\frac{1}{C+\mathrm{i}u}M_t^X(-R-\mathrm{i}u)\,\mathrm{d}u \\
&= \int_{\mathbb{R}} 1\!\!1_{\{|u|\le Y\}}e^{\mathrm{i}uZ}\frac{1}{C+\mathrm{i}u}\left(\int_{\mathbb{R}} e^{-(R+\mathrm{i}u)x}\varphi(x)\,\mathrm{d}x\right)\mathrm{d}u \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} 1\!\!1_{\{|u|\le Y\}}e^{\mathrm{i}u(Z-x)}\frac{C-\mathrm{i}u}{C^2+u^2}\,\mathrm{d}u\right)e^{-Rx}\varphi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} 2J(x,Y)e^{-Rx}\varphi(x)\,\mathrm{d}x,
\end{aligned}$$

with

$$\begin{aligned}
J(x,Y) &:= \int_0^Y \Re\left(e^{\mathrm{i}u(Z-x)}\frac{C-\mathrm{i}u}{C^2+u^2}\right)\mathrm{d}u \\
&= \int_0^Y \frac{C}{C^2+u^2}\cos(u(Z-x))\,\mathrm{d}u \\
&\qquad + \int_0^Y \frac{1}{u}\sin(u(Z-x))\,\mathrm{d}u - \int_0^Y \frac{C^2}{uC^2+u^3}\sin(u(Z-x))\,\mathrm{d}u \\
&=: J^1(x,Y) + J^2(x,Y) - J^3(x,Y).
\end{aligned}$$

If we can show that $J(x,Y)$ is bounded by a constant that does not depend on x or Y and that $\lim_{Y\to\infty} J(x,Y)$ exists for all x, this will imply the existence of $\lim_{Y\to\infty} I(Y)$ (remember that $M_t^X(-R) = \int_{\mathbb{R}} e^{-Rx}\varphi(x)\,\mathrm{d}x < \infty$ by assumption). It is clear that J^1 and J^3 have the two desired properties. Now let us have a look at J^2:

$$J^2(x,Y) = \left\{\begin{array}{ll} 0 & \text{for } x = Z \\ \int_0^{|Z-x|Y} \frac{1}{t}\sin t\,\mathrm{d}t & \text{for } x \neq Z \end{array}\right\} = \mathrm{Si}(|Z-x|Y),$$

where Si denotes the sine integral. From the properties of the sine integral we can conclude that $J^2(x,Y)$ is bounded by a constant that does not depend on x or Y and that $\lim_{Y\to\infty} J^2(x,Y)$ exists. $\square$

In a similar manner as for the call we can derive the price P_0 of a put with strike 1 and maturity t on the coupon bond:

Corollary 2.15 *Suppose the distribution of X possesses a Lebesgue-density. Choose an $R > 0$ such that $M_t^X(-R) < \infty$ and let Z be the unique zero of the strictly decreasing and continuous function*

$$g(x) := 1 - \sum_{i=1}^n D_i e^{B_i x}.$$

Then we have

$$P_0 = \frac{1}{2\pi} B(0,t) \lim_{Y\to\infty} \int_{-Y}^{Y} L[v](R+\mathrm{i}u) M_t^X(-R-\mathrm{i}u)\,\mathrm{d}u, \tag{2.26}$$

where $L[v]$ denotes the bilateral Laplace transform of $v : \mathbb{R} \to \mathbb{R}$ defined by $v(x) := (g(-x))^+$. Moreover,

$$L[v](R+\mathrm{i}u) = e^{(R+\mathrm{i}u)Z}\left(\sum_{i=1}^n \left(D_i e^{B_i Z}\frac{-1}{B_i+R+\mathrm{i}u}\right) + \frac{1}{R+\mathrm{i}u}\right) \qquad (u \in \mathbb{R})$$

and for $z \in \mathbb{C}$ with $\Re z = -R$

$$M_t^X(z) = \exp \int_0^t \Big(\theta_s(z\Sigma(s,T_n) + (1-z)\Sigma(s,t)) - \theta_s(\Sigma(s,t))\Big)\,\mathrm{d}s.$$

REMARK: We can observe a similarity to the pricing formulae for calls and puts on zero coupon bonds. The formulae for the call and the respective put on a coupon bearing bond coincide. Different are again only the permitted values for R.

2.5 Valuation of floating range notes

Turnbull (1995) as well as Navatte and Quittard-Pinon (1999) derived explicit pricing formulae for floating range notes in a one-factor Gaussian Heath–Jarrow–Morton (HJM) model. Nunes (2004) extended their results to a multifactor Gaussian HJM framework. The main aim of this section is to generalize their results by providing explicit valuation formulae for floating range notes in the Lévy term structure model.

Range notes are structured products, convenient for investors with a strong belief that interest rates will stay within a certain corridor. They provide interest payments which are proportional to the time in which a reference index rate (most commonly the Libor rate) lies inside that range. In return for the drawback that no interest will be paid for the time the corridor is left, they offer higher rates than comparable standard products, like e.g. floating rate notes. *Floating range notes* pay coupon rates which are linked to some reference index rate (e.g. 3-month Libor plus 100 basis points) whereas the coupon rates of *fixed range notes* are specified in advance. Let us stress that coupon payments of both products depend on the path of the reference index rate.

Turnbull (1995) provided an explicit valuation formula for floating range notes in the one-factor Gaussian HJM framework. Using the same model and the change-of-numeraire technique developed by Geman, El Karoui, and Rochet (1995), Navatte and Quittard-Pinon (1999) derived a pricing solution in a more intuitive way. For this purpose, they introduced *double delayed digital options.* The value of each floating range note coupon is shown to be equal to the value of a portfolio of those options plus some additional term. This extra term only involves the cumulative density function of a standard normal distribution. Nunes (2004) managed to generalize the former results to a multifactor Gaussian HJM model. His valuation formula for floating range notes looks very similar, i.e. each coupon is written as a portfolio of *delayed digital options* plus some extra term. This extra term, although given in closed form, is quite complicated and stems from evaluating the joint probability law of two random variables.

One purpose of this section is to show that the calculation of the joint probability distribution can be circumvented by changing the probability measure in a suitable way. Concretely, we will use the adjusted forward measure that has been introduced in section 2.2. Proceeding this way, a much simpler pricing formula can be obtained in the multifactor Gaussian HJM model (see theorem 2.21). However, our main goal is to price range notes in the more general framework of the Lévy term structure model. Once again, we use the change-of-numeraire technique as well as a Laplace transform method. As a necessary tool for pricing range notes (and a nice side result), we begin by deriving a valuation formula for digital options.

2.5.1 Digital Options

In this section, we discuss the pricing of interest rate digital options. For convenience we adopt the notation of Nunes (2004).

A *standard European interest rate digital call (put)* with strike rate r_k is a financial security that pays an amount of one currency unit to its owner if and only if the simply compounded interest rate for the period $[T, T+\delta]$ lies above (below) r_k at maturity T of the option. More precisely, the time-T value of this option is given by

$$SD(\Theta)_T[r_n(T, T+\delta); r_k; T] := \mathbb{1}_{\{\Theta r_n(T,T+\delta) > \Theta r_k\}}$$

with

$$r_n(T, T+\delta) := \frac{1}{\delta}\left[\frac{1}{B(T, T+\delta)} - 1\right], \tag{2.27}$$

where $\Theta = 1$ for a digital call and $\Theta = -1$ for a digital put.

In accordance with Navatte and Quittard-Pinon (1999) and Nunes (2004) we call an interest rate digital option *delayed* if option maturity T and payment date T_1 differ ($T_1 > T$). The time-T_1 price of a *delayed digital option* is given by

$$DD(\Theta)_{T_1}[r_n(T, T+\delta); r_k; T_1] := \mathbb{1}_{\{\Theta r_n(T,T+\delta) > \Theta r_k\}},$$

where again $\Theta = 1$ for a delayed digital call and $\Theta = -1$ for a delayed digital put. Since a standard digital option is a special case of a delayed digital option ($T_1 = T$), we will only consider the latter in the following.

Delayed range digital options provide a terminal payoff equal to 1 paid at T_1 if and only if at option maturity T ($T \leq T_1$) the underlying interest rate lies inside a prespecified corridor. Consequently, the time-T_1 price of a delayed range digital option is

$$DRD_{T_1}[r_n(T, T+\delta); r_l; r_u; T_1] := \mathbb{1}_{\{r_n(T,T+\delta) \in [r_l, r_u]\}}.$$

By arbitrage arguments, the time-t prices ($t \in [0, T_1]$) of delayed digital calls, puts, and range options are related via

$$\begin{aligned} DRD_t[r_n(T, T+\delta); r_l; r_u; T_1] &= B(t, T_1) - DD(1)_t[r_n(T, T+\delta); r_u; T_1] \\ &\quad - DD(-1)_t[r_n(T, T+\delta); r_l; T_1]. \end{aligned}$$

Unfortunately, a call-put parity like

$$DD(1)_t[r_n(T, T+\delta); r_k; T_1] = B(t, T_1) - DD(-1)_t[r_n(T, T+\delta); r_k; T_1] \tag{2.28}$$

does not hold for all $t \in [0, T_1]$ (note that in case $r_n(T, T+\delta) = r_k$ equality fails for $t \in [T, T_1]$). However, equation (2.28) holds for $t < T$ in models in which the distribution of $B(T, T+\delta)$ does not have point masses (like e.g. in the Gaussian HJM model with a reasonable volatility structure). If L is a Poisson process, equation (2.28) might fail for $t < T$ though. The technique that we are going to present for option valuation only works for model specifications that do not produce point masses in the distribution of $B(T, T+\delta)$ (see proposition 2.8 and the discussion preceding it). In these cases, we have the call-put parity (2.28) for $t < T$ and can thus price any of the mentioned digital options if we are able to price a delayed digital call.

We calculate the value of the call by taking the $\mathbb{P}$-conditional expectation of its discounted payoff, i.e.

$$\begin{aligned}
D_t &:= DD(1)_t[r_n(T, T+\delta); r_k; T_1] \\
&= B_t \mathbb{E}\left[\frac{1}{B_{T_1}} \mathbb{1}_{\{r_n(T,T+\delta) > r_k\}} \middle| \mathcal{F}_t\right] \\
&= B(t, T_1) \mathbb{E}_{\mathbb{P}_{T_1}}\left[\mathbb{1}_{\{r_n(T,T+\delta) > r_k\}} \middle| \mathcal{F}_t\right] \\
&\overset{(2.27)}{=} B(t, T_1) \mathbb{E}_{\mathbb{P}_{T_1}}\left[\mathbb{1}_{\left\{B(T,T+\delta) < \frac{1}{\delta r_k + 1}\right\}} \middle| \mathcal{F}_t\right] \\
&\underset{(2.5)}{\overset{(2.2)}{=}} B(t, T_1) \\
&\quad \times \mathbb{E}_{\mathbb{P}_{T_1}}\left[\mathbb{1}_{\left\{\frac{B(t,T+\delta)}{B(t,T)} \exp\left[-\int_t^T A(s,T,T+\delta)\,\mathrm{d}s + \int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s\right] < \frac{1}{\delta r_k + 1}\right\}} \middle| \mathcal{F}_t\right].
\end{aligned}$$

For the change of numeraire we used equations (2.11)–(2.12) and the abstract Bayes formula. By independence of the increments of L and since $\frac{B(t,T+\delta)}{B(t,T)}$ is $\mathcal{F}_t$-measurable, we get (compare Musiela and Rutkowski (1998, lemma A.0.1.(v)))

$$D_t = B(t, T_1) h\left(\frac{B(t, T+\delta)}{B(t, T)}\right) \tag{2.29}$$

with $h : \mathbb{R} \to [0, 1]$ given by

$$h(y) := \mathbb{E}_{\mathbb{P}_{T_1}}\left[\mathbb{1}_{\left\{y \exp\left[-\int_t^T A(s,T,T+\delta)\,\mathrm{d}s + \int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s\right] < \frac{1}{\delta r_k + 1}\right\}}\right].$$

To calculate $h(y)$ for $y > 0$, observe that

$$h(y) = \int_\Omega \mathbb{1}_{\left\{e^X < \frac{K}{y}\right\}} \,\mathrm{d}\mathbb{P}_{T_1} = \int_{\mathbb{R}} \mathbb{1}_{\left\{e^x < \frac{K}{y}\right\}} \,\mathrm{d}\mathbb{P}_{T_1}^X(x), \tag{2.30}$$

where

$$\begin{aligned}
X &:= \int_t^T \Sigma(s, T, T+\delta)\,\mathrm{d}L_s, \\
K &:= \frac{1}{\delta r_k + 1} \exp \int_t^T A(s, T, T+\delta)\,\mathrm{d}s, \qquad (2.31)
\end{aligned}$$

and $\mathbb{P}^X_{T_1}$ denotes the distribution of X under $\mathbb{P}_{T_1}$. If this distribution possesses a Lebesgue-density φ in $\mathbb{R}$ then

$$\begin{aligned} h(y) &= \int_{\mathbb{R}} 1\!\!1_{\left\{e^x < \frac{K}{y}\right\}} \varphi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f_y(-x)\varphi(x)\,\mathrm{d}x \\ &= (f_y * \varphi)(0) = V(0) \end{aligned} \tag{2.32}$$

with $f_y(x) := 1\!\!1_{\left\{e^{-x} < \frac{K}{y}\right\}}(x)$ and $V(\xi) := (f_y * \varphi)(\xi)$. Denote by $M^X_{T_1}$ the moment generating function of the random variable X with respect to the measure $\mathbb{P}_{T_1}$. The next theorem gives an analytic expression for the price of the call that can numerically be evaluated fast:

Theorem 2.16 *Suppose the distribution of X possesses a Lebesgue-density. Choose an $R > 0$ such that $M^X_{T_1}(-R) < \infty$. Then*

$$D_t = \frac{1}{\pi} B(t, T_1) \int_0^\infty \Re\left(\left(\frac{B(t,T)}{B(t,T+\delta)} K \right)^{R+\mathrm{i}u} \frac{1}{R+\mathrm{i}u} M^X_{T_1}(-R-\mathrm{i}u) \right) \mathrm{d}u$$

with

$$K := \frac{1}{\delta r_k + 1} \exp \int_t^T \Big(\theta_s(\Sigma(s, T+\delta)) - \theta_s(\Sigma(s,T)) \Big)\, \mathrm{d}s.$$

Remark: It is always possible to choose an R that satisfies the prerequisites of the theorem (compare lemma 2.10). The particular choice of R does not – of course – have an impact on the option price, but it has influence on the speed at which the integral can be evaluated numerically (see Raible (2000, Section 3.7)).

Proof: We use the convolution representation (2.32) and apply theorem 2.6 to the functions $F_1(x) := f_y(x)$ and $F_2(x) := \varphi(x)$, that is we express the bilateral Laplace transform of their convolution as the product of the bilateral Laplace transforms of the convolution factors. The prerequisites of the theorem are satisfied since $x \mapsto e^{-Rx} f_y(x)$ is bounded,

$$\int_{\mathbb{R}} e^{-Rx} |f_y(x)|\,\mathrm{d}x = \frac{1}{R}\left(\frac{K}{y}\right)^R < \infty,$$

and

$$\int_{\mathbb{R}} e^{-Rx} |\varphi(x)|\,\mathrm{d}x = M^X_{T_1}(-R) < \infty$$

by assumption. The cited theorem together with (2.32) yields

$$L[V](R+\mathrm{i}u) = L[f_y](R+\mathrm{i}u) L[\varphi](R+\mathrm{i}u) \qquad (u \in \mathbb{R})$$

where $L[V]$ denotes the bilateral Laplace transform of V (analogously for f_y and φ). The theorem also yields that $\xi \mapsto V(\xi)$ is continuous and that $\int_{\mathbb{R}} e^{-R\xi} V(\xi)\,\mathrm{d}\xi$ is absolutely convergent. Therefore, applying theorem 2.7 and proceeding as in the proof of theorem 2.9 we obtain

$$V(0) = \frac{1}{2\pi} \lim_{Y\to\infty} \int_{-Y}^{Y} L[f_y](R+\mathrm{i}u) L[\varphi](R+\mathrm{i}u)\,\mathrm{d}u,$$

if this limit exists. Note that the integrand evaluated at u equals the complex conjugate of the integrand evaluated at $-u$. Therefore, using the relationship $z + \bar{z} = 2\,\Re(z)$ one arrives at

$$V(0) = \frac{1}{\pi} \lim_{Y\to\infty} \int_{0}^{Y} \Re\Big(L[f_y](R+\mathrm{i}u) L[\varphi](R+\mathrm{i}u) \Big)\,\mathrm{d}u.$$

We have

$$L[\varphi](R+\mathrm{i}u) = M_{T_1}^X(-R-\mathrm{i}u)$$

and, since $R > 0$, one obtains

$$L[f_y](R+\mathrm{i}u) = \frac{1}{(R+\mathrm{i}u)} \left(\frac{K}{y}\right)^{R+\mathrm{i}u}$$

and (after some calculations) concludes that the above limit exists. Plugging in the expressions from (2.29), (2.31), and (2.32) as well as remembering the drift condition, i.e. $A(s,T,T+\delta) = \theta_s(\Sigma(s,T+\delta)) - \theta_s(\Sigma(s,T))$, yields the claim. $\square$

Theorem 2.17 *Under the assumptions of theorem 2.16 we have an explicit expression for $M_{T_1}^X$, namely for $u \in \mathbb{R}$*

$$M_{T_1}^X(-R-\mathrm{i}u) = \exp \int_{t}^{T} \Big[\theta_s(g_s(-R-\mathrm{i}u)) - \theta_s(g_s(0))\Big]\,\mathrm{d}s \qquad (2.33)$$

with $g_s(z) := z\Sigma(s,T,T+\delta) + \Sigma(s,T_1)$.

Proof: To obtain the expression for the moment generating function of X we use equation (2.11), the independence of the increments of L, the fact that we have $\mathbb{E}[\exp \int_t^T \Sigma(s,T)\,\mathrm{d}L_s] = \exp \int_t^T A(s,T)\,\mathrm{d}s$, equation (2.8), and proposition

1.9 (in this order) and get for $z \in \mathbb{C}$ with $\Re(z) = -R$

$$\begin{aligned}
M_{T_1}^X(z) &= \mathbb{E}_{\mathbb{P}_{T_1}}\left[\exp\left(z\int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s\right)\right]\\
&= \exp\left(-\int_0^{T_1} A(s,T_1)\,\mathrm{d}s\right)\\
&\quad\times \mathbb{E}\left[\exp\left(z\int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s + \int_0^{T_1}\Sigma(s,T_1)\,\mathrm{d}L_s\right)\right]\\
&= \exp\left(-\int_t^T A(s,T_1)\,\mathrm{d}s\right)\\
&\quad\times \mathbb{E}\left[\exp\left(\int_t^T \Big(z\Sigma(s,T,T+\delta)+\Sigma(s,T_1)\Big)\,\mathrm{d}L_s\right)\right]\\
&= \exp\left(-\int_t^T \theta_s(g_s(0))\,\mathrm{d}s\right)\exp\left(\int_t^T \theta_s(g_s(z))\,\mathrm{d}s\right)
\end{aligned}$$

with $g_s(z) := z\Sigma(s,T,T+\delta)+\Sigma(s,T_1)$. Now (2.33) follows. □

Let us consider the multifactor Gaussian HJM model as a special case, i.e. L is a d-dimensional standard Brownian motion under $\mathbb{P}$. Then $\theta(x) = \frac{\langle x,x\rangle}{2}$ for $x \in \mathbb{C}^d$. From (2.33) and using (2.6) we get for $z \in \mathbb{C}$

$$\begin{aligned}
M_{T_1}^X(z) &= \exp\int_t^T\left(\frac{\langle z\Sigma(s,T,T+\delta)+\Sigma(s,T_1), z\Sigma(s,T,T+\delta)+\Sigma(s,T_1)\rangle}{2}\right.\\
&\qquad\left.-\frac{\langle\Sigma(s,T_1),\Sigma(s,T_1)\rangle}{2}\right)\mathrm{d}s\\
&= \exp\left(\frac{z^2}{2}\int_t^T ||\Sigma(s,T+\delta)-\Sigma(s,T)||^2\,\mathrm{d}s\right.\\
&\qquad\left.+z\int_t^T\langle\Sigma(s,T+\delta)-\Sigma(s,T),\Sigma(s,T_1)\rangle\,\mathrm{d}s\right).
\end{aligned}$$

Consequently, X is normally distributed under $\mathbb{P}_{T_1}$ with mean

$$m(t,T,T+\delta,T_1) := \int_t^T\langle\Sigma(s,T+\delta)-\Sigma(s,T),\Sigma(s,T_1)\rangle\,\mathrm{d}s$$

and variance

$$g(t,T,T+\delta) := \int_t^T ||\Sigma(s,T+\delta) - \Sigma(s,T)||^2 \,\mathrm{d}s.$$

From (2.30) we get

$$h(y) = \int_{-\infty}^{\log \frac{K}{y}} \mathrm{d}\mathbb{P}_{T_1}^X(x) = \mathbb{P}_{T_1}\left(X \le \log \frac{K}{y}\right) = \Phi\left(\frac{\log \frac{K}{y} - m(t,T,T+\delta,T_1)}{\sqrt{g(t,T,T+\delta)}}\right),$$

where Φ denotes the cumulative density function of a standard normal distribution. Plugging in the expression for K from (2.31) and using (2.29) we arrive at

$$D_t = B(t,T_1)\Phi\left(\frac{\log \frac{B(t,T)}{B(t,T+\delta)(\delta r_k+1)} + \frac{1}{2}g(t,T,T+\delta) - l(t,T,T+\delta,T_1)}{\sqrt{g(t,T,T+\delta)}}\right),$$

where

$$l(t,T,T+\delta,T_1) := \int_t^T \langle \Sigma(s,T+\delta) - \Sigma(s,T), \Sigma(s,T_1) - \Sigma(s,T)\rangle \,\mathrm{d}s.$$

This formula coincides with the one derived in Nunes (2004, Proposition 3.3). Note that for a standard digital call ($T_1 = T$) one gets $l(t,T,T+\delta,T_1) = 0$.

2.5.2 Range Notes

The purpose of this section is to derive a formula for pricing range notes in the Lévy term structure model. As a special case, we will also consider the multifactor Gaussian HJM model and obtain a pricing formula that is simpler than the one provided by Nunes (2004). Once again, his notation is adopted.

In the following, we put ourselves at time t, the valuation date of the range note. Consider a bond with bullet redemption having had its previous coupon payment date at T_0 ($\le t$) and having its N future coupons paid at times T_{j+1} ($j = 0, \ldots, N-1$). Based on some day count convention, let n_j (δ_j) denote the number of days (years) between the times T_j and T_{j+1}. For the current period we split up n_0 into the sum of n_0^- and n_0^+, representing the number of days between T_0 and t and between t and T_1 respectively. Furthermore, denote by $T_{j,i}$ the date that corresponds to i days after T_j and by $\delta_{j,i}$ the length (in years) of the compounding period starting at time $T_{j,i}$.

For the multifactor Gaussian HJM model, Nunes (2004) shows that the value of a fixed range note equals the value of a portfolio of delayed range digital options. Although the Lévy term structure model is more general, the same arguments apply since they do not depend on the driving process. We refer the reader to Nunes (2004, Proposition 4.1) and concentrate on floating range notes in what follows.

To value floating range notes we will first switch from the spot measure to a suitable forward measure. Afterwards, another change of measure from the forward measure to an adjusted forward measure will be performed. Proceeding this way, we will not have to deal with a joint probability distribution of two random variables.

We cite the following definition from Nunes (2004, Definition 4.2):

Definition 2.18 *For a floating range note, the value of the $(j+1)^{th}$ coupon, at time T_{j+1}, is equal to*

$$\nu_{j+1}(T_{j+1}) := \frac{r_n(T_j, T_j + \delta_j) + s_j}{D_j} H(T_j, T_{j+1}),$$

where s_j represents the spread over the reference interest rate paid by the bond during the $(j+1)^{th}$ compounding period, D_j is the number of days in a year for the $(j+1)^{th}$ compounding period, and

$$H(T_j, T_{j+1}) := \sum_{i=1}^{n_j} \mathbb{1}_{\{r_l(T_{j,i}) \le r_n(T_{j,i}, T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}}$$

denotes the number of days, in the $(j+1)^{th}$ compounding period, that the reference interest rate lies inside a prespecified range, which is equal to $[r_l(T_{j,i}), r_u(T_{j,i})]$ for the i^{th} day of the $(j+1)^{th}$ compounding period.

Consequently, the time-t value of the floating range note is given by

$$FlRN(t) := B(t, T_N) + \sum_{j=0}^{N-1} \nu_{j+1}(t)$$

where $B(t, T_N)$ corresponds to the discounted value of the final payment of 1.

For the valuation of the first coupon we follow Nunes (2004) and get, since $r_n(T_0, T_0 + \delta_0)$ is already known at time t or, mathematically speaking, measurable with respect to $\mathcal{F}_t$,

$$\begin{aligned}
\nu_1(t) &= B_t \mathbb{E}\left[\frac{1}{B_{T_1}} \frac{r_n(T_0, T_0+\delta_0) + s_0}{D_0} H(T_0, T_1) \middle| \mathcal{F}_t\right] \\
&= \frac{r_n(T_0, T_0+\delta_0) + s_0}{D_0} B(t, T_1) \mathbb{E}_{\mathbb{P}_{T_1}}\left[H(T_0, T_1) \middle| \mathcal{F}_t\right] \\
&= \frac{r_n(T_0, T_0+\delta_0) + s_0}{D_0} \Bigg(B(t, T_1) H(T_0, t) \\
&\quad + \sum_{i=n_0^-+1}^{n_0} B(t, T_1) \mathbb{E}_{\mathbb{P}_{T_1}}\left[\mathbb{1}_{\{r_l(T_{0,i}) \le r_n(T_{0,i}, T_{0,i}+\delta_{0,i}) \le r_u(T_{0,i})\}} \middle| \mathcal{F}_t\right] \Bigg) \\
&= \frac{r_n(T_0, T_0+\delta_0) + s_0}{D_0} \Bigg(B(t, T_1) H(T_0, t) \\
&\quad + \sum_{i=n_0^-+1}^{n_0} DRD_t\left[r_n(T_{0,i}, T_{0,i}+\delta_{0,i}); r_l(T_{0,i}); r_u(T_{0,i}); T_1\right] \Bigg)
\end{aligned}$$

with

$$H(T_0,t) := \sum_{i=1}^{n_0^-} \mathbb{1}_{\{r_l(T_{0,i}) \le r_n(T_{0,i},T_{0,i}+\delta_{0,i}) \le r_u(T_{0,i})\}}.$$

For the subsequent coupons, we get

$$\begin{aligned}
&\nu_{j+1}(t) \\
&= B_t \mathbb{E}\left[\frac{1}{B_{T_{j+1}}} \frac{r_n(T_j,T_{j+1})+s_j}{D_j} H(T_j,T_{j+1}) \Big| \mathcal{F}_t\right] \\
&= B(t,T_{j+1}) \mathbb{E}_{\mathbb{P}_{T_{j+1}}}\left[\frac{r_n(T_j,T_{j+1})+s_j}{D_j} H(T_j,T_{j+1}) \Big| \mathcal{F}_t\right] \\
&\overset{(2.27)}{=} \left(\frac{s_j}{D_j} - \frac{1}{\delta_j D_j}\right) B(t,T_{j+1}) \sum_{i=1}^{n_j} \mathbb{E}_{\mathbb{P}_{T_{j+1}}}\left[\mathbb{1}_{\{r_l(T_{j,i}) \le r_n(T_{j,i},T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&\quad + \frac{B(t,T_{j+1})}{\delta_j D_j} \sum_{i=1}^{n_j} \mathbb{E}_{\mathbb{P}_{T_{j+1}}}\left[\frac{1}{B(T_j,T_{j+1})} \mathbb{1}_{\{r_l(T_{j,i}) \le r_n(T_{j,i},T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&=: \nu_{j+1}^1(t) + \nu_{j+1}^2(t).
\end{aligned}$$

To evaluate $\nu_{j+1}^1(t)$ we proceed as before and get

$$\nu_{j+1}^1(t) = \left(\frac{s_j}{D_j} - \frac{1}{\delta_j D_j}\right) \sum_{i=1}^{n_j} DRD_t[r_n(T_{j,i},T_{j,i}+\delta_{j,i}); r_l(T_{j,i}); r_u(T_{j,i}); T_{j+1}].$$

For the evaluation of $\nu_{j+1}^2(t)$ we switch from the forward measure $\mathbb{P}_{T_{j+1}}$ to the adjusted forward measure $\mathbb{P}_{T_j,T_{j+1}}$. This procedure has the advantage that we do not have to deal with the joint distribution of the two random variables $B(T_j,T_j+\delta_j)$ and $r_n(T_{j,i},T_{j,i}+\delta_{j,i})$. Using the abstract Bayes formula together with (2.13)–(2.14) and denoting by $\mathbb{E}_{T_j,T_{j+1}}$ the expectation with respect to $\mathbb{P}_{T_j,T_{j+1}}$ yields

$$\begin{aligned}
\nu_{j+1}^2(t) &= \sum_{i=1}^{n_j} \frac{B(t,T_{j+1})}{\delta_j D_j} \\
&\qquad \times \mathbb{E}_{\mathbb{P}_{T_{j+1}}}\left[\frac{1}{B(T_j,T_{j+1})} \mathbb{1}_{\{r_l(T_{j,i}) \le r_n(T_{j,i},T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{n_j} \frac{B(t,T_{j+1})}{\delta_j D_j} \frac{B(0,T_j)}{B(0,T_{j+1})} \\
&\qquad \times \mathbb{E}_{\mathbb{P}_{T_{j+1}}}\left[\frac{F(T_j,T_j,T_{j+1})}{F(0,T_j,T_{j+1})} \mathbb{1}_{\{r_l(T_{j,i}) \le r_n(T_{j,i},T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{n_j} \frac{B(t,T_{j+1})}{\delta_j D_j} \frac{B(0,T_j)}{B(0,T_{j+1})} \frac{B(0,T_{j+1})B(t,T_j)}{B(0,T_j)B(t,T_{j+1})} \\
&\qquad \times \mathbb{E}_{T_j,T_{j+1}}\left[\mathbb{1}_{\{r_l(T_{j,i}) \le r_n(T_{j,i},T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{n_j} \frac{B(t,T_j)}{\delta_j D_j} \mathbb{E}_{T_j,T_{j+1}}\left[\mathbb{1}_{\{r_l(T_{j,i}) \le r_n(T_{j,i},T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right].
\end{aligned}$$

The summands on the right hand side look (except for a multiplicative constant) very similar to the time-t value of a range digital option, the only difference being that the expectation is taken under the adjusted forward measure. We can proceed in the same way as we did for digital options and use the independence of the increments of L to obtain

$$\nu_{j+1}^2(t) = \frac{B(t,T_j)}{\delta_j D_j} \sum_{i=1}^{n_j} D_t^{j,i}.$$

Here,

$$\begin{aligned}
D_t^{j,i} &\overset{(2.27)}{=} \mathbb{E}_{T_j,T_{j+1}}\left[\mathbb{1}_{\left\{\frac{1}{\delta_{j,i} r_u(T_{j,i})+1} \leq B(T_{j,i},T_{j,i}+\delta_{j,i}) \leq \frac{1}{\delta_{j,i} r_l(T_{j,i})+1}\right\}} \big| \mathcal{F}_t \right] \\
&\overset{(2.9)}{=} \mathbb{E}_{T_j,T_{j+1}}\left[\mathbb{1}_{\left\{\underline{K}^{j,i} \leq \frac{B(t,T_{j,i}+\delta_{j,i})}{B(t,T_{j,i})} \exp(X^{j,i}) \leq \overline{K}^{j,i}\right\}} \big| \mathcal{F}_t \right] \\
&= h^{j,i}\left(\frac{B(t,T_{j,i}+\delta_{j,i})}{B(t,T_{j,i})} \right) \qquad (2.34)
\end{aligned}$$

with $h^{j,i} : \mathbb{R}^+ \to [0,1]$ given by

$$h^{j,i}(y) = \int_{\mathbb{R}} \mathbb{1}_{\left\{\frac{1}{y}\underline{K}^{j,i} \leq e^x \leq \frac{1}{y}\overline{K}^{j,i}\right\}} \, \mathrm{d}\mathbb{P}_{T_j,T_{j+1}}^{X^{j,i}}(x), \qquad (2.35)$$

where

$$\begin{aligned}
X^{j,i} &:= \int_t^{T_{j,i}} \Sigma(s,T_{j,i},T_{j,i}+\delta_{j,i}) \, \mathrm{d}L_s, \\
\overline{K}^{j,i} &:= \frac{1}{\delta_{j,i} r_l(T_{j,i})+1} \exp \int_t^{T_{j,i}} \Big(\theta_s(\Sigma(s,T_{j,i}+\delta_{j,i})) - \theta_s(\Sigma(s,T_{j,i})) \Big) \, \mathrm{d}s, \\
\underline{K}^{j,i} &:= \frac{1}{\delta_{j,i} r_u(T_{j,i})+1} \exp \int_t^{T_{j,i}} \Big(\theta_s(\Sigma(s,T_{j,i}+\delta_{j,i})) - \theta_s(\Sigma(s,T_{j,i})) \Big) \, \mathrm{d}s,
\end{aligned}$$

and $\mathbb{P}_{T_j,T_{j+1}}^{X^{j,i}}$ denotes the distribution of $X^{j,i}$ with respect to $\mathbb{P}_{T_j,T_{j+1}}$.

To improve readability, let us simplify notation and fix j and i. In what follows, we omit the sub– and superscripts j,i and write T, δ, D_t, h, X, $\overline{K}$ and $\underline{K}$ for short. Denote by $M_{T_j,T_{j+1}}^X$ the moment generating function of the random variable X with respect to $\mathbb{P}_{T_j,T_{j+1}}$. Then we have the following pricing formula for D_t, which immediately gives us a formula for the value of the floating range note:

Theorem 2.19 *Suppose the distribution of X possesses a Lebesgue-density. Choose an $R>0$ such that $M^X_{T_j,T_{j+1}}(-R)<\infty$. Then*

$$D_t = \frac{1}{\pi}\int_0^\infty \Re\left(\left(\frac{B(t,T)}{B(t,T+\delta)}\overline{K}\right)^{R+\mathrm{i}u}\frac{1}{R+\mathrm{i}u}M^X_{T_j,T_{j+1}}(-R-\mathrm{i}u)\right)\mathrm{d}u$$
$$-\frac{1}{\pi}\int_0^\infty \Re\left(\left(\frac{B(t,T)}{B(t,T+\delta)}\underline{K}\right)^{R+\mathrm{i}u}\frac{1}{R+\mathrm{i}u}M^X_{T_j,T_{j+1}}(-R-\mathrm{i}u)\right)\mathrm{d}u$$

with

$$\overline{K} := \frac{1}{\delta r_l(T)+1}\exp\int_t^T\Big(\theta_s(\Sigma(s,T+\delta))-\theta_s(\Sigma(s,T))\Big)\,\mathrm{d}s, \tag{2.36}$$
$$\underline{K} := \frac{1}{\delta r_u(T)+1}\exp\int_t^T\Big(\theta_s(\Sigma(s,T+\delta))-\theta_s(\Sigma(s,T))\Big)\,\mathrm{d}s. \tag{2.37}$$

Proof: Observe that

$$h(y) = \int \mathbb{1}_{\left\{e^x\le\frac{\overline{K}}{y}\right\}}\,\mathrm{d}\mathbb{P}^X_{T_j,T_{j+1}}(x) - \int \mathbb{1}_{\left\{e^x<\frac{\underline{K}}{y}\right\}}\,\mathrm{d}\mathbb{P}^X_{T_j,T_{j+1}}(x).$$

Applying exactly the same arguments as in the proof of theorem 2.16 yields the claim. The only difference is that we consider the moment generating function of X with respect to an adjusted forward measure and not with respect to a forward measure. □

The next theorem gives an expression for $M^X_{T_j,T_{j+1}}$:

Theorem 2.20 *Under the assumptions of theorem 2.19 we have for $u\in\mathbb{R}$*

$$M^X_{T_j,T_{j+1}}(-R-\mathrm{i}u) = \exp\int_t^T\Big[\theta_s(g_s(-R-\mathrm{i}u))-\theta_s(g_s(0))\Big]\,\mathrm{d}s, \tag{2.38}$$

with $g_s(z) := z\Sigma(s,T,T+\delta)+\Sigma(s,T_j)\mathbb{1}_{\{s\le T_j\}}+\Sigma(s,T_{j+1})\mathbb{1}_{\{T_j<s\}}$.

Proof: Equations (2.15)–(2.17) yield for $z\in\mathbb{C}$ with $\Re(z)=-R$

$$\begin{aligned} M^X_{T_j,T_{j+1}}(z) &= \mathbb{E}_{T_j,T_{j+1}}\left[z\int_t^T\Sigma(s,T,T+\delta)\,\mathrm{d}L_s\right] \\ &= \exp\left(-\int_0^{T_{j+1}}A_{T_j,T_{j+1}}(s)\,\mathrm{d}s\right) \\ &\quad\times\mathbb{E}\left[\exp\left(z\int_t^T\Sigma(s,T,T+\delta)\,\mathrm{d}L_s+\int_0^{T_{j+1}}\Sigma_{T_j,T_{j+1}}(s)\,\mathrm{d}L_s\right)\right]. \end{aligned}$$

Using the independence of the increments of L plus the fact that for $s_1 < s_2$

$$\mathbb{E}\left[\exp \int_{s_1}^{s_2} \Sigma_{T_j,T_{j+1}}(s)\,\mathrm{d}L_s\right] = \exp \int_{s_1}^{s_2} \theta_s(\Sigma_{T_j,T_{j+1}}(s))\,\mathrm{d}s = \exp \int_{s_1}^{s_2} A_{T_j,T_{j+1}}(s)\,\mathrm{d}s \tag{2.39}$$

(which is a consequence of proposition 1.9 and the drift condition (2.8)) yields

$$M^X_{T_j,T_{j+1}}(z) = \exp\left(-\int_t^T A_{T_j,T_{j+1}}(s)\,\mathrm{d}s\right) \times \mathbb{E}\left[\exp \int_t^T \big(z\Sigma(s,T,T+\delta) + \Sigma_{T_j,T_{j+1}}(s)\big)\,\mathrm{d}L_s\right].$$

Making use of proposition 1.9 again and keeping in mind (2.39) as well as the definition of $\Sigma_{T_j,T_{j+1}}$ in (2.17), one arrives at equation (2.38). □

Once again, let us consider the special case of a multifactor Gaussian HJM model. We have $\theta(x) = \frac{\langle x,x\rangle}{2}$ for $x \in \mathbb{C}^d$ and from (2.38) we get for $z \in \mathbb{C}$

$$M^X_{T_j,T_{j+1}}(z) = \exp\left(\frac{z^2}{2}\int_t^T ||\Sigma(s,T,T+\delta)||^2\,\mathrm{d}s + z\int_t^T \langle \Sigma(s,T,T+\delta), \Sigma(s,T_j)\mathbb{1}_{\{s\le T_j\}} + \Sigma(s,T_{j+1})\mathbb{1}_{\{T_j<s\}}\rangle\,\mathrm{d}s\right).$$

Consequently, X is normally distributed under $\mathbb{P}_{T_j,T_{j+1}}$ with mean

$$m(t,T,T+\delta,T_j,T_{j+1}) := \int_t^T \langle \Sigma(s,T,T+\delta), \Sigma(s,T_j)\mathbb{1}_{\{s\le T_j\}} + \Sigma(s,T_{j+1})\mathbb{1}_{\{T_j<s\}}\rangle\,\mathrm{d}s$$

and variance

$$g(t,T,T+\delta) := \int_t^T ||\Sigma(s,T,T+\delta)||^2\,\mathrm{d}s. \tag{2.40}$$

From (2.35) we get

$$
\begin{aligned}
h(y) &= \int_{\log \frac{\underline{K}}{y}}^{\log \frac{\overline{K}}{y}} \mathrm{d}\mathbb{P}^X_{T_j,T_{j+1}}(x) \\
&= \mathbb{P}_{T_j,T_{j+1}}\left(\log \frac{\underline{K}}{y} \le X \le \log \frac{\overline{K}}{y}\right) \\
&= \Phi\left(\frac{\log \frac{\overline{K}}{y} - m(t,T,T+\delta,T_j,T_{j+1})}{\sqrt{g(t,T,T+\delta)}}\right) \\
&\qquad -\Phi\left(\frac{\log \frac{\underline{K}}{y} - m(t,T,T+\delta,T_j,T_{j+1})}{\sqrt{g(t,T,T+\delta)}}\right).
\end{aligned}
$$

Plugging in the expression for $\overline{K}$ and $\underline{K}$ from (2.36)–(2.37) and using (2.34) we arrive at

$$
\begin{aligned}
D_t &= \Phi\left(\frac{\log \frac{B(t,T)}{B(t,T+\delta)(\delta r_l(T)+1)} + \frac{1}{2}g(t,T,T+\delta) - l(t,T,T+\delta,T_j,T_j+1)}{\sqrt{g(t,T,T+\delta)}}\right) \\
&\quad -\Phi\left(\frac{\log \frac{B(t,T)}{B(t,T+\delta)(\delta r_u(T)+1)} + \frac{1}{2}g(t,T,T+\delta) - l(t,T,T+\delta,T_j,T_j+1)}{\sqrt{g(t,T,T+\delta)}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&l(t,T,T+\delta,T_j,T_{j+1}) := \\
&\quad \int_t^T \langle \Sigma(s,T,T+\delta), \Sigma(s,T_j)\mathbb{1}_{\{s\le T_j\}} + \Sigma(s,T_{j+1})\mathbb{1}_{\{T_j<s\}} - \Sigma(s,T)\rangle \,\mathrm{d}s.
\end{aligned}
\tag{2.41}
$$

Putting pieces together, we obtain the following result:

Theorem 2.21 *Using the notation introduced above, the time-t price of a floating range note in the multifactor Gaussian HJM model is equal to*

$$
FlRN(t) = B(t,T_N) + \sum_{j=0}^{N-1} \nu_{j+1}(t)
$$

with

$$
\begin{aligned}
\nu_1(t) = \frac{r_n(T_0,T_0+\delta_0)+s_0}{D_0}\Bigg(& B(t,T_1)H(T_0,t) \\
&+ \sum_{i=n_0^-+1}^{n_0} DRD_t\left[r_n(T_{0,i},T_{0,i}+\delta_{0,i}); r_l(T_{0,i}); r_u(T_{0,i}); T_1\right]\Bigg)
\end{aligned}
$$

and

$$\nu_{j+1}(t) = \left(\frac{s_j}{D_j} - \frac{1}{\delta_j D_j}\right) \sum_{i=1}^{n_j} DRD_t[r_n(T_{j,i}, T_{j,i} + \delta_{j,i}); r_l(T_{j,i}); r_u(T_{j,i}); T_{j+1}]$$
$$+ \frac{B(t,T_j)}{\delta_j D_j} \sum_{i=1}^{n_j} \Big(\Phi(\eta_{j,i}(r_l(T_{j,i}))) - \Phi(\eta_{j,i}(r_u(T_{j,i})))\Big)$$

where

$$\eta_{j,i}(r) := \frac{\log \frac{B(t,T_{j,i})}{B(t,T_{j,i}+\delta_{j,i})(\delta_{j,i}r+1)} + \frac{1}{2} g(t, T_{j,i}, T_{j,i}+\delta_{j,i}) - l(t, T_{j,i}, T_{j,i}+\delta_{j,i}, T_j, T_{j+1})}{\sqrt{g(t, T_{j,i}, T_{j,i}+\delta_{j,i})}}$$

and g and l are defined as in (2.40) and (2.41).

2.6 An example of calibration to market data

To test the ability of the Lévy term structure model to reproduce observed option prices, we calibrate it to market data. Our data set consists of caplet and swaption prices (quoted via their implied volatilities) of February 19, 2002 as well as of the zero coupon bond yield curve observed at the same day. The calibration is done for a driving homogeneous as well as for a non-homogeneous Lévy process.

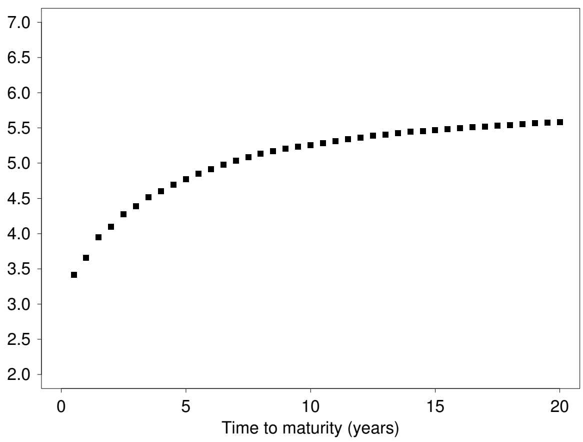

Figure 2.1: Euro yield curve on February 19, 2002

Before being able to start a calibration, we first have to specify two ingredients to the model, namely the volatility structure σ and the driving process L. We choose the Vasiček volatility structure, that is $\sigma(s,T) = e^{-a(T-s)}$ for a real a. Note that we set $\widehat{\sigma} = 1$ in example 2.12 since this multiplicative constant can be included in the process L and is therefore redundant in the volatility structure. Two cases are considered for the driving process:

T	0.5 Y	1 Y	1.5 Y	2 Y	2.5 Y
B(0,T)	0.9833630	0.9647388	0.9435826	0.9228903	0.9006922
T	3 Y	3.5 Y	4 Y	4.5 Y	5 Y
B(0,T)	0.8790279	0.8568412	0.8352144	0.8133497	0.7920573
T	5.5 Y	6 Y	6.5 Y	7 Y	7.5 Y
B(0,T)	0.7706813	0.7498822	0.7292111	0.7091098	0.6893282
T	8 Y	8.5 Y	9 Y	9.5 Y	10 Y
B(0,T)	0.6700983	0.6514969	0.6334118	0.6159873	0.5990420
T	10.5 Y	11 Y	11.5 Y	12 Y	12.5 Y
B(0,T)	0.5822022	0.5658357	0.5497905	0.5342002	0.5189684
T	13 Y	13.5 Y	14 Y	14.5 Y	15 Y
B(0,T)	0.5041710	0.4898851	0.4760039	0.4626807	0.4497303
T	15.5 Y	16 Y	16.5 Y	17 Y	17.5 Y
B(0,T)	0.4369867	0.4246043	0.4126901	0.4011102	0.3897380
T	18 Y	18.5 Y	19 Y	19.5 Y	20 Y
B(0,T)	0.3786882	0.3678357	0.3572942	0.3472375	0.3374637

Table 2.1: Euro zero coupon bond prices on February 19, 2002

- First, the model is driven by a (homogeneous) Lévy process. Concretely, we model increments of length 1 to be NIG-distributed (see e.g. Barndorff-Nielsen (1998)) with parameters μ, α, β, δ. It can be shown that parameter μ does not have an impact on option prices. Thus, μ can be set equal to zero. This leads to a model with a total number of 4 parameters.

- For the second calibration a non-homogeneous Lévy process is used. Motivated by the observation that traders act differently on the short term bond market (maturities up to one year), the middle (one to five years), and the long term market (greater than five years), we use a "piecewise Lévy process" as driving motion, i.e. a non-homogeneous Lévy process whose increments on $[0, 1]$ are stationary as well as the increments on $[1, 5]$ and $[5, T^*]$. Again, increments of length 1 are modelled to be NIG-distributed. We end up with a model consisting of a total number of 10 parameters, one for the volatility structure and 3×3 parameters for the process L.

Note that in both cases the martingale measure is unique and the short rate follows a Markov process. Of course, these particular choices are just two out of many possibilities in our modelling framework.

Let us consider the calibration to prices of caplets first. We use market prices for caplets with maturities ranging from one to ten years and 12 different strike rates from 2.5 % to 10 %. They are given via their implied volatilities (in %) as shown in figure 2.2 and table 2.2. Note that the shape of the implied volatility surface is quite typical, i.e. for the short maturities there is a smile in the implied volatilities whereas for the long maturities a skewed shape can be observed. All caplets are linked to the 6-month EURIBOR. Let us stress that all except for the one year caplet are in fact portfolios of two caplets. The two

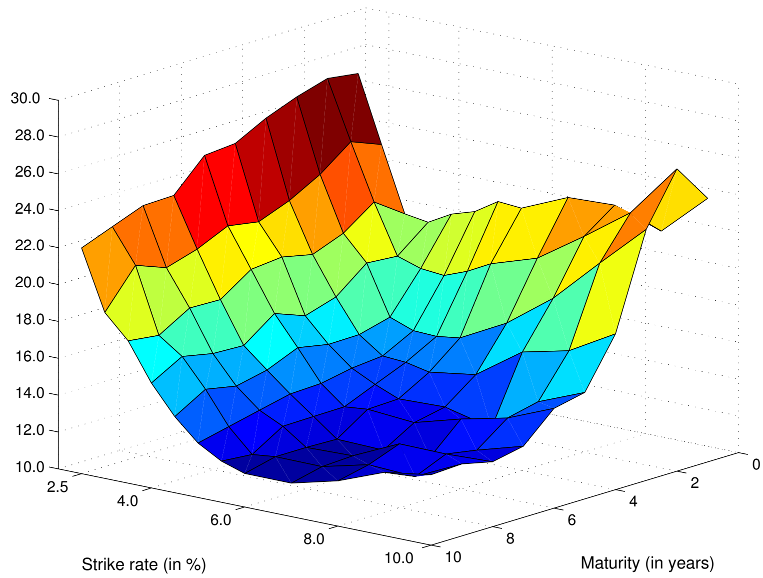

Figure 2.2: Euro caplet implied volatility surface on February 19, 2002

	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	7.0	8.0	9.0	10.0
1 Y	27.2	23.6	20.1	19.9	20.6	21.0	21.8	21.7	22.8	22.9	22.0	24.3
2 Y	27.4	24.3	21.3	18.6	18.3	18.1	18.6	19.2	20.0	21.7	23.5	26.4
3 Y	26.9	22.9	20.0	18.7	16.1	15.6	15.5	15.7	17.0	18.9	21.3	23.6
4 Y	26.3	22.0	19.4	17.4	15.6	14.7	14.6	14.3	14.7	16.4	17.0	18.4
5 Y	25.4	21.4	19.8	16.8	15.6	14.5	13.9	13.4	13.5	12.8	14.8	15.8
6 Y	25.2	21.7	19.6	17.5	15.9	14.2	13.2	13.3	13.1	13.8	14.4	15.4
7 Y	23.6	20.9	18.4	16.2	15.2	14.1	13.2	12.2	12.1	12.2	13.1	13.8
8 Y	23.5	20.4	18.5	16.3	14.8	13.7	13.1	12.3	12.3	13.5	13.5	13.6
9 Y	22.9	21.0	17.5	16.6	15.1	13.3	12.1	12.2	12.2	12.9	12.7	13.9
10 Y	22.2	19.0	17.7	15.7	14.1	13.0	12.2	11.8	11.8	12.5	13.4	13.8

Table 2.2: Euro caplet implied volatilities on February 19, 2002

year "caplet" for example consists of a caplet with option maturity date in one year as well as of a caplet that matures in one and a half years.

The calibration is done by minimizing the sum of the squared errors between theoretical and market price relative to the at-the-money caplet market price for the respective maturity. That is, we minimize the sum of

$$\left(\frac{\text{model price - market price}}{\text{ATM market price for the respective maturity}}\right)^2.$$

The differences (resp. absolute differences) between implied volatility of the model price and implied volatility of the market price are shown in table 2.3 (figure 2.3) for the homogeneous and in table 2.4 (figure 2.4) for the non-homogeneous Lévy process. To improve the readability of figure 2.3 and to make the two pictures comparable, we truncated differences in the implied volatilities that exceed 10 %. In the tables, the two strike rates that are closest to the at-the-money strike rate are highlighted in red.

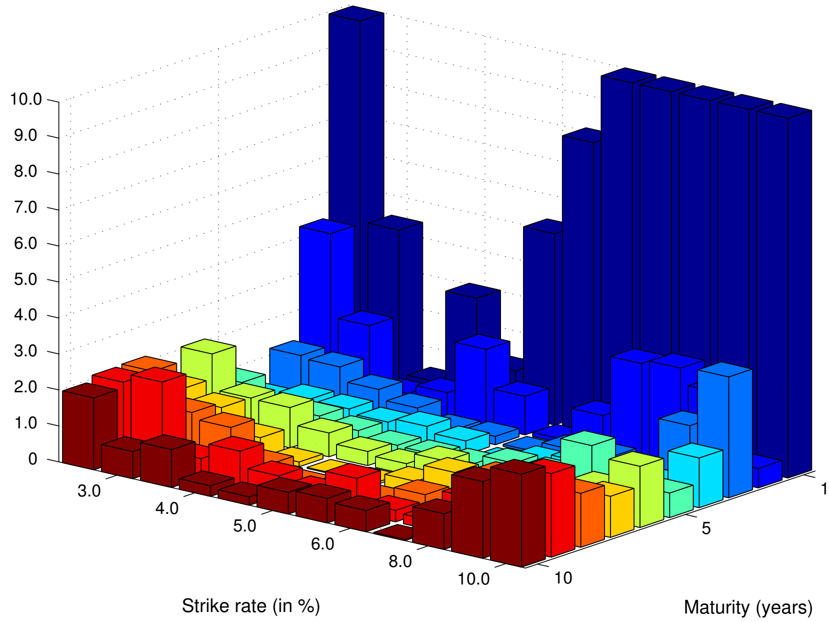

Figure 2.3: Absolute errors in caplet calibration for the driving homogeneous Lévy process

	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	7.0	8.0	9.0	10.0
1 Y	10.4	4.4	-0.5	-3.0	1.3	5.3	8.1	11.1	14.5	17.8	21.2	20.9
2 Y	4.4	2.0	0.4	-0.7	-2.1	-1.1	0.1	1.0	2.7	2.8	2.4	0.5
3 Y	1.3	1.2	0.8	-0.5	0.2	-0.2	-0.1	0.3	0.4	-0.3	-1.8	-3.4
4 Y	-0.4	0.6	0.5	0.4	0.5	0.4	0.0	0.3	0.5	-0.5	-0.5	-1.4
5 Y	-1.0	0.1	-0.6	0.5	0.3	0.3	0.3	0.5	0.5	1.5	0.0	-0.7
6 Y	-2.1	-1.1	-1.1	-0.7	-0.4	0.3	0.6	0.1	0.0	-0.5	-0.9	-1.7
7 Y	-1.4	-1.2	-0.6	0.1	0.0	0.1	0.3	0.8	0.5	0.3	-0.5	-1.1
8 Y	-2.2	-1.3	-1.1	-0.3	0.0	0.2	0.1	0.5	-0.1	-1.5	-1.5	-1.5
9 Y	-2.2	-2.4	-0.5	-1.0	-0.5	0.4	1.0	0.3	-0.3	-1.1	-1.1	-2.3
10 Y	-2.0	-0.7	-1.1	-0.3	0.2	0.6	0.7	0.6	0.0	-1.0	-2.1	-2.6

Table 2.3: Errors in caplet calibration for the driving homogeneous Lévy process

The model parameters that lead to these results are

$$a = 0.0504489$$

and

$$\alpha = 48.9992, \qquad \beta = -5.47554, \qquad \delta = 0.00417802$$

for the driving homogeneous Lévy process.

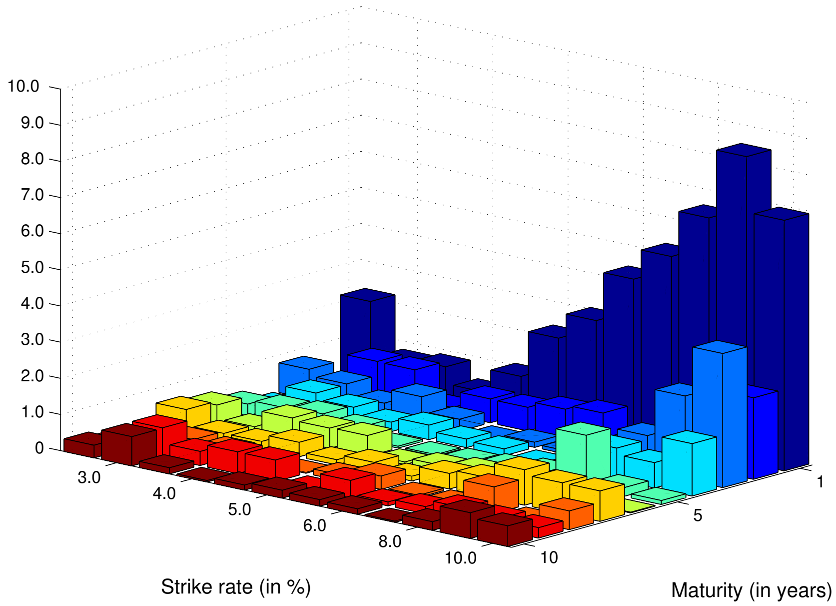

Figure 2.4: Absolute errors in caplet calibration for the driving non-homogeneous Lévy process

	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	7.0	8.0	9.0	10.0
1 Y	2.2	0.8	0.9	0.4	1.0	2.3	3.0	4.4	5.3	6.6	8.5	6.9
2 Y	0.1	-1.0	-1.0	-0.2	-0.7	-0.6	-0.8	-0.9	-0.3	-0.4	-0.7	-2.2
3 Y	0.8	0.6	0.3	-0.7	0.3	0.0	-0.1	-0.1	-0.4	-1.0	-2.3	-3.7
4 Y	-0.1	0.6	0.4	0.3	0.4	0.2	-0.2	0.0	0.1	-0.9	-0.7	-1.5
5 Y	-0.3	0.4	-0.5	0.4	0.0	0.0	-0.1	0.2	0.3	1.5	0.2	-0.1
6 Y	-0.5	-0.2	-0.6	-0.6	-0.6	-0.1	0.2	-0.3	0.1	0.0	0.2	0.0
7 Y	0.6	0.2	0.2	0.4	-0.1	-0.3	-0.1	0.4	0.5	1.0	0.8	0.8
8 Y	0.1	0.3	-0.2	0.0	0.0	-0.2	-0.4	0.1	-0.1	-0.7	0.0	0.5
9 Y	0.2	-0.8	0.4	-0.6	-0.6	0.0	0.5	-0.1	-0.2	-0.4	0.3	-0.3
10 Y	0.4	0.8	-0.2	0.0	0.1	0.2	0.2	0.1	0.0	-0.2	-0.7	-0.6

Table 2.4: Errors in caplet calibration for the driving non-homogeneous Lévy process

In case of the piecewise Lévy process we get

$$a = 0.0864322$$

and

$$\begin{aligned}
\alpha &= 179.818, & \beta &= -34.6837, & \delta &= 0.01282933 & &\text{on } [0,1],\\
\alpha &= \ \ 37.531, & \beta &= -\ \ 2.1500, & \delta &= 0.00386291 & &\text{on } [1,5],\\
\alpha &= \ \ 14.132, & \beta &= -\ \ 3.0837, & \delta &= 0.00220217 & &\text{on } [5,10].
\end{aligned}$$

From these results we can draw some conclusions:

1. Except for the short maturities (i.e. the one and two year caplets), the model driven by a homogeneous Lévy process produces very good results. In a large interval around the at-the-money strike rates differences in the implied volatilities are well within a 1 % range. Only for far out-of-the-money options errors exceed the 1 % range. Remember that the model is driven by four parameters only.

2. The model driven by a piecewise Lévy process produces excellent results across all maturities and strike rates. For the one year caplet this may not be obvious from the picture since we observe large differences in implied volatilities away-from-the-money. Nevertheless, the respective differences between model and market price are very small. For the one year caplet with a strike rate of 9 % for example, we notice the largest difference (8.5 %) in implied volatilities, but model and market price differ only by 6×10^{-8} (less than 1/1000 of a basis point).

3. It is not surprising that a model driven by 10 parameters outperforms a model that depends on 4 parameters only. However, not only the smaller number of parameters makes the model driven by a homogeneous Lévy process inferior to its non-homogeneous counterpart. It is well known that exponential Lévy models for stock prices allow for an excellent calibration to implied volatility patterns for single maturities and also for a certain range of maturities, but fail to reproduce option prices with the same accuracy over the full range of different maturities because the smile in the implied volatilities flattens as time to maturity increases. In other words, a model that produces the correct smile for short maturities will usually not give rise to enough smile for the long maturities. Conversely, a model that gets the long maturities right will produce a smile that is too strong in the short end. The same observation can be made in the Lévy term structure model as figure 2.5 shows. The model driven by a homogeneous Lévy process produces an accurate smile for the 5 year caplet but too much of a smile in the short end and too few for the 10 year caplet. Note that the model driven by a non-homogeneous Lévy process is flexible enough to reproduce the smiles observed in the market across all maturities with high accuracy.

4. The calibration results for the piecewise Lévy model can be improved by increasing the number of parameters in the model. For example, when a driving process is used whose increments are stationary and NIG-distributed on $[0,1],[1,2],\ldots,[9,T^*]$ (that makes 31 parameters in total) the average error (model minus market price divided by ATM market price) reduces by 24 %. We leave the question to the reader whether or not this higher calibration accuracy justifies the larger number of parameters.

The second model calibration is done to market prices of swaptions. For this purpose, we use prices of swaptions with maturities of one, two, three, four, five, seven, and ten years. The tenors of the underlying swaps reach from one to ten years. The implied volatilities of these market prices are shown in table 2.5.

Unfortunately, our data set consists of at-the-money options only. Therefore, we will not be able to discuss any effects on the smile and we are not going to give any pictures.

	1 Y	2 Y	3 Y	4 Y	5 Y	6 Y	7 Y	8 Y	9 Y	10 Y
1 Y Opt	18.0	16.6	15.4	14.4	13.7	13.3	12.8	12.5	12.2	11.8
2 Y Opt	15.1	14.3	13.4	13.0	12.6	12.3	12.0	11.7	11.5	11.2
3 Y Opt	14.5	13.3	12.7	12.3	12.0	11.8	11.5	11.3	11.1	10.9
4 Y Opt	13.7	12.5	12.1	11.8	11.6	11.4	11.2	10.9	10.7	10.5
5 Y Opt	13.0	12.0	11.6	11.4	11.3	11.1	10.9	10.7	10.5	10.3
7 Y Opt	12.3	11.2	10.9	10.7	10.6	10.4	10.3	10.1	10.0	9.9
10 Y Opt	11.4	10.3	10.0	9.8	9.6	9.5	9.5	9.4	9.3	9.3

Table 2.5: Euro at-the-money swaptions: Implied volatilities on 19 February 2002 (in %)

The calibration is done by minimizing the sum of the squared relative errors, that is the sum of the squared differences between theoretical and market price relative to the market price. The differences in the implied volatilities of model and market prices are shown in table 2.6 for the homogeneous and in table 2.7 for the non-homogeneous Lévy process. Both models reproduce market prices with high accuracy.

	1 Y	2 Y	3 Y	4 Y	5 Y	6 Y	7 Y	8 Y	9 Y	10 Y
1 Y Opt	-1.02	-0.78	-0.42	-0.11	-0.01	-0.13	-0.08	-0.15	-0.16	-0.07
2 Y Opt	0.05	0.14	0.43	0.28	0.21	0.09	0.05	0.06	-0.03	0.00
3 Y Opt	-0.38	0.24	0.32	0.26	0.16	0.04	0.07	-0.01	-0.08	-0.13
4 Y Opt	-0.39	0.31	0.27	0.19	0.08	0.02	-0.06	-0.02	-0.06	-0.08
5 Y Opt	-0.36	0.21	0.25	0.16	0.01	-0.06	-0.12	-0.17	-0.18	-0.18
7 Y Opt	-0.59	0.27	0.35	0.26	0.09	0.03	-0.10	-0.09	-0.19	-0.27
10 Y Opt	-0.48	0.32	0.35	0.32	0.33	0.21	0.03	-0.06	-0.15	-0.31

Table 2.6: Error of swaption calibration (in %) for the homogeneous Lévy process

	1 Y	2 Y	3 Y	4 Y	5 Y	6 Y	7 Y	8 Y	9 Y	10 Y
1 Y Opt	-0.14	-0.17	0.00	0.18	0.19	0.00	0.01	-0.08	-0.11	-0.03
2 Y Opt	0.00	0.05	0.31	0.15	0.08	-0.03	-0.05	-0.02	-0.09	-0.04
3 Y Opt	-0.54	0.08	0.18	0.14	0.07	-0.02	0.03	-0.02	-0.05	-0.07
4 Y Opt	-0.52	0.20	0.19	0.14	0.07	0.04	0.00	0.08	0.07	0.08
5 Y Opt	-0.52	0.09	0.15	0.09	-0.02	-0.06	-0.09	-0.11	-0.10	-0.06
7 Y Opt	-0.73	0.14	0.25	0.19	0.04	0.01	-0.09	-0.06	-0.13	-0.18
10 Y Opt	-0.60	0.22	0.27	0.26	0.29	0.20	0.04	-0.03	-0.10	-0.24

Table 2.7: Error of swaption calibration (in %) for the non-homogeneous Lévy process

The model parameters that lead to these results are

$$a = 0.0486698$$

and

$$\alpha = 2730.651, \qquad \beta = -230.663, \qquad \delta = 0.161142$$

for the (homogeneous) Lévy process. In case of the piecewise Lévy process we get

$$a = 0.0413190$$

and

$$\begin{aligned}
\alpha &= 12.0637, & \beta &= -\ \ 6.9660, & \delta &= 0.0024161 & &\text{on } [0,1],\\
\alpha &= 41.4422, & \beta &= -\ \ 0.1376, & \delta &= 0.0025918 & &\text{on } [1,5],\\
\alpha &= 64.9670, & \beta &= -28.4413, & \delta &= 0.0027980 & &\text{on } [5,20].
\end{aligned}$$

2.7 Conclusion

As a generalization of the Lévy term structure model introduced in Eberlein and Raible (1999) we discussed a term structure model driven by non-homogeneous Lévy processes. For deterministic volatility structures, pricing formulae have been derived for caps and floors as well as for digital options and for a derivative with a path-dependent payoff, namely a floating range note. As a side result, a valuation formula derived by Nunes (2004) for floating range notes in the multifactor Gaussian HJM model has been simplified. A formula for swaption valuation has also been established under an additional restriction on the volatility structure which still allows for some well known examples as the Ho–Lee or Vasiček volatility function. An advantage of all of these pricing formulae is the speed at which they can be evaluated numerically. This gave us the opportunity to calibrate the model to market data of the most liquid interest rate derivatives, namely caps, floors, and swaptions. Calibrations were done for a driving homogeneous Lévy process as well as for a driving process with independent and piecewise stationary increments. This led to models with 4 and 10 parameters respectively. Both models proved to be flexible enough to reproduce the given derivatives' prices with high accuracy, although the latter clearly outperformed the other (which is not too surprising since it has more parameters). The model driven by a homogeneous Lévy process revealed a weakness which also occurs when modelling stock prices with Lévy processes, namely that smiles in the implied volatilities of option prices flatten too much as option maturity increases. This drawback can be removed by working with non-homogeneous Lévy processes.

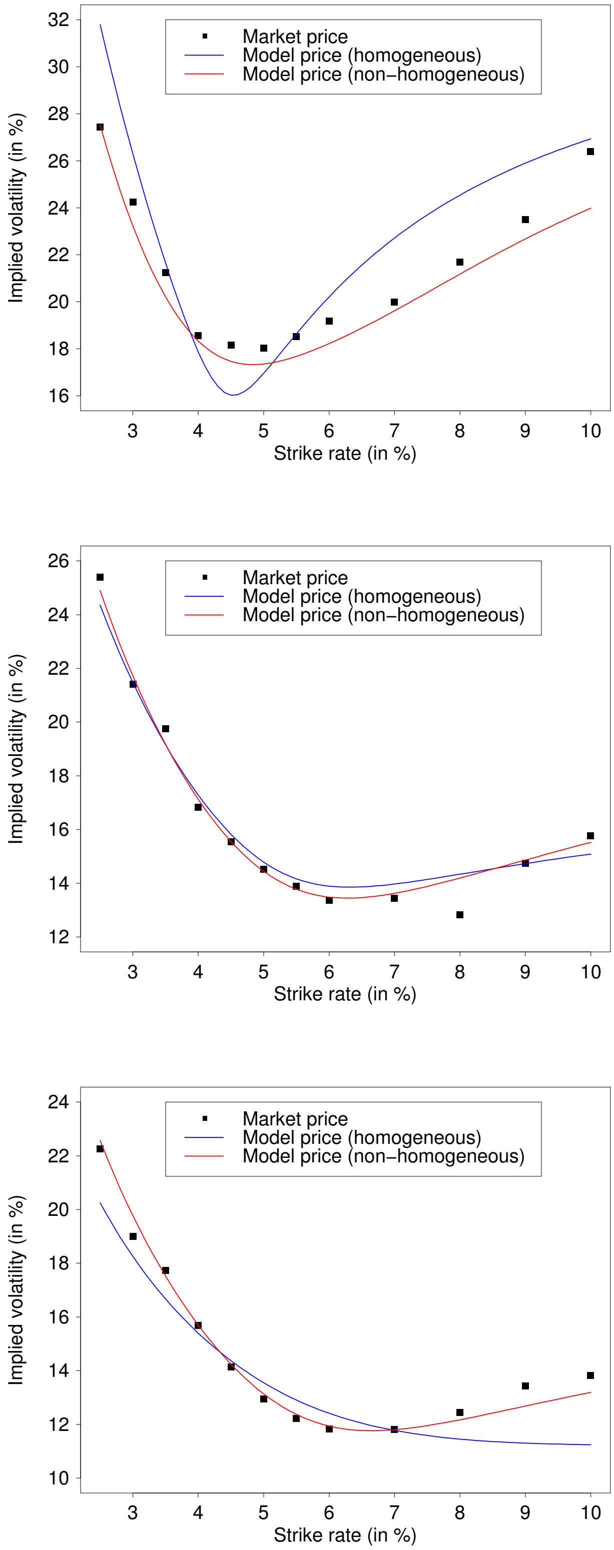

Figure 2.5: Implied volatility curves for the 2 year, 5 year, and 10 year caplet

Chapter 3

Lévy models for effective rates

Short rate models as well as term structure (Heath–Jarrow–Morton) models, like the Lévy term structure model discussed in the previous chapter, specify the dynamics of continuously compounded interest rates. From a mathematical point of view, these rates that apply for an infinitesimal time interval are very convenient for modelling purposes. However, interest rates quoted in real markets are usually effective (simply compounded). Sandmann, Sondermann, and Miltersen (1995), Miltersen, Sandmann, and Sondermann (1997), and Brace, Gatarek, and Musiela (1997) managed to incorporate a model for effective rates into an HJM-framework. Their *Forward Libor Model* which is often referred to as the *Libor market model* became a very popular approach among practitioners since it is consistent with the market practice of pricing caps and floors. Jamshidian (1999) generalized this model by considering semimartingales as driving processes, but pricing of caps and floors was not discussed in this setup. A model that lies in between Jamshidians's approach and the Libor market model as far as generality is concerned has recently been developed by Eberlein and Özkan (2005). Their *Lévy Libor model* is a lot more flexible than the Libor market model since it uses general (time-inhomogeneous) Lévy processes as drivers instead of the special case of a Brownian motion. Moreover, explicit pricing formulae for caps and floors can be obtained.

Eberlein and Özkan (2005) take two different approaches to model forward Libor rates in a discrete tenor setting. Both approaches have in common that they do not specify zero coupon bond prices directly (it is only assumed that the processes describing the evolution of bond prices are special semimartingales whose values as well as all left hand limits are strictly positive; moreover, the terminal value of each bond equals one). Instead, ratios of bond prices are specified. Eberlein and Özkan consider a fixed time horizon T^* as well as a discrete tenor structure $0 = T_0 < T_1 < \ldots < T_n = T^*$ and build up the model in one of the two following ways:

- The first approach uses the (ordinary) exponential of a non-homogeneous Lévy process to model forward Libor rates directly, which are defined by

$$L(t, T_k) := \frac{1}{\delta_k} \left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1 \right) \qquad (k \in \{1, \ldots, n-1\}),$$

where $\delta_k := T_{k+1} - T_k$. This specification generalizes the Libor market model and we will refer to it as the *Lévy Libor model*.

- In the second approach forward price processes, i.e.

$$F(t, T_k, T_{k+1}) := \frac{B(t, T_k)}{B(t, T_{k+1})} \qquad (k \in \{1, \dots, n-1\}),$$

  are specified as starting value times the exponential of a non-homogeneous Lévy process. Of course, this immediately provides a model for forward Libor rates since $L(t, T_k) = \frac{1}{\delta_k}(F(t, T_k, T_{k+1}) - 1)$. In order to be able to distinguish this ansatz from the other one we call it the *Lévy forward price model*.

Although $L(t, T_k)$ and $F(t, T_k, T_{k+1})$ differ only by an additive and a multiplicative constant, the two specifications lead to models that are quite different. If in a small time interval from t_1 to t_2 the driving process in the exponent changes its value by a small amount Δ, then in the Lévy Libor model we have

$$L(t_2, T_k) = L(t_1, T_k) \exp \Delta \approx L(t_1, T_k) + \Delta L(t_1, T_k)$$

whereas in the Lévy forward price model

$$L(t_2, T_k) = \frac{1}{\delta_k}((1 + \delta_k L(t_1, T_k)) \exp \Delta - 1) \approx L(t_1, T_k) + \frac{\Delta}{\delta_k}$$

for reasonable values of $L(t_1, T_k)$. In other words, (small) changes in the driving process have different impact on the forward Libor rates. In the Lévy Libor model, forward Libor rates change by an amount that is relative to their level while the change in the Lévy forward price model does not depend on the actual level.

The two models also differ in tractability. The Lévy forward price model is very pleasant from an analytical point of view. The driving process remains a non-homogeneous Lévy process under all forward measures, a fact that simplifies the valuation of derivatives considerably. It might be seen as a drawback that this model allows for negative Libor rates. In the Lévy Libor model, Libor rates are always positive. The driving process is usually only a Lévy process with respect to one forward measure. This makes option pricing more complicated and forces us to work with approximations.

The aim of this chapter is to push further the derivation of option prices within the Lévy Libor model as well as within the Lévy forward price model. Our focus lies on the most common interest rate derivatives, i.e. caps, floors, and swaptions.

In the Lévy forward price model, exact pricing formulae can be obtained. We prove that this model can be regarded as a special case of the Lévy term structure model. Consequently, the pricing formulae for caps, floors, and swaptions from chapter 2 can be used.

Eberlein and Özkan (2005) provide an approximate pricing formula for caps and floors in the Lévy Libor model. Valuation of swaptions is not considered.

Our goal is twofold: first, we derive an alternative valuation formula for caps and floors based on a different approximation. The advantage of our formula is the much higher speed at which cap and floor prices can be computed. Second, we present a method to price swaptions.

The chapter is organized as follows: Section 3.1 reviews the construction of the Lévy forward price model which is done only very briefly in Eberlein and Özkan (2005). In section 3.1.1 we prove that this model can be embedded into the Lévy term structure model. A brief presentation of the Lévy Libor model is given in section 3.2. The remaining sections are devoted to the pricing of caps, floors, and swaptions within this model.

3.1 The Lévy forward price model

The Lévy forward price model is constructed by backward induction. It is driven by a non-homogeneous Lévy process L^{T^*} on a complete stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_s)_{0\leq s\leq T^*}, \mathbb{P}_{T^*})$. The measure $\mathbb{P}_{T^*}$ plays the role of the forward measure associated with the settlement day T^*. The process L^{T^*} is supposed to have exponential moments in the sense of assumption ($\mathbb{EM}$) from chapter 1. Two of the characteristics (b^{T^*}, c, F^{T^*}) of L^{T^*} can be chosen freely, namely c and F^{T^*}, whereas the drift characteristic b^{T^*} will be derived later. Since we proceed by backward induction, let us denote $T_i^* := T_{n-i}$ and $\delta_i^* := \delta_{n-i}$ for $i \in \{0, \ldots, n\}$. The following assumptions are made:

($\mathbb{FP}$.1): For any maturity T_i there is a bounded deterministic volatility function $\lambda(\cdot, T_i) : [0, T^*] \to \mathbb{R}^d$ which represents the volatility of the forward price process $F(\cdot, T_i, T_{i+1})$. In particular, for all $k \in \{1, \ldots, n-1\}$

$$\left| \sum_{i=1}^{k} \lambda^j(s, T_i) \right| \leq M \qquad (s \in [0, T^*],\ j \in \{1, \ldots, d\}), \tag{3.1}$$

where M is the constant from assumption ($\mathbb{EM}$) and $\lambda(s, T_i) = 0$ for $s > T_i$.

($\mathbb{FP}$.2): The initial term structure of zero coupon bond prices $B(0, T_i)$ is strictly positive ($i \in \{1, \cdots, n\}$).

We begin by constructing the forward price with the longest maturity and postulate that

$$F(t, T_1^*, T^*) = F(0, T_1^*, T^*) \exp\left(\int_0^t \lambda(s, T_1^*)\, \mathrm{d}L_s^{T^*} \right) \tag{3.2}$$

subject to the initial condition

$$F(0, T_1^*, T^*) = \frac{B(0, T_1^*)}{B(0, T^*)}.$$

Another way to write (3.2) is in terms of the forward Libor rate

$$1 + \delta_1^* L(t, T_1^*) = (1 + \delta_1^* L(0, T_1^*)) \exp\left(\int_0^t \lambda(s, T_1^*)\, \mathrm{d}L_s^{T^*}\right).$$

Our goal is to specify the drift characteristic b^{T^*} in such a way that the forward price process $F(\cdot, T_1^*, T^*)$ (or equivalently the forward Libor rate $L(\cdot, T_1^*)$) is a martingale with respect to $\mathbb{P}_{T^*}$. For this purpose, we choose b^{T^*} such that

$$\begin{aligned}\int_0^t \langle \lambda(s, T_1^*), b_s^{T^*}\rangle\, \mathrm{d}s = &-\frac{1}{2}\int_0^t \langle \lambda(s, T_1^*), c_s \lambda(s, T_1^*)\rangle\, \mathrm{d}s \\ &- \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s, T_1^*), x\rangle} - 1 - \langle \lambda(s, T_1^*), x\rangle\right) \nu^{T^*}(\mathrm{d}s, \mathrm{d}x),\end{aligned}$$

where $\nu^{T^*}(\mathrm{d}s, \mathrm{d}x) := F_s^{T^*}(\mathrm{d}x)\, \mathrm{d}s$ is the compensator of the random measure μ^L that is associated with the jumps of L^{T^*}. Lemma 2.6 in Kallsen and Shiryaev (2002) yields that the forward price $F(\cdot, T_1^*, T^*)$ can then be expressed as the stochastic exponential of a local martingale, namely

$$F(t, T_1^*, T^*) = F(0, T_1^*, T^*)\mathcal{E}_t(H(\cdot, T_1^*))$$

with

$$\begin{aligned}H(t, T_1^*) = &\int_0^t \sqrt{c_s}\lambda(s, T_1^*)\, \mathrm{d}W_s^{T^*} \\ &+ \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s, T_1^*), x\rangle} - 1\right) (\mu^L - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x).\end{aligned} \tag{3.3}$$

Note that $H(\cdot, T_1^*)$ is also a non-homogeneous Lévy process. The stochastic exponential of a process that is a local martingale as well as a non-homogeneous Lévy process is not only a local martingale, but in fact a martingale (see e.g. Eberlein, Jacod, and Raible (2005) for a proof). Hence, $F(\cdot, T_1^*, T^*)$ and thus also $L(\cdot, T_1^*)$ are martingales.

We define the forward martingale measure associated with the date T_1^* by setting

$$\frac{\mathrm{d}\mathbb{P}_{T_1^*}}{\mathrm{d}\mathbb{P}_{T^*}} := \frac{F(T_1^*, T_1^*, T^*)}{F(0, T_1^*, T^*)} = \mathcal{E}_{T_1^*}(H(\cdot, T_1^*)).$$

From equation (3.3) we can immediately identify the two predictable processes β and Y in Girsanov's Theorem for semimartingales (see Jacod and Shiryaev (2003, Theorem III.3.24)) that describe the change of measure, namely

$$\beta(s) = \lambda(s, T_1^*) \qquad \text{and} \qquad Y(s, x) = \exp\langle \lambda(s, T_1^*), x\rangle.$$

In particular, $W_t^{T_1^*} := W_t^{T^*} - \int_0^t \sqrt{c_s}\lambda(s,T_1^*)\,\mathrm{d}s$ is a standard Brownian motion with respect to $\mathbb{P}_{T_1^*}$ and $\nu^{T_1^*}(\mathrm{d}t,\mathrm{d}x) := \exp\langle\lambda(s,T_1^*),x\rangle\nu^{T^*}(\mathrm{d}t,\mathrm{d}x)$ is the $\mathbb{P}_{T_1^*}$-compensator of μ^L. We have the following $\mathbb{P}_{T_1^*}$-canonical representation of L^{T^*}:

$$L_t^{T^*} = \int_0^t \widehat{b}_s\,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T_1^*} + \int_0^t\int_{\mathbb{R}^d} x(\mu^L - \nu^{T_1^*})(\mathrm{d}s,\mathrm{d}x)$$

with a deterministic drift coefficient $\widehat{b}$ which can be calculated using Girsanov's Theorem.

Now we are ready to construct the forward price $F(\cdot,T_2^*,T_1^*)$ by postulating that

$$F(t,T_2^*,T_1^*) = F(0,T_2^*,T_1^*)\exp\left(\int_0^t \lambda(s,T_2^*)\,\mathrm{d}L_s^{T_1^*}\right),$$

where

$$L_t^{T_1^*} = \int_0^t b_s^{T_1^*}\,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T_1^*} + \int_0^t\int_{\mathbb{R}^d} x(\mu^L - \nu^{T_1^*})(\mathrm{d}s,\mathrm{d}x).$$

In order to ensure that $F(\cdot,T_2^*,T_1^*)$ is a $\mathbb{P}_{T_1^*}$-martingale, we choose the drift characteristic $b^{T_1^*}$ appropriately, namely such that

$$\begin{aligned}\int_0^t \langle\lambda(s,T_2^*),b_s^{T_1^*}\rangle\,\mathrm{d}s = &-\frac{1}{2}\int_0^t \langle\lambda(s,T_2^*),c_s\lambda(s,T_2^*)\rangle\,\mathrm{d}s \\ &-\int_0^t\int_{\mathbb{R}^d}\left(e^{\langle\lambda(s,T_2^*),x\rangle} - 1 - \langle\lambda(s,T_2^*),x\rangle\right)\nu^{T_1^*}(\mathrm{d}s,\mathrm{d}x).\end{aligned}$$

Note that $L^{T_1^*}$ differs from L^{T^*} only by a deterministic drift term. In particular, both processes are non-homogeneous Lévy processes under $\mathbb{P}_{T^*}$ and $\mathbb{P}_{T_1^*}$. Again, we can express the forward price process $F(\cdot,T_2^*,T_1^*)$ as the stochastic exponential of a non-homogeneous Lévy process and local martingale $H(\cdot,T_2^*)$ and use the martingale $\left(\frac{F(t,T_2^*,T_1^*)}{F(0,T_2^*,T_1^*)}\right)_{0\le t\le T_2^*}$ to define the forward martingale measure associated with the date T_2^* by setting

$$\frac{\mathrm{d}\mathbb{P}_{T_2^*}}{\mathrm{d}\mathbb{P}_{T_1^*}} := \frac{F(T_2^*,T_2^*,T_1^*)}{F(0,T_1^*,T_1^*)}.$$

Proceeding as before, forward prices $F(\cdot,T_i^*,T_{i-1}^*)$ for $i=3,\dots,n-1$ and forward measures $\mathbb{P}_{T_i^*}$ for $i=3,\dots,n-2$ are defined inductively. We obtain a model where the forward price $F(\cdot,T_i^*,T_{i-1}^*)$ is given by

$$F(t,T_i^*,T_{i-1}^*) = F(0,T_i^*,T_{i-1}^*)\exp\left(\int_0^t \lambda(s,T_i^*)\,\mathrm{d}L_s^{T_{i-1}^*}\right) \tag{3.4}$$

with

$$L_t^{T^*_{i-1}} = \int_0^t b_s^{T^*_{i-1}}\,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s^{T^*_{i-1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu^{T^*_{i-1}})(\mathrm{d}s, \mathrm{d}x). \tag{3.5}$$

$W^{T^*_{i-1}}$ is a $\mathbb{P}_{T^*_{i-1}}$-standard Brownian motion and $\nu^{T^*_{i-1}}$ is the $\mathbb{P}_{T^*_{i-1}}$-compensator of μ^L. It is given by

$$\nu^{T^*_{i-1}}(\mathrm{d}t, \mathrm{d}x) = \exp\left(\sum_{j=1}^{i-1} \langle \lambda(t, T^*_j), x\rangle\right) F_t^{T^*}(\mathrm{d}x)\,\mathrm{d}t. \tag{3.6}$$

The characteristic $b^{T^*_{i-1}}$ satisfies

$$\begin{aligned}\int_0^t \langle \lambda(s, T^*_i), b_s^{T^*_{i-1}}\rangle\,\mathrm{d}s \; = & -\frac{1}{2}\int_0^t \langle \lambda(s, T^*_i), c_s \lambda(s, T^*_i)\rangle\,\mathrm{d}s \\ & - \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s,T^*_i), x\rangle} - 1 - \langle \lambda(s, T^*_i), x\rangle\right) \nu^{T^*_{i-1}}(\mathrm{d}s, \mathrm{d}x).\end{aligned} \tag{3.7}$$

Observe that the driving processes $L^{T^*_i}$ differ only by deterministic drift terms. Hence, all of them are non-homogeneous Lévy processes with respect to each forward measure.

3.1.1 The Lévy forward price model as a special case of the Lévy term structure model

The aim of this section is to show that the Lévy forward price model can be seen as a special case of the Lévy term structure model. By a special case we mean that the model parameters in the term structure model can be chosen in such a way that we end up with forward price processes as given in (3.4)–(3.7). Let us briefly recall those properties of the Lévy term structure model that will be needed to embed the forward price model.

In the Lévy term structure model, the price $B(\cdot, T)$ of a zero coupon bond with maturity T is given by

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(\int_0^t \left(\widetilde{\theta}_s(\Sigma(s,t)) - \widetilde{\theta}_s(\Sigma(s,T))\right)\mathrm{d}s + \int_0^t \Sigma(s,t,T)\,\mathrm{d}\widetilde{L}_s\right), \tag{3.8}$$

where

$$\Sigma(s,t,T) := \Sigma(s,T) - \Sigma(s,t)$$

and $\widetilde{L}$ is a non-homogeneous Lévy process with characteristics $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ under the spot martingale measure $\mathbb{P}$. $\widetilde{L}$ satisfies assumption ($\mathbb{EM}$) and $\widetilde{\theta}_s$ denotes the cumulant associated with the infinitely divisible distribution characterized by the Lévy–Khintchine-triplet $(\widetilde{b}_s, \widetilde{c}_s, \widetilde{F}_s)$. The volatility structure is given by

$$\Sigma(s,T) := \int_{s\wedge T}^{T} \sigma(s,u)\,\mathrm{d}u$$

for a measurable, bounded and deterministic function σ. The forward martingale measure $\mathbb{P}_{T_i^*}$ for the settlement day T_i^* is related to $\mathbb{P}$ via the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}_{T_i^*}}{\mathrm{d}\mathbb{P}} = \exp\Big(- \int_0^{T_i^*} \widetilde{\theta}_s(\Sigma(s,T_i^*))\,\mathrm{d}s + \int_0^{T_i^*} \Sigma(s,T_i^*)\,\mathrm{d}\widetilde{L}_s \Big).$$

$\widetilde{L}$ is also a non-homogeneous Lévy process with respect to $\mathbb{P}_{T_i^*}$ and its $\mathbb{P}_{T_i^*}$-characteristics $(\widetilde{b}^{T_i^*}, \widetilde{c}^{T_i^*}, \widetilde{F}^{T_i^*})$ are given by

$$\begin{aligned} \widetilde{b}_s^{T_i^*} &= \widetilde{b}_s + \widetilde{c}_s \Sigma(s,T_i^*) + \int_{\mathbb{R}^d} \Big(e^{\langle \Sigma(s,T_i^*), x\rangle} - 1 \Big)\, x\, \widetilde{F}_s(\mathrm{d}x), &(3.9)\\ \widetilde{c}_s^{T_i^*} &= \widetilde{c}_s, \\ \widetilde{F}_s^{T_i^*}(\mathrm{d}x) &= e^{\langle \Sigma(s,T_i^*), x\rangle}\, \widetilde{F}_s(\mathrm{d}x). &(3.10) \end{aligned}$$

Since $\widetilde{L}$ is also a $\mathbb{P}_{T_i^*}$-special semimartingale, it can be written in its $\mathbb{P}_{T_i^*}$-canonical representation as

$$\widetilde{L}_t = \int_0^t \widetilde{b}_s^{T_i^*}\,\mathrm{d}s + \int_0^t \sqrt{\widetilde{c}_s}\,\mathrm{d}W_s^{T_i^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu^{\widetilde{L}} - \widetilde{\nu}^{T_i^*})(\mathrm{d}s,\mathrm{d}x),$$

where $W^{T_i^*}$ is a $\mathbb{P}_{T_i^*}$-standard Brownian motion and $\widetilde{\nu}^{T_i^*}(\mathrm{d}s,\mathrm{d}x) := \widetilde{F}_s^{T_i^*}(\mathrm{d}x)\,\mathrm{d}s$ is the $\mathbb{P}_{T_i^*}$-compensator of $\mu^{\widetilde{L}}$, the random measure associated with the jumps of the process $\widetilde{L}$.

From (3.8) we can deduce the forward price process

$$
\begin{aligned}
&F(t,T^*_{i+1},T^*_i)\\
&\qquad= \frac{B(t,T^*_{i+1})}{B(t,T^*_i)}\\
&\qquad= \frac{B(0,T^*_{i+1})}{B(0,T^*_i)}\exp\Bigg(\int_0^t \Big(\widetilde{\theta}_s(\Sigma(s,T^*_i)) - \widetilde{\theta}_s(\Sigma(s,T^*_{i+1}))\Big)\,\mathrm{d}s\\
&\qquad\qquad + \int_0^t \Sigma(s,T^*_i,T^*_{i+1})\,\mathrm{d}\widetilde{L}_s\Bigg)\\
&\qquad= F(0,T^*_{i+1},T^*_i)\exp\Bigg(I^1_t + I^2_t + \int_0^t \sqrt{\widetilde{c}_s}\Sigma(s,T^*_i,T^*_{i+1})\,\mathrm{d}W^{T^*_i}_s\\
&\qquad\qquad + \int_0^t\int_{\mathbb{R}^d} \langle \Sigma(s,T^*_i,T^*_{i+1}),x\rangle \left(\mu^{\widetilde{L}} - \widetilde{\nu}^{T^*_i}\right)(\mathrm{d}s,\mathrm{d}x)\Bigg),
\end{aligned}
$$

with

$$
\begin{aligned}
I^1_t &:= \int_0^t \Big(\widetilde{\theta}_s(\Sigma(s,T^*_i)) - \widetilde{\theta}_s(\Sigma(s,T^*_{i+1}))\Big)\,\mathrm{d}s\\
&= \int_0^t \Bigg[- \langle \Sigma(s,T^*_i,T^*_{i+1}),\widetilde{b}_s\rangle\\
&\qquad + \frac{1}{2}\langle \Sigma(s,T^*_i),\widetilde{c}_s\Sigma(s,T^*_i)\rangle - \frac{1}{2}\langle \Sigma(s,T^*_{i+1}),\widetilde{c}_s\Sigma(s,T^*_{i+1})\rangle\\
&\qquad + \int_{\mathbb{R}^d} \Big(e^{\langle \Sigma(s,T^*_i),x\rangle} - e^{\langle \Sigma(s,T^*_{i+1}),x\rangle} + \langle \Sigma(s,T^*_i,T^*_{i+1}),x\rangle\Big)\widetilde{F}_s(\mathrm{d}x)\Bigg]\,\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
I^2_t &:= \int_0^t \langle \Sigma(s,T^*_i,T^*_{i+1}),\widetilde{b}^{T^*_i}_s\rangle\,\mathrm{d}s\\
&\overset{(3.9)}{=} \int_0^t \Bigg[\langle \Sigma(s,T^*_i,T^*_{i+1}),\widetilde{b}_s\rangle + \langle \Sigma(s,T^*_i,T^*_{i+1}),\widetilde{c}_s\Sigma(s,T^*_i)\rangle\\
&\qquad + \int_{\mathbb{R}^d} \langle \Sigma(s,T^*_i,T^*_{i+1}),x\rangle \left(e^{\langle \Sigma(s,T^*_i),x\rangle} - 1\right)\widetilde{F}_s(\mathrm{d}x)\Bigg]\,\mathrm{d}s.
\end{aligned}
$$

Summing up I^1 and I^2 yields

$$\begin{aligned} I_t^1 + I_t^2 = & -\frac{1}{2}\int_0^t \langle \Sigma(s,T_i^*,T_{i+1}^*), \widetilde{c}_s \Sigma(s,T_i^*,T_{i+1}^*)\rangle \, \mathrm{d}s \\ & - \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \Sigma(s,T_i^*,T_{i+1}^*),x\rangle} - 1 - \langle \Sigma(s,T_i^*,T_{i+1}^*),x\rangle \right) \widetilde{F}_s^{T_i^*}(\mathrm{d}x)\,\mathrm{d}s. \end{aligned}$$

Hence, the forward price process in the Lévy term structure model is given by

$$\begin{aligned} & F(t,T_{i+1}^*,T_i^*) \\ & \quad = F(0,T_{i+1}^*,T_i^*) \times \exp\Bigg(-\frac{1}{2}\int_0^t \langle \Sigma(s,T_i^*,T_{i+1}^*), \widetilde{c}_s \Sigma(s,T_i^*,T_{i+1}^*)\rangle \, \mathrm{d}s \\ & \qquad - \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \Sigma(s,T_i^*,T_{i+1}^*),x\rangle} - 1 - \langle \Sigma(s,T_i^*,T_{i+1}^*),x\rangle \right) \widetilde{F}_s^{T_i^*}(\mathrm{d}x)\,\mathrm{d}s \\ & \qquad + \int_0^t \sqrt{\widetilde{c}_s}\Sigma(s,T_i^*,T_{i+1}^*)\,\mathrm{d}W_s^{T_i^*} \\ & \qquad + \int_0^t \int_{\mathbb{R}^d} \langle \Sigma(s,T_i^*,T_{i+1}^*),x\rangle \left(\mu^{\widetilde{L}} - \widetilde{\nu}^{T_i^*}\right)(\mathrm{d}s,\mathrm{d}x)\Bigg). \end{aligned}$$

The next step is to specify the model parameters, that is the volatility structure σ and the characteristics $(\widetilde{b},\widetilde{c},\widetilde{F})$ of $\widetilde{L}$, in such a way that these forward price dynamics match the dynamics given in (3.4)-(3.7). First, we choose the volatility function such that

$$\Sigma(s,T_i^*,T_{i+1}^*) = \lambda(s,T_{i+1}^*).$$

This can be reached by setting

$$\sigma(s,u) := -\sum_{i=0}^{n} \frac{1}{\delta_{i+1}^*}\lambda(s,T_{i+1}^*)\mathbb{1}_{[T_{i+1}^*,T_i^*)}(u)$$

since

$$\Sigma(s,T_i^*,T_{i+1}^*) = \Sigma(s,T_{i+1}^*) - \Sigma(s,T_i^*) = -\int_{T_{i+1}^*}^{T_i^*} \sigma(s,u)\,\mathrm{d}u = \lambda(s,T_{i+1}^*).$$

Of course there are many other possibilities to specify σ. It is also possible to choose a volatility structure σ that is continuous or smooth in the second variable.

Next, we specify the characteristics $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ of the driving process $\widetilde{L}$. For $s \in [0, T^*]$ let $\widetilde{b}_s$ be arbitrary, $\widetilde{c}_s = c_s$ and $\widetilde{F}_s$ such that

$$\widetilde{F}_s(\mathrm{d}x) = \exp\langle -\Sigma(s, T^*), x\rangle F_s^{T^*}(\mathrm{d}x). \tag{3.11}$$

Remember that F^{T^*} is the third characteristic of the driving process L^{T^*} in the Lévy forward price model. Then

$$\begin{aligned}
\widetilde{F}_s^{T_i^*}(\mathrm{d}x) &\overset{(3.10)}{=} \exp\langle \Sigma(s, T_i^*) - \Sigma(s, T^*), x\rangle F_s^{T^*}(\mathrm{d}x) \\
&= \exp\left(\sum_{j=1}^{i} \langle \Sigma(s, T_j^*) - \Sigma(s, T_{j-1}^*), x\rangle\right) F_s^{T^*}(\mathrm{d}x) \\
&= \exp\left(\sum_{j=1}^{i} \langle \lambda(s, T_j^*), x\rangle\right) F_s^{T^*}(\mathrm{d}x)
\end{aligned}$$

and we arrive at the forward price process

$$F(t, T_{i+1}^*, T_i^*) = F(0, T_{i+1}^*, T_i^*) \exp\left(\int_0^t \lambda(s, T_{i+1}^*)\, \mathrm{d}\widetilde{L}_s^{T_i^*}\right),$$

where

$$\widetilde{L}_t^{T_i^*} = \int_0^t b_s^{T_i^*}\, \mathrm{d}s + \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{T_i^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu^{\widetilde{L}} - \widetilde{\nu}^{T_i^*})(\mathrm{d}s, \mathrm{d}x).$$

The $\mathbb{P}_{T_i^*}$-compensator $\widetilde{\nu}^{T_i^*}$ of $\mu^{\widetilde{L}}$ is given by

$$\widetilde{\nu}^{T_i^*}(\mathrm{d}t, \mathrm{d}x) = \exp\left(\sum_{j=1}^{i} \langle \lambda(t, T_j^*), x\rangle\right) F_t^{T^*}(\mathrm{d}x)\, \mathrm{d}t$$

and $(b_s^{T_i^*})$ satisfies

$$\begin{aligned}
\int_0^t \langle \lambda(s, T_{i+1}^*), b_s^{T_i^*}\rangle\, \mathrm{d}s &= -\frac{1}{2} \int_0^t \langle \lambda(s, T_{i+1}^*), \widetilde{c}_s \lambda(s, T_{i+1}^*)\rangle\, \mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s, T_{i+1}^*), x\rangle} - 1 - \langle \lambda(s, T_{i+1}^*), x\rangle\right) \widetilde{\nu}^{T_i^*}(\mathrm{d}s, \mathrm{d}x).
\end{aligned}$$

Hence, we obtain forward price dynamics in the Lévy term structure model as given in (3.4)–(3.7) for the forward price model. Consequently, we can regard the Lévy forward price model as a special case of the Lévy term structure model. In particular, the option pricing formulae developed in chapter 2 can be used.

Remark: This embedding only works for driving processes that are non-homogenous Lévy processes. If both models are driven by a process with stationary increments, that is $F_s^{T^*}$ and $\widetilde{F}_s$ do not depend on s, we usually cannot embed the forward price model into the term structure model. Due to equation (3.11) this only works if $F_s^{T^*} = \widetilde{F}_s = 0$ or in the pathetic case that $\Sigma(s, T^*)$ does not depend on s (which implies that $\Sigma(\cdot, T^*)$ is equal to zero).

3.2 The Lévy Libor model

The goal of this section is to give a short overview over the Lévy Libor model. We are not going to present a construction of the model since this is done in Eberlein and Özkan (2005) in detail. Instead, we list some of the model properties that will be needed for option pricing in the subsequent sections as well as for the Lévy Libor model with default risk which will be discussed in chapter 4.

The model is constructed by backward induction and driven by a non-homogeneous Lévy process L^{T^*} on a complete stochastic basis $(\Omega, \mathcal{F} = \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_s)_{0 \le s \le T^*}, \mathbb{P}_{T^*})$. As in the Lévy forward price model, $\mathbb{P}_{T^*}$ should be regarded as the forward measure associated with the settlement day T^*. L^{T^*} is required to satisfy assumption $(\mathbb{EM})$ and can be written in its canonical decomposition as

$$L_t^{T^*} = \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x).$$

Here, W^{T^*} denotes a standard Brownian motion, μ is the random measure associated with the jumps of L^{T^*}, and $\nu^{T^*}(\mathrm{d}t, \mathrm{d}x) = F_s^{T^*}(\mathrm{d}x)\,\mathrm{d}t$ is the compensator of μ. The characteristics of L^{T^*} are given by $(0, c, F^{T^*})$. Note that without loss of generality we assume L^{T^*} to be driftless. The following assumptions are made:

$(\mathbb{LR}.1)$: For any maturity T_i there is a deterministic function $\lambda(\cdot, T_i) : [0, T^*] \to \mathbb{R}^d$, which represents the volatility of the forward Libor rate process $L(\cdot, T_i)$. In addition,

$$\sum_{i=1}^{n-1} |\lambda^j(s, T_i)| \le M \qquad \text{for all } s \in [0, T^*] \text{ and } j \in \{1, \dots, d\}, \tag{3.12}$$

where M is the constant from assumption $(\mathbb{EM})$ and $\lambda(s, T_i) = 0$ for $s > T_i$.

$(\mathbb{LR}.2)$: The initial term structure $B(0, T_i)$ $(i \in \{1, \dots, n\})$ is strictly positive and strictly decreasing (in i).

The dynamics of the forward Libor rates are specified as

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b^L(s, T_k, T_{k+1})\,\mathrm{d}s + \int_0^t \lambda(s, T_k)\,\mathrm{d}L_s^{T_{k+1}} \right) \tag{3.13}$$

with initial condition

$$L(0,T_k) = \frac{1}{\delta_k}\left(\frac{B(0,T_k)}{B(0,T_{k+1})} - 1\right).$$

$L^{T_{k+1}}$ equals L^{T^*} plus some – in general non-deterministic – drift term which is chosen in such a way that $L^{T_{k+1}}$ is driftless under the forward measure associated with the settlement day T_{k+1}, henceforth denoted by $\mathbb{P}_{T_{k+1}}$. More precisely,

$$L_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x), \tag{3.14}$$

where $W^{T_{k+1}}$ is a standard Brownian motion with respect to $\mathbb{P}_{T_{k+1}}$ and $\nu^{T_{k+1}}$ is the $\mathbb{P}_{T_{k+1}}$-compensator of μ. The drift term $b^L(s,T_k,T_{k+1})$ is specified in such a way that $L(\cdot,T_k)$ becomes a $\mathbb{P}_{T_{k+1}}$-martingale, i.e.

$$\begin{aligned} b^L(s,T_k,T_{k+1}) \;=\; & -\frac{1}{2}\langle \lambda(s,T_k), c_s\lambda(s,T_k)\rangle \\ & - \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s,T_k),x\rangle} - 1 - \langle \lambda(s,T_k), x\rangle\right) F_s^{T_{k+1}}(\mathrm{d}x). \end{aligned} \tag{3.15}$$

The connection between different forward measures is given by

$$\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}} = \prod_{l=k+1}^{n-1} \frac{1+\delta_l L(T_{k+1},T_l)}{1+\delta_l L(0,T_l)} = \frac{B(0,T^*)}{B(0,T_{k+1})} \prod_{l=k+1}^{n-1} (1+\delta_l L(T_{k+1},T_l)). \tag{3.16}$$

Once restricted to the σ-field $\mathcal{F}_t$ this becomes

$$\left.\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t} = \frac{B(0,T^*)}{B(0,T_{k+1})} \prod_{l=k+1}^{n-1} (1+\delta_l L(t,T_l)) \qquad (t \in [0,T_{k+1}]). \tag{3.17}$$

The Brownian motions and compensators with respect to the different measures are connected via

$$W_t^{T_{k+1}} = W_t^{T^*} - \int_0^t \sqrt{c_s}\left(\sum_{l=k+1}^{n-1} \alpha(s,T_l,T_{l+1})\right) \mathrm{d}s \tag{3.18}$$

with

$$\alpha(s,T_l,T_{l+1}) := \frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}\lambda(s,T_l) \tag{3.19}$$

and

$$\nu^{T_{k+1}}(\mathrm{d}t,\mathrm{d}x) = \left(\prod_{l=k+1}^{n-1} \beta(s,x,T_l,T_{l+1})\right) \nu^{T^*}(\mathrm{d}t,\mathrm{d}x) =: F_s^{T_{k+1}}(\mathrm{d}x)\,\mathrm{d}s, \tag{3.20}$$

where

$$\beta(s,x,T_l,T_{l+1}) := \frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}\left(e^{\langle \lambda(s,T_l),x\rangle} - 1\right) + 1. \tag{3.21}$$

Note that $L^{T_{k+1}}$ is usually not a (non-homogeneous) Lévy process under any of the measures $\mathbb{P}_{T_i}$ (except for $k = n-1$, since L^{T^*} is by definition a PIIAC under $\mathbb{P}_{T^*}$). The construction by backward induction guarantees that $\frac{B(\cdot,T_j)}{B(\cdot,T_k)}$ is a $\mathbb{P}_{T_k}$-martingale for all $j,k \in \{1,\dots,n\}$.

3.2.1 Valuation of caps and floors

Eberlein and Özkan (2005) present a valuation formula for caps (and, therewith, via the cap-floor-parity, also for floors) based on an approximation and Laplace transform methods. In this section, we derive an alternative pricing formula by making use of the same methods but a different approximation. The advantage of our formula is the much higher speed at which it can be evaluated numerically. Both approximations yield exact prices if the driving process is continuous. At the end of this section, we consider an example to compare the two approximations.

Remember that a *cap* (resp. *floor*) is a series of call (resp. put) options on subsequent Libor rates. These single options are called *caplets* (resp. *floorlets*). The time-t price of a caplet with strike K and maturity T_i is given by

$$C_t(K,T_i) := \delta_i B(t,T_{i+1}) \mathbb{E}_{\mathbb{P}_{T_{i+1}}}[(L(T_i,T_i)-K)^+ \,|\, \mathcal{F}_t].$$

A problem in evaluating the expression on the right-hand side arises from the fact that $L(\cdot,T_i)$ is not – generally – driven by a non-homogeneous Lévy process under $\mathbb{P}_{T_{i+1}}$. To put it differently, the random measure associated with the jumps of the driving process does not possess a deterministic $\mathbb{P}_{T_{i+1}}$-compensator (except for the case $i = n-1$, i.e. $\mathbb{P}_{T_{i+1}} = \mathbb{P}_{T^*}$). Eberlein and Özkan (2005) solve this problem by approximating the non-deterministic compensator with a deterministic one. Concretely, they replace the stochastic term $\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}$ in (3.21) by its deterministic initial value $\frac{\delta_l L(0,T_l)}{1+\delta_l L(0,T_l)}$. Combined with Laplace transformation techniques, this leads to the following approximation for the price of the caplet (compare Eberlein and Özkan (2005, Theorem 5.1)):

Proposition 3.1 *The price of a caplet is approximately given by*

$$\begin{aligned} C_0(K,T_i) &= \delta_i B(0,T_{i+1}) K \frac{1}{\pi} \qquad (3.22)\\ &\quad \times \int_0^\infty \Re\left(\left(\frac{K}{L(0,T_i)}\right)^{R+\mathrm{i}u} \frac{1}{(R+\mathrm{i}u)(R+1+\mathrm{i}u)}\,\chi(\mathrm{i}R-u)\right)\mathrm{d}u, \end{aligned}$$

where

$$\begin{aligned} \chi(z) &= \exp\Bigg(-\frac{1}{2}\int_0^{T_i} (z^2+\mathrm{i}z)\langle \lambda(s,T_i), c_s\lambda(s,T_i)\rangle\,\mathrm{d}s \\ &\qquad + \int_0^{T_i}\int_{\mathbb{R}^d} \left(e^{\mathrm{i}z\langle\lambda(s,T_i),x\rangle} - 1 - \mathrm{i}z e^{\langle\lambda(s,T_i),x\rangle} + \mathrm{i}z\right) \widetilde{\nu}^{T_{i+1}}(\mathrm{d}s,\mathrm{d}x)\Bigg) \end{aligned}$$

and $\widetilde{\nu}^{T_{i+1}}$ *is an approximation for* $\nu^{T_{i+1}}$ *given by*

$$\widetilde{\nu}^{T_{i+1}}(\mathrm{d}s,\mathrm{d}x) = \prod_{k=i+1}^{n-1} \left(\frac{\delta_k L(0,T_k)}{1+\delta_k L(0,T_k)} \left(e^{\langle \lambda(s,T_k),x\rangle} - 1 \right) + 1 \right) \nu^{T^*}(\mathrm{d}s,\mathrm{d}x).$$

Here, $R < -1$ *has to be chosen in such a way that* $\chi(\mathrm{i}R) < \infty$ *and it is assumed that* $\int_{-\infty}^{\infty} |\chi(u)|\,\mathrm{d}u < \infty$.

To evaluate the expression in (3.22) one usually has to deal with a triple integral, whose numerical evaluation is time consuming. Using a different approximation, we can get a pricing formula that only involves a double integral, provided that the characteristic exponent θ_s of the infinitely divisible distribution associated with the Lévy triplet $(0, c_s, F_s^{T^*})$ is known in closed form. Note that

$$\begin{aligned}
C_0(K,T_i) &= \delta_i B(0,T_{i+1}) \mathbb{E}_{\mathbb{P}_{T_{i+1}}}[(L(T_i,T_i)-K)^+] \\
&\overset{(3.16)}{=} \delta_i B(0,T_{i+1}) \frac{B(0,T^*)}{B(0,T_{i+1})} \\
&\qquad \mathbb{E}_{\mathbb{P}_{T^*}}\left[\prod_{k=i+1}^{n-1} (1+\delta_k L(T_i,T_k))(L(T_i,T_i)-K)^+ \right] \\
&= \delta_i B(0,T^*) K\, \mathbb{E}_{\mathbb{P}_{T^*}}[(M^1_{T_i} - M^2_{T_i})^+],
\end{aligned}$$

where the $\mathbb{P}_{T^*}$-martingales $(M^1_t)_{0\le t\le T_i}$ and $(M^2_t)_{0\le t\le T_{i+1}}$ are given by

$$\begin{aligned}
M^1_t &:= \prod_{k=i+1}^{n-1} (1+\delta_k L(t,T_k)) \frac{L(t,T_i)}{K} \\
&= \prod_{k=i+1}^{n-1} \left(1 + \delta_k L(0,T_k) \exp\left(\int_0^t b^L(s,T_k,T_{k+1})\,\mathrm{d}s + \int_0^t \lambda(s,T_k)\,\mathrm{d}L_s^{T_{k+1}} \right) \right) \\
&\qquad \times \frac{L(0,T_i)}{K} \exp\left(\int_0^t b^L(s,T_i,T_{i+1})\,\mathrm{d}s + \int_0^t \lambda(s,T_i)\,\mathrm{d}L_s^{T_{i+1}} \right) \\
&= \prod_{k=i+1}^{n-1} \left(1 + \delta_k L(0,T_k) \exp\left(\int_0^t \lambda(s,T_k)\,\mathrm{d}L_s^{T^*} + \text{drift} \right) \right) \\
&\qquad \times \frac{L(0,T_i)}{K} \exp\left(\int_0^t \lambda(s,T_i)\,\mathrm{d}L_s^{T^*} + \text{drift} \right)
\end{aligned}$$

since, for all i, L^{T_i} and L^{T^*} differ only by a (non-deterministic) drift; similarly,

$$\begin{aligned}
M^2_t &:= \prod_{k=i+1}^{n-1} (1+\delta_k L(t,T_k)) \\
&= \prod_{k=i+1}^{n-1} \left(1 + \delta_k L(0,T_k) \exp\left(\int_0^t \lambda(s,T_k)\,\mathrm{d}L_s^{T^*} + \text{drift} \right) \right).
\end{aligned}$$

In the following, we approximate M^1 and M^2 by two processes for which we require that they remain martingales with respect to $\mathbb{P}_{T^*}$. Let us exploit the fact that $1+\varepsilon\exp(x) \approx (1+\varepsilon)\exp\left(\frac{\varepsilon}{1+\varepsilon}x\right)$ for small absolute values of x and approximate

$$1+\delta_k L(0,T_k)\exp\left(\int_0^t \lambda(s,T_k)\,\mathrm{d}L_s^{T^*} + \text{drift}\right)$$

by

$$(1+\delta_k L(0,T_k))\exp\left(\int_0^t \frac{\delta_k L(0,T_k)}{1+\delta_k L(0,T_k)}\lambda(s,T_k)\,\mathrm{d}L_s^{T^*} + \text{new drift}\right).$$

We obtain the approximations

$$\widetilde{M}_t^1 := \frac{L(0,T_i)}{K}\frac{B(0,T_{i+1})}{B(0,T^*)}\exp\left(\int_0^t (f^i(s)+\lambda(s,T_i))\,\mathrm{d}L_s^{T^*} + \text{drift}\right)$$

and

$$\widetilde{M}_t^2 := \frac{B(0,T_{i+1})}{B(0,T^*)}\exp\left(\int_0^t f^i(s)\,\mathrm{d}L_s^{T^*} + \text{drift}\right),$$

where

$$f^i(s) := \sum_{k=i+1}^{n-1}\frac{\delta_k L(0,T_k)}{1+\delta_k L(0,T_k)}\lambda(s,T_k). \tag{3.23}$$

We derive the drift terms from our requirement that $\widetilde{M}^1$ and $\widetilde{M}^2$ have to be $\mathbb{P}_{T^*}$-martingales and get

$$\begin{aligned}
\widetilde{M}_t^1 &= \frac{L(0,T_i)}{K}\frac{B(0,T_{i+1})}{B(0,T^*)}\exp\left(\int_0^t (f^i(s)+\lambda(s,T_i))\,\mathrm{d}L_s^{T^*} + D_t^1\right),\\
\widetilde{M}_t^2 &= \frac{B(0,T_{i+1})}{B(0,T^*)}\exp\left(\int_0^t f^i(s)\,\mathrm{d}L_s^{T^*} + D_t^2\right)
\end{aligned} \tag{3.24}$$

with

$$\begin{aligned}
D_t^1 &:= \log\left(\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^t (f^i(s)+\lambda(s,T_i))\,\mathrm{d}L_s^{T^*}\right]^{-1}\right),\\
D_t^2 &:= \log\left(\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^t f^i(s)\,\mathrm{d}L_s^{T^*}\right]^{-1}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}C_0(K,T_i) &\approx \delta_i B(0,T^*)K\,\mathbb{E}_{\mathbb{P}_{T^*}}[(\widetilde{M}^1_{T_i} - \widetilde{M}^2_{T_i})^+]\\ &= \delta_i B(0,T^*)K\,\mathbb{E}_{\mathbb{P}_{T^*}}\left[\widetilde{M}^2_{T_i}\left(\frac{\widetilde{M}^1_{T_i}}{\widetilde{M}^2_{T_i}} - 1\right)^+\right].\end{aligned}$$

Next, we make use of the change of numeraire technique and define a new measure $\widetilde{\mathbb{P}}_{T_{i+1}}$ on $(\Omega, \mathcal{F}_{T_{i+1}})$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{T_{i+1}}}{\mathrm{d}\mathbb{P}_{T^*}} := \frac{\widetilde{M}^2_{T_{i+1}}}{\widetilde{M}^2_0} = \exp\left(\int_0^{T_{i+1}} f^i(s)\,\mathrm{d}L^{T^*}_s + D^2_{T_{i+1}}\right). \tag{3.25}$$

Then, denoting

$$X_t := \log\frac{\widetilde{M}^1_t}{\widetilde{M}^2_t} = \log\frac{L(0,T_i)}{K} + \int_0^t \lambda(s,T_i)\,\mathrm{d}L^{T^*}_s + D^1_t - D^2_t$$

and assuming that the distribution of X_{T_i} with respect to $\widetilde{\mathbb{P}}_{T_{i+1}}$ possesses a Lebesgue-density φ, we get

$$\begin{aligned}C_0(K,T_i) &\approx \delta_i B(0,T_{i+1})K\mathbb{E}_{\widetilde{\mathbb{P}}_{T_{i+1}}}\left[\left(e^{X_{T_i}} - 1\right)^+\right]\\ &= \delta_i B(0,T_{i+1})K(g*\varphi)(0),\end{aligned} \tag{3.26}$$

where $g(x) := (e^{-x} - 1)^+$. We obtain the following approximation for the caplet price:

Proposition 3.2 *Suppose that the distribution of X_{T_i} possesses a Lebesgue-density. Denote by $\widetilde{M}^{X_{T_i}}_{T_{i+1}}$ the $\widetilde{\mathbb{P}}_{T_{i+1}}$-moment generating function of X_{T_i}. Choose an $R < -1$ such that $\widetilde{M}^{X_{T_i}}_{T_{i+1}}(-R) < \infty$. Then approximately*

$$\begin{aligned}C_0(K,T_i) = \delta_i B(0,T_{i+1})K\frac{1}{\pi}\int_0^\infty \Re\Bigg(\left(\frac{K}{L(0,T_i)}\right)^{R+\mathrm{i}u} &\frac{1}{(R+\mathrm{i}u)(R+1+\mathrm{i}u)}\\ \times\exp\int_0^{T_i}\Big(\theta_s(f^i(s) - (R+\mathrm{i}u)\lambda(s,T_i))&\\ +(R+\mathrm{i}u)\,\theta_s(f^i(s)+\lambda(s,T_i)) - (R+1+\mathrm{i}u)\,\theta_s(f^i(s))\Big)\,\mathrm{d}s&\Bigg)\,\mathrm{d}u\end{aligned}$$

with f^i given by equation (3.23).

Proof: Proceeding in the same way as in the proof of theorem 2.16 we obtain

$$\begin{aligned}C_0(K,T_i) = \delta_i B(0,T_{i+1})K\frac{1}{\pi}&\\ \times\int_0^\infty \Re\left(\frac{1}{(R+\mathrm{i}u)(R+1+\mathrm{i}u)}\widetilde{M}^{X_{T_i}}_{T_{i+1}}(-R-\mathrm{i}u)\right)\mathrm{d}u.&\end{aligned}$$

The claim now follows from the fact that for $z \in \mathbb{C}$ with $\Re z = -R$ we have

$$\begin{aligned}
\widetilde{M}^{X_{T_i}}_{T_{i+1}}(z) &= \mathbb{E}_{\widetilde{\mathbb{P}}_{T_{i+1}}}\left[\left(\frac{L(0,T_i)}{K}\right)^z \exp\left(\int_0^{T_i} z\lambda(s,T_i)\,\mathrm{d}L_s^{T^*} + z(D^1_{T_i} - D^2_{T_i})\right)\right] \\
&= \left(\frac{L(0,T_i)}{K}\right)^z \mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^{T_i} f^i(s)\,\mathrm{d}L_s^{T^*}\right]^{-1} \\
&\quad \times \mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^{T_i} (f^i(s) + z\lambda(s,T_i))\,\mathrm{d}L_s^{T^*}\right] \\
&\quad \times \left(\frac{\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^{T_i} f^i(s)\,\mathrm{d}L_s^{T^*}\right]}{\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^{T_i} (f^i(s) + \lambda(s,T_i))\,\mathrm{d}L_s^{T^*}\right]}\right)^z \\
&= \left(\frac{L(0,T_i)}{K}\right)^z \exp\int_0^{T_i}\Big(\theta_s(f^i(s) + z\lambda(s,T_i)) \\
&\qquad\qquad -z\,\theta_s(f^i(s) + \lambda(s,T_i)) + (z-1)\,\theta_s(f^i(s))\Big)\,\mathrm{d}s,
\end{aligned}$$

where the second equality follows from (3.25) and the last equality results from proposition 1.9. □

The pricing formula of Eberlein and Özkan (2005) is exact if the model is driven by a Brownian motion since in this case there is no compensator which has to be approximated. The above approximation is also exact for a driving Brownian motion as the following proposition shows:

Proposition 3.3 *The above approximation yields the exact price for the caplet if the driving process does not have jumps.*

PROOF: Equation (3.26) yields the approximate caplet price

$$\delta_i B(0,T_{i+1})\mathbb{E}_{\widetilde{\mathbb{P}}_{T_{i+1}}}\left[\left(L(0,T_i)\exp\left(\int_0^{T_i}\sqrt{c_s}\lambda(s,T_i)\,\mathrm{d}\widetilde{W}_s^{T_{i+1}} + \text{drift}\right) - K\right)^+\right],$$

where $\widetilde{W}^{T_{i+1}}$ denotes a $\widetilde{\mathbb{P}}_{T_{i+1}}$-standard Brownian motion. From the construction above it is clear that $(\exp X_t)_{0\le t\le T_i}$ is a $\widetilde{\mathbb{P}}_{T_{i+1}}$-martingale. We can thus determine the drift and get

$$\begin{aligned}
C_0(K,T_i) \approx \delta_i B(0,T_{i+1})\mathbb{E}_{\widetilde{\mathbb{P}}_{T_{i+1}}}\Bigg[\Bigg(L(0,T_i)\exp\Bigg(&\int_0^{T_i}\sqrt{c_s}\lambda(s,T_i)\,\mathrm{d}\widetilde{W}_s^{T_{i+1}} \\
&-\frac{1}{2}\int_0^{T_i}\langle\lambda(s,T_i), c_s\lambda(s,T_i)\rangle\,\mathrm{d}s\Bigg) - K\Bigg)^+\Bigg].
\end{aligned}$$

On the other hand, the exact price for the caplet is given by

$$\begin{aligned} C_0(K,T_i) &= \delta_i B(0,T_{i+1}) \mathbb{E}_{\mathbb{P}_{T_{i+1}}}[(L(T_i,T_i)-K)^+] \\ &= \delta_i B(0,T_{i+1}) \mathbb{E}_{\mathbb{P}_{T_{i+1}}}\Bigg[\Bigg(L(0,T_i)\exp\Bigg(\int_0^{T_i}\sqrt{c_s}\lambda(s,T_i)\,\mathrm{d}W_s^{T_{i+1}} \\ &\qquad -\frac{1}{2}\int_0^{T_i}\langle\lambda(s,T_i),c_s\lambda(s,T_i)\rangle\,\mathrm{d}s\Bigg)-K\Bigg)^+\Bigg]. \end{aligned}$$

Hence, the approximate caplet price is exact. □

In the remaining part of this section, let us consider an example to compare the caplet price approximation suggested by Eberlein and Özkan (2005) with ours. Suppose that

$$T_0 = 0, \quad T_1 = 0.5, \quad T_2 = 1, \quad T_3 = 1.5, \quad \ldots, \quad T_{10} = 5 = T^*.$$

We take discount factors (zero coupon bond prices) as quoted on February 19, 2002 (see table 2.1) and constant volatilities

$$\begin{aligned} &\lambda(s,T_1) = 0.20 \qquad \lambda(s,T_2) = 0.19 \qquad \lambda(s,T_3) = 0.18 \\ &\lambda(s,T_4) = 0.17 \qquad \lambda(s,T_5) = 0.16 \qquad \lambda(s,T_6) = 0.15 \\ &\lambda(s,T_7) = 0.14 \qquad \lambda(s,T_8) = 0.13 \qquad \lambda(s,T_9) = 0.12. \end{aligned}$$

For the driving process, three different cases are considered:

(d1): The driving process is a standard Brownian motion. In this case we are in the market model of Brace, Gatarek, and Musiela (1997) (henceforth BGM model).

(d2): The driving process is a Lévy process generated by the NIG-distribution with parameters $\alpha = \delta = 100$, $\mu = \beta = 0$. This distribution has zero mean, variance one, and it is very close to the standard normal distribution (see Eberlein and v. Hammerstein (2004) for a survey of limiting cases of generalized hyperbolic distributions).

(d3): The driving process is a Lévy process generated by the NIG-distribution with parameters $\alpha = \delta = 1.5$, $\mu = \beta = 0$. This distribution has zero mean and variance one, but it has much fatter tails than the standard normal distribution. Keep in mind that the parameters have to be chosen in such a way that they satisfy (3.12). For an NIG-distribution with $\beta = 0$ this means we have to have $\alpha > \sum_{i=1}^{9}|\lambda(s,T_i)| = 1.44$.

We price caplets with maturities ranging from T_1 to T_9 and strike rates ranging from 2.5 % to 7 %. To calculate the option prices for (d1) the market formula can be employed. In cases (d2) and (d3) we use the approximation of Eberlein and Özkan (2005) and the approximation developed above.

The results can be found at the end of this chapter. Table 3.1 gives the caplet prices for scenario (d1). Using distribution (d2) and any of the two approximations leads to exactly the same caplet prices (up to truncation of 1/1000 of a basis point) as in table 3.1 and hence to the BGM-implied volatilities as given in table 3.2. Tables 3.3 and 3.5 show caplet prices for scenario (d3) using the approximation of Eberlein and Özkan (2005) and the approximation developed above. The corresponding BGM-implied volatilities are given in tables 3.4 and 3.6. From these results we can draw some conclusions:

1. Not only are both approximations exact in the case of a driving Brownian motion, they also (at least in this example) produce caplet prices for a (purely discontinuous) driving Lévy process whose underlying distribution is close to standard normal (case (d2)) that virtually perfectly fit the BGM-prices.
2. In case (d3) a smile in the implied volatilities can be observed. This is not surprising since the underlying distribution has fat tails. Note that the two approximations produce almost the same caplet prices.
3. In case (d3) a phenomenon can be observed that already occurred in the Lévy term structure model. For this driving homogeneous Lévy process the smile in the implied volatilities flattens as time to maturity increases.

Let us shortly comment on the time that is needed to calculate the prices for this set of 90 caplets. Naturally, the time depends on many factors as e.g. on the choice of R, the upper limit of integration in the infinite integral as well as on the numerical integration algorithm or the error bound in the numerical integration. For choices that we consider to be reasonable, the following amounts of time were needed on a personal computer: using the approximation of Eberlein and Özkan (2005), the computation of the caplet prices lasted 385,04 and 436.31 seconds in the scenarios (d2) and (d3) respectively. The calculation using our approximation lasted 0.12 and 0.09 seconds respectively. Besides the additional integral, the fact that the Lebesgue-density of the Lévy measure of an NIG-distribution contains a Bessel function made the first approximation slower.

3.2.2 Swaption pricing

The aim of this section is to provide a swaption pricing formula whose numerical evaluation can be done fast. We use an approximation that has already been employed by Brace, Gatarek, and Musiela (1997) in the Libor market model. Moreover, we put the following restriction on the volatility structure which is similar to assumption ($\mathbb{VOL}$) in the Lévy term structure model:

Assumption ($\mathbb{LR}.\mathbb{VOL}$). *The volatility structure factorizes, i.e.*

$$\lambda(s, T_i) = \lambda_i \lambda(s) \qquad (0 \leq s \leq T_i,\ i \in \{1, \ldots, n-1\})$$

where λ_i is a positive constant and $\lambda : [0, T^*] \to \mathbb{R}^d$ does not depend on i.

Examples of volatility structures for Libor models that satisfy this assumption can be found in the book of Brigo and Mercurio (2001, Section 6.3.1).

Remember that a payer (resp. receiver) swaption can be seen as a put (resp. call) option on a coupon bond with an exercise price of 1 (compare Musiela and Rutkowski (1998, Section 16.3.2)). Consider a payer swaption with strike rate K where the underlying is a swap that starts at option maturity T_i and matures at T_m ($i < m \le n$). It's time-T_i value is given by

$$\begin{aligned}\pi_{T_i}(K, T_i, T_m) &:= \left(1 - \sum_{k=i+1}^{m} c_k B(T_i, T_k)\right)^+ \\ &= \left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} (1 + \delta_l L(T_i, T_l))^{-1}\right)\right)^+\end{aligned}$$

with $c_k := K$ for $i + 1 \le k \le m - 1$ and $c_m := 1 + K$.

To calculate todays value we proceed as before, that is, in order to be able to apply Laplace transform methods, we derive a convolution representation for the option price first. The value of the swaption is obtained by taking the $\mathbb{P}_{T_i}$-expectation of its time-T_i value. More precisely,

$$\begin{aligned}\pi_0 &:= \pi_0(K, T_i, T_m) \\ &= B(0, T_i)\mathbb{E}_{\mathbb{P}_{T_i}}\left[\left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} (1 + \delta_l L(T_i, T_l))^{-1}\right)\right)^+\right] \\ &\overset{(3.16)}{=} B(0, T^*) \\ &\quad \times \mathbb{E}_{\mathbb{P}_{T^*}}\left[\prod_{l=i}^{n-1} (1 + \delta_l L(T_i, T_l)) \left(1 - \sum_{k=i+1}^{m} \left(c_k \prod_{l=i}^{k-1} (1 + \delta_l L(T_i, T_l))^{-1}\right)\right)^+\right] \\ &= B(0, T^*)\mathbb{E}_{\mathbb{P}_{T^*}}\left[\left(- \sum_{k=i}^{m} \left(c_k \prod_{l=k}^{n-1} (1 + \delta_l L(T_i, T_l))\right)\right)^+\right]\end{aligned}$$

with $c_i := -1$. Combining (3.13), (3.14), and (3.15) with (3.18)–(3.21) yields

$$\begin{aligned}L(t, T_l) = L(0, T_l) \exp\Bigg(&\int_0^t b^L(s, T_l, T^*)\, \mathrm{d}s + \int_0^t \sqrt{c_s}\lambda(s, T_l)\, \mathrm{d}W_s^{T^*} \\ &+ \int_0^t \int_{\mathbb{R}^d} \langle \lambda(s, T_l), x\rangle (\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x)\Bigg)\end{aligned}$$

where

$$\begin{aligned}&b^L(s, T_l, T^*) := \\ &\quad -\frac{1}{2}\langle \lambda(s, T_l), c_s\lambda(s, T_l)\rangle - \sum_{j=l+1}^{n-1} \frac{\delta_j L(s-, T_j)}{1 + \delta_j L(s-, T_j)} \langle \lambda(s, T_j), c_s\lambda(s, T_l)\rangle \\ &\quad - \int_{\mathbb{R}^d} \left(\left(e^{\langle \lambda(s, T_l), x\rangle} - 1\right) \prod_{j=l+1}^{n-1} \beta(s, x, T_j, T_{j+1}) - \langle \lambda(s, T_l), x\rangle\right) F_s^{T^*}(\mathrm{d}x).\end{aligned}$$

We approximate the stochastic term $\frac{\delta_j L(s-,T_j)}{1+\delta_j L(s-,T_j)}$ in the drift by its starting value $\frac{\delta_j L(0,T_j)}{1+\delta_j L(0,T_j)}$ (remember that $\frac{\delta_j L(s-,T_j)}{1+\delta_j L(s-,T_j)}$ is also contained in $\beta(s,x,T_j,T_{j+1})$) and call the resulting approximation for the drift term $b_0^L(s,T_l,T^*)$. Similar approximations have already been employed by Brace, Gatarek, and Musiela (1997), Rebonato (1998), and Schlögl (2002). Using the assumption ($\mathbb{LR}.\mathbb{VOL}$) on the volatility structure yields

$$\begin{aligned}\pi_0 &\approx B(0,T^*)\\ &\quad\times \mathbb{E}_{\mathbb{P}_{T^*}}\left[\left(-\sum_{k=i}^{m}\left(c_k\prod_{l=k}^{n-1}\left(1+\delta_l L(0,T_l)\exp\left(\frac{\lambda_l}{\lambda_{\mathrm{sum}}}X_{T_i}+B_l\right)\right)\right)\right)^+\right]\end{aligned}$$

with

$$\begin{aligned}\lambda_{\mathrm{sum}} &:= \sum_{l=i}^{n-1}\lambda_l,\\ X_t &:= \int_0^t\sum_{l=i}^{n-1}\lambda(s,T_l)\,\mathrm{d}L_s^{T^*} = \lambda_{\mathrm{sum}}\int_0^t\lambda(s)\,\mathrm{d}L_s^{T^*},\\ B_l &:= \int_0^{T_i} b_0^L(s,T_l,T^*)\,\mathrm{d}s.\end{aligned}$$

Note that, due to the assumption on the volatility structure, we have derived a representation for the price of the swaption that depends only on the distribution of one random variable, namely on the distribution of X_{T_i} with respect to $\mathbb{P}_{T^*}$. Assume that this distribution possesses a Lebesgue-density φ, then

$$\begin{aligned}\pi_0 &\approx B(0,T^*)\\ &\quad\times\int_{\mathbb{R}}\left(-\sum_{k=i}^{m}\left(c_k\prod_{l=k}^{n-1}\left(1+\delta_l L(0,T_l)\exp\left(\frac{\lambda_l}{\lambda_{\mathrm{sum}}}x+B_l\right)\right)\right)\right)^+\varphi(x)\,\mathrm{d}x\\ &= B(0,T^*)(g*\varphi)(0)\end{aligned}\tag{3.27}$$

with $g(x) := (v(x))^+$ and

$$v(x) := -\sum_{k=i}^{m}\left(c_k\prod_{l=k}^{n-1}\left(1+\delta_l L(0,T_l)\exp\left(-\frac{\lambda_l}{\lambda_{\mathrm{sum}}}x+B_l\right)\right)\right),\tag{3.28}$$

i.e. we have derived a convolution representation for the price of the swaption.

The next step is to determine the bilateral Laplace transform of g. Observe that v has a unique zero; let us write v in a more complicated form as

$$\begin{aligned}v(x) &= \prod_{l=i}^{n-1}\left(1+\delta_l L(0,T_l)\exp\left(-\frac{\lambda_l}{\lambda_{\mathrm{sum}}}x+B_l\right)\right)\\ &\quad\times\left(1-\sum_{k=i+1}^{m}\left(c_k\prod_{l=i}^{k-1}\left(1+\delta_l L(0,T_l)\exp\left(-\frac{\lambda_l}{\lambda_{\mathrm{sum}}}x+B_l\right)\right)^{-1}\right)\right).\end{aligned}$$

Since the first $(n-i)$ factors on the right hand side are strictly positive and the last factor is continuous, strictly decreasing, and takes positive as well as negative values, v has a unique zero Z. Consequently,

$$g(x) = v(x)\mathbb{1}_{(-\infty,Z]}(x).$$

Note that $v(x)$ can be written (compare (3.28)) as a finite sum of expressions of the type "$c_1 \exp(-c_2 x)$" with $c_1 \in \mathbb{R}$ and $c_2 \in [0,1]$. For $z \in \mathbb{C}$ with $\Re z < -1$ we get

$$\int_{\mathbb{R}} e^{-zx} \left(c_1 e^{-c_2 x}\right) \mathbb{1}_{(-\infty,Z]}(x)\,\mathrm{d}x = \frac{-c_1}{z+c_2} e^{-Z(z+c_2)}.$$

Hence, the Laplace transform of g exists for all $z \in \mathbb{C}$ with $\Re z < -1$ and a closed form expression (depending on Z) can be derived. However, since the number of summands of the above form in v increases exponentially as $(n-i)$ increases, a numerical evaluation of the Laplace transform is (at least for large values of $(n-i)$) more appropriate. Note that we can save computational time by applying the following multiplication scheme to $v(x)$:

$$\sum_{k=i}^{m} c_k \prod_{l=k}^{n-1} d_l = \Big(c_m + d_{m-1}(c_{m-1} + d_{m-2}(c_{m-2} + d_{m-3}(\ldots(c_{i+1} + d_i c_i))))\Big) \prod_{l=m}^{n-1} d_l.$$

Putting pieces together, we obtain the following formula for the swaption price:

Proposition 3.4 *Suppose that the distribution of X_{T_i} possesses a Lebesgue-density. Denote by $M_{T^*}^{X_{T_i}}$ the $\mathbb{P}_{T^*}$-moment generating function of X_{T_i}. Choose an $R < -1$ such that $M_{T^*}^{X_{T_i}}(-R) < \infty$. Then approximately*

$$\pi_0(K, T_i, T_m) = B(0,T^*)\frac{1}{\pi}\int_0^\infty \Re\Big(L[g](R+\mathrm{i}u) \times \exp\int_0^{T_i} \theta_s(z\lambda_{sum}\lambda(s))\,\mathrm{d}s\Big).$$

Proof: Using the convolution representation (3.27) and performing Laplace and inverse Laplace transformations as usual, we get

$$\pi_0 = B(0,T^*)\frac{1}{\pi}\int_0^\infty \Re\Big(L[g](R+\mathrm{i}u) M_{T^*}^{X_{T_i}}(-R-\mathrm{i}u)\Big)\,\mathrm{d}u.$$

It remains to derive an expression for the moment generating function. For $z \in \mathbb{C}$ with $\Re z = -R$ we have

$$\begin{aligned} M_{T^*}^{X_{T_i}}(z) &= \mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\left(z\lambda_{\mathrm{sum}}\int_0^{T_i} \lambda(s)\,\mathrm{d}L_s^{T^*}\right)\right] \\ &= \exp\int_0^{T_i} \theta_s(z\lambda_{\mathrm{sum}}\lambda(s))\,\mathrm{d}s, \end{aligned}$$

where the last line follows from proposition 1.9. □

Remark: We can also use the approximation employed in this section to price caplets and floorlets. However, volatility structures that do not satisfy assumption ($\mathbb{LR}.\mathbb{VOL}$) can then not be considered.

strike	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %
T_1	65.656	41.876	21.027	7.667	2.015
T_2	93.623	70.228	48.032	29.218	15.671
T_3	91.603	68.989	48.043	30.599	17.832
T_4	109.422	87.172	65.916	46.932	31.411
T_5	106.807	85.187	64.677	46.481	31.617
T_6	114.792	93.622	73.276	54.745	39.003
T_7	111.898	91.275	71.474	53.461	38.164
T_8	116.996	96.827	77.249	59.083	43.236
T_9	113.930	94.266	75.135	57.332	41.774
strike	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	0.398	0.063	0.008	0.001	0.000
T_2	7.461	3.202	1.261	0.463	0.161
T_3	9.590	4.817	2.288	1.040	0.456
T_4	19.834	11.896	6.830	3.783	2.035
T_5	20.441	12.641	7.531	4.352	2.453
T_6	26.569	17.389	10.997	6.758	4.057
T_7	26.074	17.131	10.885	6.726	4.062
T_8	30.318	20.450	13.333	8.447	5.224
T_9	29.100	19.453	12.542	7.841	4.778

Table 3.1: Scenarios (d1) and (d2): Caplet prices in basis points.

strike	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.01	20.01
T_2	19.00	19.00	19.00	19.00	19.00	19.00	19.00	19.00	19.00	19.00
T_3	18.00	18.00	18.00	18.00	18.00	18.00	18.00	18.00	18.00	18.00
T_4	17.00	17.00	17.00	17.00	17.00	17.00	17.00	17.00	17.00	17.00
T_5	16.00	16.00	16.00	16.00	16.00	16.00	16.00	16.00	16.00	16.00
T_6	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00
T_7	14.00	14.00	14.00	14.00	14.00	14.00	14.00	14.00	14.00	14.00
T_8	13.00	13.00	13.00	13.00	13.00	13.00	13.00	13.00	13.00	13.00
T_9	12.00	12.00	12.00	12.00	12.00	12.00	12.00	12.00	12.00	12.00

Table 3.2: Scenario (d2): BGM-implied volatilities of the caplet prices (in %) using any of the two approximations.

strike	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %
T_1	65.717	42.047	20.617	7.054	2.296
T_2	93.666	70.364	48.072	28.768	15.041
T_3	91.663	69.103	47.990	30.191	17.348
T_4	109.462	87.269	65.968	46.750	30.990
T_5	106.848	85.271	64.704	46.307	31.261
T_6	114.822	93.693	73.328	54.658	38.736
T_7	111.924	91.337	71.519	53.384	37.934
T_8	117.014	96.876	77.304	59.058	43.075
T_9	113.943	94.307	75.186	57.318	41.640
strike	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	0.867	0.374	0.179	0.092	0.051
T_2	7.305	3.542	1.781	0.938	0.518
T_3	9.425	5.035	2.718	1.504	0.857
T_4	19.420	11.729	6.967	4.134	2.477
T_5	20.095	12.489	7.623	4.630	2.824
T_6	26.231	17.142	10.940	6.895	4.329
T_7	25.783	16.914	10.826	6.831	4.285
T_8	30.063	20.212	13.207	8.466	5.367
T_9	28.879	19.243	12.428	7.853	4.897

Table 3.3: Scenario (d3): Caplet prices in basis points using the approximation of Eberlein and Özkan.

strike	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	25.33	21.62	18.94	18.81	20.87	23.28	25.49	27.45	29.19	30.74
T_2	22.48	20.56	19.12	18.32	18.25	18.79	19.64	20.58	21.52	22.41
T_3	19.89	18.69	17.89	17.51	17.52	17.82	18.30	18.85	19.43	19.99
T_4	18.67	17.77	17.14	16.76	16.62	16.67	16.86	17.14	17.47	17.82
T_5	17.19	16.52	16.06	15.80	15.71	15.75	15.88	16.08	16.32	16.58
T_6	16.08	15.52	15.13	14.89	14.78	14.77	14.84	14.96	15.12	15.30
T_7	14.88	14.42	14.10	13.91	13.82	13.81	13.86	13.96	14.09	14.23
T_8	13.85	13.44	13.15	12.97	12.87	12.84	12.86	12.92	13.01	13.12
T_9	12.76	12.39	12.14	11.98	11.89	11.87	11.88	11.93	12.01	12.10

Table 3.4: Scenario (d3): BGM-implied volatilities of the caplet prices (in %) using the approximation of Eberlein and Özkan.

strike	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %
T_1	65.716	42.044	20.609	7.041	2.285
T_2	93.665	70.362	48.066	28.756	15.024
T_3	91.663	69.100	47.982	30.178	17.331
T_4	109.461	87.268	65.964	46.740	30.976
T_5	106.848	85.269	64.700	46.300	31.250
T_6	114.822	93.692	73.326	54.653	38.729
T_7	111.924	91.336	71.517	53.381	37.930
T_8	117.014	96.876	77.304	59.057	43.073
T_9	113.943	94.307	75.186	57.318	41.640
strike	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	0.859	0.369	0.175	0.090	0.049
T_2	7.287	3.527	1.769	0.930	0.512
T_3	9.407	5.020	2.706	1.494	0.850
T_4	19.404	11.713	6.952	4.122	2.467
T_5	20.082	12.476	7.612	4.620	2.816
T_6	26.222	17.133	10.931	6.886	4.322
T_7	25.777	16.908	10.821	6.826	4.281
T_8	30.061	20.209	13.204	8.463	5.365
T_9	28.879	19.243	12.428	7.853	4.897

Table 3.5: Scenario (d3): Caplet prices in basis points using the approximation above.

strike	2.5 %	3.0 %	3.5 %	4.0 %	4.5 %	5.0 %	5.5 %	6.0 %	6.5 %	7.0 %
T_1	25.29	21.59	18.92	18.79	20.84	23.23	25.43	27.38	29.10	30.65
T_2	22.45	20.54	19.10	18.30	18.23	18.77	19.61	20.55	21.48	22.37
T_3	19.87	18.67	17.87	17.49	17.50	17.81	18.28	18.83	19.40	19.96
T_4	18.65	17.75	17.12	16.75	16.61	16.66	16.85	17.12	17.45	17.81
T_5	17.18	16.51	16.05	15.79	15.70	15.74	15.87	16.07	16.31	16.57
T_6	16.07	15.51	15.12	14.89	14.78	14.77	14.83	14.95	15.11	15.29
T_7	14.88	14.42	14.10	13.91	13.82	13.81	13.86	13.96	14.08	14.23
T_8	13.85	13.44	13.15	12.97	12.87	12.84	12.86	12.92	13.01	13.12
T_9	12.76	12.39	12.14	11.98	11.90	11.87	11.88	11.93	12.01	12.10

Table 3.6: Scenario (d3): BGM-implied volatilities of the caplet prices (in %) using the approximation above.

Chapter 4

The Lévy Libor model with default risk

In this chapter we present a credit risk model that extends the Lévy Libor model to defaultable market rates. Since the Libor market model is a special case of the Lévy Libor model, our approach can also be seen as an extension of the market model.

The first extension of the Libor market model to defaultable contracts was done by Lotz and Schlögl (2000). They use a deterministic hazard rate to construct a default time and then price defaultable forward rate agreements with unilateral as well as with bilateral default risk. Their assumption of a deterministic hazard rate is rather restrictive. In particular, it implies that the pre-default value of a defaultable bond is a deterministic multiple of the respective default-free bond price.

A second approach to extend the Libor market model was presented in Schönbucher (1999a). Default-free forward Libor rates are modelled according to the market model. In addition, the dynamics of defaultable forward Libor rates, i.e. ratios of pre-default values of defaultable zero coupon bonds, are specified. This specification is not done directly but via forward credit spreads or, alternatively, by modelling forward default intensities. A problem in this approach arises from the fact that their evolution is just specified; however, dynamics of defaultable forward Libor rates (or forward credit spreads or forward default intensities) cannot be modelled freely in this context. In fact, they follow by arbitrage arguments from the specification of the default time and the default-free forward Libor rates (for more details on this point we refer to section 4.1). Consequently, dynamics of defaultable forward Libor rates can only be "specified" by giving a pre-specification and then constructing a default time that implies these dynamics. In the same way, Bielecki and Rutkowski (1999, 2000) and Eberlein and Özkan (2003) have already extended the Heath–Jarrow–Morton model to defaultable bonds, for driving Brownian motions and Lévy processes respectively.

The model presented here is an extension of the Lévy Libor model. We follow the idea of Schönbucher (1999a) and specify, in addition to default-free forward Libor rates, the evolution of defaultable forward Libor rates, not directly but via forward default intensities. A pre-specification for their evolution is given and a default time is then constructed that implies this evolution. The resulting model is a generalization of Schönbucher's *Libor market model with default risk* to driving (non-homogeneous) Lévy processes. A formula for derivative pricing is presented that uses two analogons to forward measures, namely *defaultable forward measures* (or *survival measures*) and *restricted defaultable forward measures*. Finally, we price some of the most popular credit derivatives.

The chapter is organized as follows. Section 4.1 presents the details of the model. The time of default is constructed in section 4.2. A condition which guarantees that the evolution of the forward default intensities implied by the time of default matches its pre-specification is established in section 4.3. Section 4.4 introduces defaultable forward measures as well as restricted defaultable forward measures and presents a formula for derivative valuation. This formula is used in the remaining sections to price some of the most heavily traded credit derivatives, namely *credit default swaps*, *total rate of return swaps*, *options on defaultable bonds*, *credit spread options* and *credit default swaptions*.

4.1 Presentation of the model

Let us introduce some notation first. We consider a fixed time horizon T^* and a discrete tenor structure $0 = T_0 < T_1 < \ldots < T_n = T^*$ with $\delta_k := T_{k+1} - T_k$ for $k = 0, \ldots, n-1$. We assume that default-free as well as defaultable zero coupon bonds with maturities $T_1, \ldots, T_n$ are traded on the market. By $B(t, T_k)$ (resp. $B^0(t, T_k)$) we denote the time-t price of a default-free zero coupon bond (resp. a defaultable zero coupon bond with zero recovery) with maturity T_k. Indicate the time of default by τ and the pre-default values of the defaultable bonds by $\overline{B}(\cdot, \cdot)$, then we have

$$B^0(t, T_i) = 1\!\!1_{\{\tau > t\}} \overline{B}(t, T_i) \qquad \text{and} \qquad \overline{B}(T_i, T_i) = 1 \quad \text{for } i \in \{1, \ldots, n\}.$$

In what follows we are not going to model bond prices directly (it is only assumed that the processes describing the evolution of the bond prices $B(\cdot, T_i)$ and of the pre-default prices $\overline{B}(\cdot, T_i)$ are special semimartingales whose values as well as all left hand limits are strictly positive). Instead, we are going to specify the dynamics of forward Libor rates. The following notation (taken from Schönbucher (1999a)) will be used:

- The *default-free forward Libor rates* are given by
$$L(t, T_k) := \frac{1}{\delta_k} \left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1 \right) \qquad (k \in \{1, \ldots, n-1\}).$$
- The *defaultable forward Libor rates* are given by
$$\overline{L}(t, T_k) := \frac{1}{\delta_k} \left(\frac{\overline{B}(t, T_k)}{\overline{B}(t, T_{k+1})} - 1 \right) \qquad (k \in \{1, \ldots, n-1\}).$$
- The *forward Libor spreads* are given by
$$S(t, T_k) := \overline{L}(t, T_k) - L(t, T_k) \qquad (k \in \{1, \ldots, n-1\}).$$

- The *default risk factors* or *forward survival processes* are given by

$$D(t,T_k) := \frac{\overline{B}(t,T_k)}{B(t,T_k)} \qquad (k \in \{1,\dots,n\}).$$

- The discrete-tenor *forward default intensities* are given by

$$H(t,T_k) := \frac{1}{\delta_k}\left(\frac{D(t,T_k)}{D(t,T_{k+1})} - 1\right) \qquad (k \in \{1,\dots,n-1\}).$$

We begin by building up the default-free part of the model. The dynamics of default-free forward Libor rates are specified in the same way as in the Lévy Libor model. Once again, we refer to Eberlein and Özkan (2005) for a detailed construction. The most important properties of this model including those that we will need in this chapter can be found in section 3.2.

The model is driven by a d-dimensional non-homogeneous Lévy process L^{T^*} with characteristics $(0, c, F^{T^*})$ on a complete stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_{T^*}, \widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_s)_{0\le s\le T^*}, \mathbb{P}_{T^*})$. Moreover, L^{T^*} is assumed to satisfy assumption $(\mathbb{SUP})$ from section 1.3. The only difference to section 3.2 is that we put a slightly stronger restriction on L^{T^*} (remember we only assumed $(\mathbb{EM})$ before) in this chapter and that we use the notation $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \mathbb{P}_{T^*})$ instead of $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_{T^*})$. All the properties of the Lévy Libor model that have been established in section 3.2 remain valid (of course, we have to replace $\mathcal{F}_t$ by $\widetilde{\mathcal{F}}_t$ in (3.17)). Our goal in what follows is to include defaultable forward Libor rates in the Lévy Libor model.

At first sight, an evident way to build up the defaultable part of the model is to specify the dynamics of defaultable forward Libor rates by an expression similar to (3.13). However, $\overline{L}(T_k,T_k) < L(T_k,T_k)$ implies $\overline{B}(T_k,T_{k+1}) > B(T_k,T_{k+1})$ in which case there is an arbitrage opportunity in the market, provided that $B^0(\cdot,T_{k+1})$ has not defaulted until T_k. It seems thus natural to specify the model in such a way that defaultable forward Libor rates are always higher than their default-free counterparts. This can be achieved by modelling forward Libor spreads or forward default intensities as positive processes, instead of specifying defaultable forward Libor rates directly. We can then get the defaultable forward Libor rates through

$$\overline{L}(t,T_k) = S(t,T_k) + L(t,T_k)$$

or

$$\overline{L}(t,T_k) = H(t,T_k)(1 + \delta_k L(t,T_k)) + L(t,T_k). \tag{4.1}$$

Unfortunately, there is a problem in specifying the dynamics of H or S directly. Suppose that we have already constructed a time τ describing the time of default. Let us for a moment assume that τ is a stopping time with respect to the filtration $\widetilde{\mathbb{F}}$ (in the explicit construction which will follow τ will only be a stopping time with respect to some larger filtration, but for the point we are making here, this can be ignored). The terminal value of a defaultable bond is given by

$$B^0(T_k,T_k) = 1\!\!1_{\{\tau > T_k\}}\overline{B}(T_k,T_k) = 1\!\!1_{\{\tau > T_k\}}.$$

On the other hand, in the model for the default-free Libor rates the time-t price of a contingent claim X paying $1\!\!1_{\{\tau>T_k\}}$ at T_k is given by

$$X_t := B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[1\!\!1_{\{\tau>T_k\}}|\widetilde{\mathcal{F}}_t] = 1\!\!1_{\{\tau>t\}}B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[1\!\!1_{\{\tau>T_k\}}|\widetilde{\mathcal{F}}_t].$$

To have a consistent model, we thus have to have

$$B^0(t,T_k) = 1\!\!1_{\{\tau>t\}}B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[1\!\!1_{\{\tau>T_k\}}|\widetilde{\mathcal{F}}_t]$$

and consequently (at least on $\{\tau > t\}$)

$$\overline{B}(t,T_k) = B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[1\!\!1_{\{\tau>T_k\}}|\widetilde{\mathcal{F}}_t]$$

or equivalently

$$D(t,T_k) = \mathbb{E}_{\mathbb{P}_{T_k}}[1\!\!1_{\{\tau>T_k\}}|\widetilde{\mathcal{F}}_t],$$

which immediately provides a formula for H (and also for S). In other words, as soon as τ is specified we cannot freely choose the dynamics of H (or S). What we can and will do in the sequel is the following: We give a pre-specification for H and then construct τ in such a way that the dynamics of H implied by τ will match this pre-specification. The following assumptions are made in addition to ($\mathbb{LR}$.1) and ($\mathbb{LR}$.2):

($\mathbb{DLR}$.1): For any maturity T_i there is a deterministic function $\gamma(\cdot,T_i) : [0,T^*] \to \mathbb{R}^d_+$, which represents the volatility of the forward default intensity $H(\cdot,T_i)$. We suppose that $\gamma(s,T_k) = 0$ for $T_k < s \le T^*$. Moreover, we require that the functions $\lambda(\cdot,T_i)$ from ($\mathbb{LR}$.1) map to $\mathbb{R}^d_+$ and condition (3.12) is tightened by assuming that

$$\sum_{i=1}^{n-1}(|\lambda^j(s,T_i)| + |\gamma^j(s,T_i)|) \le M \qquad \text{for all } s \in [0,T^*] \text{ and } j \in \{1,\dots,d\}. \tag{4.2}$$

($\mathbb{DLR}$.2): The initial term structure $\overline{B}(0,T_i)$ ($i \in \{1,\dots,n\}$) of defaultable zero coupon bond prices satisfies $0 < \overline{B}(0,T_i) \le B(0,T_i)$ for all T_i as well as $\overline{L}(0,T_i) \ge L(0,T_i)$, i.e.

$$\frac{\overline{B}(0,T_i)}{\overline{B}(0,T_{i+1})} \ge \frac{B(0,T_i)}{B(0,T_{i+1})}.$$

To avoid confusion, let us denote by $\widehat{H}$ the pre-specified forward default intensities, which we postulate to be given by

$$\begin{aligned}\widehat{H}(t,T_k) = H(0,T_k)\exp\Bigg(\int_0^t b^H(s,T_k,T_{k+1})\,\mathrm{d}s + \int_0^t \sqrt{c_s}\gamma(s,T_k)\,\mathrm{d}W_s^{T_{k+1}}\\ + \int_0^t\int_{\mathbb{R}^d}\langle\gamma(s,T_k),x\rangle(\mu-\nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\Bigg)\end{aligned} \tag{4.3}$$

subject to the initial condition

$$H(0,T_k) = \frac{1}{\delta_k}\left(\frac{\overline{B}(0,T_k)B(0,T_{k+1})}{B(0,T_k)\overline{B}(0,T_{k+1})} - 1\right).$$

$W^{T_{k+1}}$ and $\nu^{T_{k+1}}$ are defined in (3.18) and (3.20). The drift term $b^H(\cdot,T_k,T_{k+1})$ will be specified later. For the moment we only assume $b^H(s,T_k,T_{k+1}) = 0$ for $T_k < s \le T^*$, i.e. we require that $\widehat{H}(t,T_k) = \widehat{H}(T_k,T_k)$ for $t \in [T_k,T^*]$.

4.2 Construction of the time of default

The construction of the default time will be done in the canonical way, that is for a given $\widetilde{\mathbb{F}}$-hazard process Γ a stopping time τ on an enlarged probability space will be constructed. We will do the construction for a general Γ first. The key question then will be which particular hazard process to choose to make H match $\widehat{H}$. For more details on the canonical construction we refer to Bielecki and Rutkowski (2002), from whom the notation is adopted.

Let Γ be an $\widetilde{\mathbb{F}}$-adapted, right-continuous, increasing process on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\mathbb{P}_{T^*})$ satisfying $\Gamma_0 = 0$ and $\lim_{t\to\infty}\Gamma_t = \infty$. Furthermore, let η be a random variable on some probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{P}})$ that is uniformly distributed on $[0,1]$. Consider the product space $(\Omega,\mathcal{G},\mathbb{Q}_{T^*})$ defined by

$$\Omega := \widetilde{\Omega}\times\widehat{\Omega}, \qquad \mathcal{G} := \widetilde{\mathcal{F}}\otimes\widehat{\mathcal{F}}, \qquad \mathbb{Q}_{T^*} := \mathbb{P}_{T^*}\otimes\widehat{\mathbb{P}}$$

and denote by $\mathbb{F}$ the trivial extension of $\widetilde{\mathbb{F}}$ to the enlarged probability space $(\Omega,\mathcal{G},\mathbb{Q}_{T^*})$, i.e. each $A\in\mathcal{F}_t$ is of the form $\widetilde{A}\times\widehat{\Omega}$ for some $\widetilde{A}\in\widetilde{\mathcal{F}}_t$. We extend all stochastic processes from the default-free part of the model to the extended probability space (by setting $L^{T^*}(\widetilde{\omega},\widehat{\omega}) := L^{T^*}(\widetilde{\omega})$ and similarly for all other processes).

Define a random variable $\tau:\Omega\to\mathbb{R}_+$ by

$$\tau := \inf\{t\in\mathbb{R}_+ : e^{-\Gamma_t}\le\eta\}.$$

and denote $\mathcal{H}_t := \sigma\left(1\!\!1_{\{\tau\le u\}} \,|\, 0\le u\le t\right)$ and $\mathcal{G}_t := \mathcal{F}_t\vee\mathcal{H}_t$ for $t\in[0,T^*]$. Then τ is a stopping time with respect to the filtration $\mathbb{G} := (\mathcal{G}_s)_{0\le s\le T^*}$ since $\{\tau\le t\}\in\mathcal{H}_t\subset\mathcal{G}_t$. Moreover, for $0\le s\le t\le T^*$ we have (compare Bielecki and Rutkowski (2002, (8.14)))

$$\mathbb{Q}_{T^*}\{\tau>s|\mathcal{F}_{T^*}\} = \mathbb{Q}_{T^*}\{\tau>s|\mathcal{F}_t\} = \mathbb{Q}_{T^*}\{\tau>s|\mathcal{F}_s\} = e^{-\Gamma_s}, \tag{4.4}$$

i.e. Γ is the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_{T^*}$.

A question that arises naturally is whether or not L^{T^*} is a non-homogeneous Lévy process with respect to $\mathbb{Q}_{T^*}$ and the enlarged filtration $\mathbb{G}$. To answer it, we make use of the following lemma:

Lemma 4.1 *Equation (4.4) implies each of the following equivalent conditions:*

1. *$\mathcal{F}_{T^*}$ and $\mathcal{H}_s$ are conditionally independent given $\mathcal{F}_s$ under $\mathbb{Q}_{T^*}$, i.e. for any bounded $\mathcal{F}_{T^*}$-measurable random variable X and any bounded $\mathcal{H}_s$-measurable random variable Y we have*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[XY|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s]\,\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s] \qquad (s\in[0,T^*]).$$

2. For any bounded, $\mathcal{F}_{T^*}$-measurable random variable X

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{G}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s] \qquad (s \in [0, T^*]) \tag{4.5}$$

PROOF: Let us show the equivalence first. Suppose condition 2 holds, then

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[XY|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{G}_s]\, Y|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s]\, \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s]$$

for bounded random variables X and Y that are $\mathcal{F}_{T^*}$-measurable and $\mathcal{H}_s$-measurable respectively.

Now assume that condition 1 holds. Let X be a bounded $\mathcal{F}_{T^*}$-measurable random variable and $\mathcal{A} := \{A_1 \cap A_2 : A_1 \in \mathcal{H}_s, A_2 \in \mathcal{F}_s\}$, then by assumption for $A \in \mathcal{A}$

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[X 1\!\!1_A|\mathcal{F}_s] = 1\!\!1_{A_2}\mathbb{E}_{\mathbb{Q}_{T^*}}[X 1\!\!1_{A_1}|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s]\, \mathbb{E}_{\mathbb{Q}_{T^*}}[1\!\!1_A|\mathcal{F}_s]$$

and consequently

$$\begin{aligned}\int_A \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s]\, \mathrm{d}\mathbb{Q}_{T^*} &= \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbb{E}_{\mathbb{Q}_{T^*}}[1\!\!1_A|\mathcal{F}_s]\mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s]] \\ &= \mathbb{E}_{\mathbb{Q}_{T^*}}[X 1\!\!1_A] = \int_A X\, \mathrm{d}\mathbb{Q}_{T^*}.\end{aligned}$$

Since $\mathcal{A}$ is a generator of $\mathcal{G}_s = \mathcal{F}_s \vee \mathcal{H}_s$ that is closed under the formation of finite intersections, a uniqueness result as e.g. Billingsley (1979, Theorem 34.1) yields condition 2.

It remains to show that equation (4.4) implies any of the two conditions. We show that it implies the first one. Since $\mathcal{E} := \{\{\tau \leq u\} : 0 \leq u \leq s\}$ is a generator of $\mathcal{H}_s$ that is closed under the formation of finite intersections, equation (4.4) together with the usual uniqueness result for σ-finite measures yields for any $B \in \mathcal{H}_s$

$$\mathbb{Q}_{T^*}\{B|\mathcal{F}_{T^*}\} = \mathbb{Q}_{T^*}\{B|\mathcal{F}_s\}.$$

By algebraic induction, we get for any bounded $\mathcal{H}_s$-measurable random variable Y

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_{T^*}] = \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s].$$

Hence,

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[XY|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_{T^*}]\,|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_s]\, \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_s]$$

for each bounded $\mathcal{F}_{T^*}$-measurable random variable X. □

Proposition 4.2 *L^{T^*} is a non-homogeneous Lévy process on the stochastic basis $(\Omega, \mathcal{G}_{T^*}, \mathbb{G}, \mathbb{Q}_{T^*})$ with characteristics $(0, c, F^{T^*})$.*

Proof: L^{T^*} is clearly an adapted, càdlàg process and satisfies $L_0^{T^*} = 0$. Its characteristic function is given by

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_{T^*}}[\exp(\mathrm{i}uL_t^{T^*})] &= \int\limits_{\widetilde{\Omega}\times\widehat{\Omega}} \exp(\mathrm{i}uL_t^{T^*}(\widetilde{\omega},\widehat{\omega}))\,\mathrm{d}(\mathbb{P}_{T^*}\otimes\widehat{\mathbb{P}})(\widetilde{\omega},\widehat{\omega})) \\
&= \int\limits_{\widetilde{\Omega}} \exp(\mathrm{i}uL_t^{T^*}(\widetilde{\omega}))\,\mathrm{d}\mathbb{P}_{T^*}(\widetilde{\omega}) \\
&= \mathbb{E}_{\mathbb{P}_{T^*}}[\exp(\mathrm{i}uL_t^{T^*})].
\end{aligned}$$

Hence, the characteristic function of $L_t^{T^*}$ and thus also the characteristics of L^{T^*} are preserved. It remains to show that $L_t^{T^*} - L_s^{T^*}$ is independent of $\mathcal{G}_s$ for $s < t$. Let $B \in \mathcal{B}^d$ and $A \in \mathcal{G}_s$, then using condition 2 of lemma 4.1 with $X := \mathbb{1}_B(L_t^{T^*} - L_s^{T^*})$ and the fact that $L_t^{T^*} - L_s^{T^*}$ is independent of $\mathcal{F}_s$ we get

$$\begin{aligned}
\mathbb{Q}_{T^*}(A \cap \{(L_t^{T^*} - L_s^{T^*}) \in B\}) &= \int\limits_A \mathbb{1}_B(L_t^{T^*} - L_s^{T^*})\,\mathrm{d}\mathbb{Q}_{T^*} \\
&= \int\limits_A \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbb{1}_B(L_t^{T^*} - L_s^{T^*})|\mathcal{F}_s]\,\mathrm{d}\mathbb{Q}_{T^*} \\
&= \int\limits_A \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbb{1}_B(L_t^{T^*} - L_s^{T^*})]\,\mathrm{d}\mathbb{Q}_{T^*} \\
&= \mathbb{Q}_{T^*}(A)\mathbb{Q}_{T^*}(\{(L_t^{T^*} - L_s^{T^*}) \in B\}). \qquad \square
\end{aligned}$$

In particular, each forward Libor rate $L(t, T_k)_{0\le t\le T_k}$ is a martingale with respect to the filtration $(\mathcal{G}_s)_{0\le s\le T_k}$ and the measure $\mathbb{Q}_{T_{k+1}}$, which is constructed from $\mathbb{Q}_{T^*}$ in the same way as $\mathbb{P}_{T_{k+1}}$ is constructed from $\mathbb{P}_{T^*}$.

Γ is not only the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_{T^*}$, but also the $\mathbb{F}$-hazard process of τ under all other forward measures, as the following lemma shows:

Lemma 4.3 *Γ is the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_{T_k}$ for all $k \in \{1, \dots, n\}$.*

Proof: Fix a k and denote by ψ the ($\mathcal{F}_{T_k}$-measurable) Radon–Nikodym derivative of $\mathbb{Q}_{T_k}$ with respect to $\mathbb{Q}_{T^*}$. From lemma 4.1 we know that $\mathcal{F}_{T^*}$ and $\mathcal{H}_s$ are conditionally independent given $\mathcal{F}_s$ under $\mathbb{Q}_{T^*}$. Using the abstract Bayes rule and this conditional independence (plus a dominated convergence argument) we get

$$\begin{aligned}
\mathbb{Q}_{T_k}\{\tau > s|\mathcal{F}_s\} &= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi|\mathcal{F}_s]} \\
&= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi|\mathcal{F}_s]\,\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\psi|\mathcal{F}_s]} \\
&= e^{-\Gamma_s}. \qquad \square
\end{aligned}$$

Let us now turn to the question which hazard process Γ to choose to make H match its pre-specification. As pointed out at the beginning of this chapter, to have a consistent model we have to have

$$\begin{aligned} B^0(t,T_k) &= B(t,T_k)\mathbb{Q}_{T_k}\{\tau > T_k | \mathcal{G}_t\} \\ &= B(t,T_k)\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbb{1}_{\{\tau>T_k\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]}, \end{aligned} \tag{4.6}$$

where the last equality follows from Bielecki and Rutkowski (2002, (5.2)). Let

$$\overline{B}(t,T_k) := B(t,T_k)\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbb{1}_{\{\tau>T_k\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]} = B(t,T_k)\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbb{1}_{\{\tau>T_k\}}|\mathcal{F}_t]}{e^{-\Gamma_t}}, \tag{4.7}$$

then

$$D(t,T_k) = \mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{\Gamma_t - \Gamma_{T_k}}\,\middle|\,\mathcal{F}_t\right].$$

In particular,

$$H(t,T_k) = \frac{1}{\delta_k}\left(\frac{D(t,T_k)}{D(t,T_{k+1})} - 1\right) = \frac{1}{\delta_k}\left(\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{-\Gamma_{T_k}}\,\middle|\,\mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}_{T_{k+1}}}\left[e^{-\Gamma_{T_{k+1}}}\,\middle|\,\mathcal{F}_t\right]} - 1\right). \tag{4.8}$$

It is clear from the previous equation that, in order to make H match its pre-specification $\widehat{H}$, we only need to specify the hazard process Γ at the points T_k for $k \in \{1,\dots,n\}$ in a suitable way. The values of Γ in between these points do not have an influence on the value of H. Moreover, we know that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{-\Gamma_{T_k}}\,\middle|\,\mathcal{F}_{T_{k-1}}\right] &= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_k\}}\,\middle|\,\mathcal{F}_{T_{k-1}}\right] \\ &= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_k\}}\,\middle|\,\mathcal{G}_{T_{k-1}}\right]\,\middle|\,\mathcal{F}_{T_{k-1}}\right] \\ &= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_{k-1}\}}\frac{\overline{B}(T_{k-1},T_k)}{B(T_{k-1},T_k)}\,\middle|\,\mathcal{F}_{T_{k-1}}\right] \\ &= e^{-\Gamma_{T_{k-1}}}\frac{\overline{B}(T_{k-1},T_k)}{B(T_{k-1},T_k)} \\ &= e^{-\Gamma_{T_{k-1}}}(1+\delta_{k-1}H(T_{k-1},T_{k-1}))^{-1}, \end{aligned}$$

where the third equation follows from (4.7) and Bielecki and Rutkowski (2002, (5.2)). We now define Γ recursively by setting $\Gamma_0 := 0$,

$$\begin{aligned} \Gamma_{T_k} &:= \Gamma_{T_k-1} + \log(1+\delta_{k-1}\widehat{H}(T_{k-1},T_{k-1})) \qquad (k \in \{1,\dots,n\}) \\ &= \sum_{l=0}^{k-1}\log(1+\delta_l\widehat{H}(T_l,T_l)), \end{aligned} \tag{4.9}$$

and for $t \in (T_{k-1},T_k)$

$$\Gamma_t := (1-\alpha_k(t))\Gamma_{T_{k-1}} + \alpha_k(t)\Gamma_{T_k},$$

where $\alpha_k : [T_{k-1},T_k] \to [0,1]$ is a continuous, strictly increasing function satisfying $\alpha_k(T_{k-1}) = 0$ and $\alpha_k(T_k) = 1$. Obviously Γ is a continuous, strictly increasing (since $\widehat{H}(\cdot,\cdot) > 0$ by construction), and $\widetilde{\mathbb{F}}$-adapted process (since Γ_{T_k} is $\widetilde{\mathcal{F}}_{T_{k-1}}$-measurable) and can be used for the canonical construction.

It still has to be checked whether the implied dynamics of H match those of $\widehat{H}$. Using (4.8) and (4.9) we get

$$\begin{aligned} H(t,T_1) &= \frac{1}{\delta_1}\left(\frac{1}{\mathbb{E}_{\mathbb{Q}_{T_2}}[e^{-\Gamma_{T_2}+\Gamma_{T_1}} \,|\, \mathcal{F}_t]} - 1\right) \\ &= \frac{1}{\delta_1}\left(\frac{1}{\mathbb{E}_{\mathbb{Q}_{T_2}}\left[\frac{1}{1+\delta_1 \widehat{H}(T_1,T_1)} \,|\, \mathcal{F}_t\right]} - 1\right) \end{aligned}$$

or, written differently,

$$\mathbb{E}_{\mathbb{Q}_{T_2}}\left[\frac{1}{1+\delta_1 \widehat{H}(T_1,T_1)} \Big| \mathcal{F}_t\right] = \frac{1}{1+\delta_1 H(t,T_1)}.$$

Consequently, $H(\cdot,T_1)$ meets its pre-specification if $\left(\frac{1}{1+\delta_1 \widehat{H}(t,T_1)}\right)_{0\leq t\leq T_1}$ is a $\mathbb{Q}_{T_2}$-martingale. More generally we have the following result. Recall that $\widehat{H}(t,T_i) = \widehat{H}(T_i,T_i)$ for $t \in [T_i,T^*]$.

Lemma 4.4 *$H(\cdot,T_k)$ meets its pre-specification if $\left(\prod_{i=1}^{l} \frac{1}{1+\delta_i \widehat{H}(t,T_i)}\right)_{0\leq t\leq T_l}$ is a $\mathbb{Q}_{T_{l+1}}$-martingale for all $l \in \{1,\dots,k\}$.*

Proof: The result for $k = 1$ has been proven above. Using (4.8), (4.9) and the prerequisite we get for $k > 1$

$$\begin{aligned} H(t,T_k) &= \frac{1}{\delta_k}\left(\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[\prod_{i=0}^{k-1} \frac{1}{1+\delta_i \widehat{H}(T_i,T_i)} \Big| \mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}_{T_{k+1}}}\left[\prod_{i=0}^{k} \frac{1}{1+\delta_i \widehat{H}(T_i,T_i)} \Big| \mathcal{F}_t\right]} - 1\right) \\ &= \frac{1}{\delta_k}((1+\delta_k \widehat{H}(t,T_k)) - 1) = \widehat{H}(t,T_k). \end{aligned}$$ □

Remember that we can still choose the drift coefficients $b^H(\cdot,T_k,T_{k+1})$ in (4.3) in order to satisfy the prerequisite of the previous lemma. This choice is done in the next section.

4.3 Specification of the drift

We specify the drift recursively starting with $b(\cdot,T_1,T_2)$. More precisely, we look for a process $b(\cdot,T_1,T_2)$ such that $\left((1+\delta_1 \widehat{H}(t,T_1))^{-1}\right)_{0\leq t\leq T_1}$ becomes a $\mathbb{Q}_{T_2}$-martingale first. The next step is to specify $b(\cdot,T_2,T_3)$ in such a way that $\left(((1+\delta_1 \widehat{H}(t,T_1))(1+\delta_2 \widehat{H}(t,T_2)))^{-1}\right)_{0\leq t\leq T_2}$ becomes a $\mathbb{Q}_{T_3}$-martingale, and so on. Let us begin with two lemmata:

Lemma 4.5 *Let X be a real-valued semimartingale with $X_0 = 0$ and $\Delta X > -1$. Then*

$$(\mathcal{E}(X))^{-1} = \mathcal{E}\left(-X + \langle X^c, X^c\rangle + \left(\frac{1}{1+x} - 1 + x\right) * \mu^X\right).$$

Proof: Using Kallsen and Shiryaev (2002, Lemma 2.6) we get

$$\mathcal{E}(X) = \exp\left(X - \frac{1}{2}\langle X^c, X^c\rangle + (\log(1+x) - x) * \mu^X\right)$$

Consequently, again using Kallsen and Shiryaev (2002, Lemma 2.6),

$$\begin{aligned}(\mathcal{E}(X))^{-1} &= \exp\left(-X + \frac{1}{2}\langle X^c, X^c\rangle - (\log(1+x) - x) * \mu^X\right)\\ &= \mathcal{E}\Bigg(- X + \frac{1}{2}\langle X^c, X^c\rangle - (\log(1+x) - x) * \mu^X\\ &\qquad + \frac{1}{2}\langle X^c, X^c\rangle + \left(\frac{1}{1+x} - 1 + \log(1+x)\right) * \mu^X\Bigg)\\ &= \mathcal{E}\left(-X + \langle X^c, X^c\rangle + \left(\frac{1}{1+x} - 1 + x\right) * \mu^X\right).\end{aligned}$$

□

Lemma 4.6 *For $k \in \{2, \ldots, n\}$ and $i \in \{1, \ldots, k-1\}$*

$$\begin{aligned}\widehat{H}(t, T_i) \;=\; & H(0, T_i)\mathcal{E}_t\Bigg(\int_0^{\bullet} a(s, T_i, T_k)\,\mathrm{d}s + \int_0^{\bullet} \sqrt{c_s}\gamma(s, T_i)\,\mathrm{d}W_s^{T_k}\\ &+ \int_0^{\bullet}\int_{\mathbb{R}^d} \left(e^{\langle\gamma(s,T_i),x\rangle} - 1\right)(\mu - \nu^{T_k})(\mathrm{d}s, \mathrm{d}x)\Bigg),\end{aligned}$$

where

$$\begin{aligned}a(s, T_i, T_k) \;:=\; & b^H(s, T_i, T_k) + \frac{1}{2}\langle\gamma(s, T_i), c_s\gamma(s, T_i)\rangle\\ &+ \int_{\mathbb{R}^d} \left(e^{\langle\gamma(s,T_i),x\rangle} - 1 - \langle\gamma(s, T_i), x\rangle\right) F_s^{T_k}(\mathrm{d}x).\end{aligned} \tag{4.10}$$

and $b^H(s, T_i, T_k)$ is given by (4.11).

Proof: From the default-free part of the model we know that

$$W_t^{T_{i+1}} = W_t^{T_k} - \int_0^t \sqrt{c_s}\left(\sum_{l=i+1}^{k-1} \alpha(s, T_l, T_{l+1})\right)\mathrm{d}s$$

and

$$\nu^{T_{i+1}}(\mathrm{d}t, \mathrm{d}x) = \left(\prod_{l=i+1}^{k-1} \beta(s, x, T_l, T_{l+1})\right)\nu^{T_k}(\mathrm{d}t, \mathrm{d}x)$$

with α and β given by (3.19) and (3.21). Consequently, equation (4.3) implies

$$\begin{aligned}\widehat{H}(t,T_i) \;=\; H(0,T_i)\exp\Bigg(& \int_0^t b^H(s,T_i,T_k)\,\mathrm{d}s + \int_0^t \sqrt{c_s}\gamma(s,T_i)\,\mathrm{d}W_s^{T_k} \\ & + \int_0^t\int_{\mathbb{R}^d} \langle\gamma(s,T_i),x\rangle(\mu-\nu^{T_k})(\mathrm{d}s,\mathrm{d}x)\Bigg),\end{aligned}$$

where

$$\begin{aligned} b^H(s,T_i,T_k) \;:=\; & b^H(s,T_i,T_{i+1}) \\ & - \left\langle \gamma(s,T_i), c_s\left(\sum_{l=i+1}^{k-1}\alpha(s,T_l,T_{l+1})\right)\right\rangle \\ & - \int_{\mathbb{R}^d}\langle\gamma(s,T_i),x\rangle\left(\prod_{l=i+1}^{k-1}\beta(s,x,T_l,T_{l+1})-1\right)F_s^{T_k}(\mathrm{d}x). \end{aligned} \tag{4.11}$$

The claim now follows from Kallsen and Shiryaev (2002, Lemma 2.6). □

Proposition 4.7 $\left(\frac{1}{1+\delta_1\widehat{H}(t,T_1)}\right)_{0\le t\le T_1}$ *is a $\mathbb{Q}_{T_2}$-martingale if for $s\in[0,T_1]$*

$$\begin{aligned} b^H(s,T_1,T_2) \;=\; & \left(Y_{s-}^1-\frac{1}{2}\right)\langle\gamma(s,T_1),c_s\gamma(s,T_1)\rangle \\ & + \int_{\mathbb{R}^d}\left(\langle\gamma(s,T_1),x\rangle - \frac{e^{\langle\gamma(s,T_1),x\rangle}-1}{1+Y_{s-}^1\left(e^{\langle\gamma(s,T_1),x\rangle}-1\right)}\right)F_s^{T_2}(\mathrm{d}x), \end{aligned} \tag{4.12}$$

where $Y_s^1 := \frac{\delta_1\widehat{H}(s,T_1)}{1+\delta_1\widehat{H}(s,T_1)}$.

Proof: Lemma 4.6 gives us

$$\begin{aligned}\widehat{H}(t,T_1) \;=\; H(0,T_1)\mathcal{E}_t\Bigg(& \int_0^\bullet a(s,T_1,T_2)\,\mathrm{d}s + \int_0^\bullet \sqrt{c_s}\gamma(s,T_1)\,\mathrm{d}W_s^{T_2} \\ & + \int_0^\bullet\int_{\mathbb{R}^d}\left(e^{\langle\gamma(s,T_1),x\rangle}-1\right)(\mu-\nu^{T_2})(\mathrm{d}s,\mathrm{d}x)\Bigg)\end{aligned}$$

with

$$\begin{aligned} a(s,T_1,T_2) \;=\; & b^H(s,T_1,T_2) + \frac{1}{2}\langle\gamma(s,T_1),c_s\gamma(s,T_1)\rangle \\ & + \int_{\mathbb{R}^d}\left(e^{\langle\gamma(s,T_1),x\rangle}-1-\langle\gamma(s,T_1),x\rangle\right)F_s^{T_2}(\mathrm{d}x). \end{aligned} \tag{4.13}$$

Consequently, for $X_t^1 := 1 + \delta_1 \widehat{H}(t,T_1)$ we have

$$\begin{aligned} \mathrm{d}X_t^1 &= \delta_1 \,\mathrm{d}\widehat{H}(t,T_1) \\ &= X_{t-}^1 \Bigg(Y_{t-}^1 a(t,T_1,T_2)\,\mathrm{d}t + Y_{t-}^1 \sqrt{c_t}\gamma(t,T_1)\,\mathrm{d}W_t^{T_2} \\ &\qquad + \int_{\mathbb{R}^d} Y_{t-}^1 \left(e^{\langle \gamma(t,T_1),x\rangle} - 1 \right) (\mu - \nu^{T_2})(\mathrm{d}t,\mathrm{d}x) \Bigg). \end{aligned}$$

Lemma 4.5 implies

$$\begin{aligned} (X_t^1)^{-1} &= (X_0^1)^{-1}\mathcal{E}_t\Bigg(\int_0^{\bullet} A(s,T_2)\,\mathrm{d}s - \int_0^{\bullet} Y_{s-}^1 \sqrt{c_s}\gamma(s,T_1)\,\mathrm{d}W_s^{T_2} \\ &\qquad + \int_0^{\bullet}\int_{\mathbb{R}^d} \left(\left(1 + Y_{s-}^1 \left(e^{\langle \gamma(s,T_1),x\rangle} - 1 \right) \right)^{-1} - 1 \right) (\mu - \nu^{T_2})(\mathrm{d}s,\mathrm{d}x) \Bigg), \end{aligned}$$

where

$$\begin{aligned} A(s,T_2) :=& -Y_{s-}^1 a(s,T_1,T_2) + (Y_{s-}^1)^2 \langle \gamma(s,T_1), c_s\gamma(s,T_1)\rangle \\ &+ \int_{\mathbb{R}^d} Y_{s-}^1 \left(e^{\langle \gamma(s,T_1),x\rangle} - 1 - \frac{e^{\langle \gamma(s,T_1),x\rangle} - 1}{1 + Y_{s-}^1 \left(e^{\langle \gamma(s,T_1),x\rangle} - 1 \right)} \right) F_s^{T_2}(\mathrm{d}x). \end{aligned} \tag{4.14}$$

Thus, $(1 + \delta_1 \widehat{H}(\cdot,T_1))^{-1}$ is a $\mathbb{Q}_{T_2}$-local martingale if $A(\cdot,T_2) \equiv 0$. In this case it is also a martingale since it is bounded by 0 and 1 and therefore of class $[D]$ (compare Jacod and Shiryaev (2003, I.1.47c)). Combining $A(\cdot,T_2) \equiv 0$ with (4.13) and (4.14) yields (4.12). $\square$

More generally, we get the following proposition:

Proposition 4.8 $\left(\prod_{i=1}^{k-1} \frac{1}{1+\delta_i \widehat{H}(t,T_i)} \right)_{0 \le t \le T_{k-1}}$ *is a martingale with respect to* $\mathbb{Q}_{T_k}$ *for* $k \in \{2,\ldots,n\}$ *if for all* $i \in \{1,\ldots,k-1\}$ *and* $s \in [0,T_i]$

$$\begin{aligned} b^H(s,T_i,T_{i+1}) =& \\ & \sum_{j=1}^{i} Y_{s-}^j \langle \gamma(s,T_j), c_s\gamma(s,T_i)\rangle - \frac{1}{2}\langle \gamma(s,T_i), c_s\gamma(s,T_i)\rangle \\ & + \sum_{j=1}^{i-1} \left(\frac{Y_{s-}^j}{Y_{s-}^i} \langle \gamma(s,T_j), c_s\alpha(s,T_i,T_{i+1})\rangle \right) \\ & + \int_{\mathbb{R}^d} \left(\langle \gamma(s,T_i),x\rangle - \frac{e^{\langle \gamma(s,T_i),x\rangle} - 1}{\prod_{j=1}^{i} \left(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j),x\rangle} - 1 \right) \right)} \right) F_s^{T_{i+1}}(\mathrm{d}x) \\ & + (Y_{s-}^i)^{-1} \int_{\mathbb{R}^d} (\beta(s,x,T_i,T_{i+1}) - 1) \\ & \qquad \times \left(1 - \prod_{j=1}^{i-1} \left(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j),x\rangle} - 1 \right) \right)^{-1} \right) F_s^{T_{i+1}}(\mathrm{d}x), \end{aligned} \tag{4.15}$$

where $Y_s^i := \frac{\delta_i \widehat{H}(s,T_i)}{1+\delta_i \widehat{H}(s,T_i)}$.

PROOF: The proof is done in the appendix since it is calculationally intense. □

Note that we cannot just define $b^H(s, T_i, T_{i+1})$ by (4.15) since the term on the right hand side involves Y_s^i which depends on $\widehat{H}(s, T_i)$ and thus on $b^H(\cdot, T_i, T_{i+1})$ itself. In other words, we have to deal with a stochastic differential equation. Suppose that for every $i \in \{1, \dots, k-1\}$ there is a unique solution to the SDE

$$h(t,T_i) = h(0,T_i) + \int_0^t f^i(s, h(s-,T_i))\,\mathrm{d}s + \int_0^t \sqrt{c_s}\gamma(s,T_i)\,\mathrm{d}W_s^{T_{i+1}} + \int_0^t \int_{\mathbb{R}^d} \langle \gamma(s,T_i), x\rangle (\mu - \nu^{T_{i+1}})(\mathrm{d}s, \mathrm{d}x) \qquad (4.16)$$

with

$$h(0,T_i) := \log H(0,T_i)$$

and

$$f^i(s,x) := f_1^i(s) + f_2^i(s,x) + f_3^i(s,x) + f_4^i(s,x),$$

where

$$\begin{aligned}
f_1^i(s) &:= \sum_{j=1}^{i-1} \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \langle \gamma(s,T_j), c_s \gamma(s,T_i)\rangle - \frac{1}{2}\langle \gamma(s,T_i), c_s\gamma(s,T_i)\rangle \\
&\quad - \int_{\mathbb{R}^d} \left(e^{\langle \gamma(s,T_i), y\rangle} - 1 - \langle \gamma(s,T_i), y\rangle \right) F_s^{T_{i+1}}(\mathrm{d}y), \\
f_2^i(s,x) &:= \frac{\delta_i e^x}{1+\delta_i e^x} \langle \gamma(s,T_i), c_s\gamma(s,T_i)\rangle \\
&\quad + \frac{1+\delta_i e^x}{\delta_i e^x} \sum_{j=1}^{i-1} \left(\frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \langle \gamma(s,T_j), c_s \alpha(s,T_i,T_{i+1})\rangle \right) \\
f_3^i(s,x) &:= \frac{1+\delta_i e^x}{\delta_i e^x} \int_{\mathbb{R}^d} (\beta(s,y,T_i,T_{i+1}) - 1) \qquad (4.17)\\
&\quad \left(1 - \prod_{j=1}^{i-1} \left(1 + \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \left(e^{\langle \gamma(s,T_j), y\rangle} - 1 \right) \right)^{-1} \right) F_s^{T_{i+1}}(\mathrm{d}y),
\end{aligned}$$

and

$$\begin{aligned}
f_4^i(s,x) &:= \int_{\mathbb{R}^d} \left(e^{\langle \gamma(s,T_i), y\rangle} - 1 \right) \left(1 - \left(1 + \frac{\delta_i e^x}{1+\delta_i e^x} \left(e^{\langle \gamma(s,T_i), y\rangle} - 1 \right) \right)^{-1} \right. \\
&\quad \left. \times \prod_{j=1}^{i-1} \left(1 + \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \left(e^{\langle \gamma(s,T_j), y\rangle} - 1 \right) \right)^{-1} \right) F_s^{T_{i+1}}(\mathrm{d}y).
\end{aligned}$$

Then $\widehat{H}(s,T_i) := \exp h(s,T_i)$ satisfies (4.3) with drift term $b^H(s,T_i,T_{i+1})$ given by (4.15). In this case proposition 4.8 yields that $\left(\prod_{i=1}^{k-1}\frac{1}{1+\delta_i\widehat{H}(t,T_i)}\right)_{0\le t\le T_{k-1}}$ is a $\mathbb{Q}_{T_k}$-martingale.

To prove that there is a unique solution to (4.16) we make use of the following theorem which is a direct consequence of Protter (1992, Theorem V.7) (see also Protter (1992, Theorem V.6)):

Theorem 4.9 *Assume a (one-dimensional) semimartingale Z with $Z_0 = 0$ on a complete stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ to be given and let $f : \mathbb{R}_+\times\Omega\times\mathbb{R}\to\mathbb{R}$ be such that*

1. *for fixed $x\in\mathbb{R}$, $(t,\omega)\mapsto f(t,\omega,x)$ is an adapted càglàd process, i.e. it has left-continuous paths that admit right-hand limits.*
2. *there exists a finite random variable K such that for all $t\in\mathbb{R}_+$*
$$|f(t,\omega,x)-f(t,\omega,y)|\le K(\omega)|x-y|.$$

Then the stochastic differential equation

$$X_t = X_0 + Z_t + \int_0^t f(s,\cdot,X_{s-})\,\mathrm{d}s,$$

where X_0 is a constant, has a unique (strong) solution. This solution is a semimartingale.

Unfortunately, the functions f_2^i and f_3^i in (4.16) are not globally Lipschitz, i.e. they do not satisfy condition 2 of the previous theorem. However, for the SDE in consideration we can weaken this condition by assuming that f is locally Lipschitz and satisfies a growth condition, as the following proposition shows:

Proposition 4.10 *Assume we are given a d-dimensional special semimartingale $S := \int_0^\bullet \sqrt{c_s}\,\mathrm{d}W_s + \int_0^\bullet\int_{\mathbb{R}^d} x(\mu-\nu)(\mathrm{d}s,\mathrm{d}x)$ on a complete stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where W is a standard Brownian motion, c is deterministic, and μ is the random measure associated with the jumps of S with (possibly non-deterministic) compensator $\nu(\mathrm{d}s,\mathrm{d}x) = F_s(\mathrm{d}x)\,\mathrm{d}s$. Suppose that $\sigma : \mathbb{R}_+\to\mathbb{R}^d$ is a bounded function and let $f : \mathbb{R}_+\times\Omega\times\mathbb{R}\to\mathbb{R}$ be such that*

1. *for fixed $x\in\mathbb{R}$, $(t,\omega)\mapsto f(t,\omega,x)$ is an adapted càglàd process.*
2. *for all $r>0$ there is a real number K_r such that for all (t,ω) and all $x,y\in\mathbb{R}$ with $|x|,|y|\le r$*
$$|f(t,\omega,x)-f(t,\omega,y)|\le K_r|x-y| \qquad \text{and} \qquad |f(t,\omega,x)|\le K_r.$$
3. *there is a constant B_1 such that for all (t,ω) and all $x\in\mathbb{R}$*
$$xf(t,\omega,x)\le B_1(1+x^2).$$

Suppose further that there is a constant B_2 such that for all (t,ω)

$$\langle\sigma(t), c_t\sigma(t)\rangle + \int_{\mathbb{R}^d} \langle\sigma(t), y\rangle^2 F_s(\mathrm{d}y) \le B_2. \tag{4.18}$$

Then the stochastic differential equation

$$X_t = X_0 + \int_0^t f(s,\cdot,X_{s-})\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}S_s, \tag{4.19}$$

where X_0 is a constant, has a unique (non-exploding) solution which is a semimartingale.

Proof: The proof uses ideas of the proofs of theorems 2.2.3 and 2.3.3 in Reiss (2003), where a similar statement is established for a deterministic function f and a driving Brownian motion.

Let us show uniqueness of a solution first. Suppose X^1 and X^2 are two solutions. To show that they are indistinguishable, it is enough to show that they are modifications of each other since their paths are right continuous. Fix a $t \in \mathbb{R}_+$ and define for $n \in \mathbb{N}$

$$\tau_n^1 := \inf\{s \ge 0 : |X_s^1| \ge n\}, \qquad \tau_n^2 := \inf\{s \ge 0 : |X_s^2| \ge n\}.$$

Since the usual conditions hold, τ_n^1 and τ_n^2 are stopping times. Consequently, $\tau_n := \min(\tau_n^1, \tau_n^2)$ is a stopping time that converges to infinity almost surely as $n \to \infty$. Hence,

$$\begin{aligned}
\mathbb{E}\left[|X^1_{t\wedge\tau_n} - X^2_{t\wedge\tau_n}|\right] &= \mathbb{E}\left[\left|\int_0^{t\wedge\tau_n} f(s,\cdot,X^1_{s-}) - f(s,\cdot,X^2_{s-})\,\mathrm{d}s\right|\right] \\
&\le K_n \mathbb{E}\left[\int_0^{t\wedge\tau_n} |X^1_s - X^2_s|\,\mathrm{d}s\right] \\
&= K_n \int_0^t \mathbb{E}\left[\mathbb{1}_{\{\tau_n \ge s\}} |X^1_s - X^2_s|\right]\mathrm{d}s \\
&\le K_n \int_0^t \mathbb{E}\left[|X^1_{s\wedge\tau_n} - X^2_{s\wedge\tau_n}|\right]\mathrm{d}s.
\end{aligned}$$

We can apply Gronwall's Lemma and conclude $\mathbb{E}\left[|X^1_{t\wedge\tau_n} - X^2_{t\wedge\tau_n}|\right] = 0$. Thus, $X^1_{t\wedge\tau_n} = X^2_{t\wedge\tau_n}$ almost surely for all n. Letting $n \to \infty$ yields $X^1_t = X^2_t$ almost surely.

To prove the existence statement, we use the previous theorem together with a suitable cut-off scheme. For any $R > 0$ define

$$f_R(s,\omega,x) := \begin{cases} f(s,\omega,x) & \text{for } |x| \le R \\ \left(2 - \frac{x}{R}\right) f(s,\omega,R) & \text{for } r < x < 2R \\ \left(2 + \frac{x}{R}\right) f(s,\omega,-R) & \text{for } -2R < x < R \\ 0 & \text{for } |x| \ge 2R. \end{cases}$$

Then f_R satisfies the conditions of theorem 4.9 with $K(\omega) := \max\left(K_R, \frac{K_R}{R}\right) =: \overline{K}_R$. Denote by X^R the (by theorem 4.9) unique solution of the SDE

$$\begin{aligned} X_t &= X_0 + \int_0^t f_R(s,\cdot,X_{s-})\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}S_s \\ &= X_0 + \int_0^t f_R(s,\cdot,X_{s-})\,\mathrm{d}s + \int_0^t \sqrt{c_s}\sigma(s)\,\mathrm{d}W_s + \int_0^t\int_{\mathbb{R}^d} \langle \sigma(s), x\rangle(\mu-\nu)(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

Introduce the stopping time $\tau_R := \inf\{t \geq 0 : |X_t^R| \geq R\}$ and define

$$X_t^\infty := X_t^R \qquad \text{for } t \leq \tau_R.$$

To check that X^∞ is well defined let $0 < R_1 < R_2$ and $\tau := \min(\tau_{R_1}, \tau_{R_2})$, then (similarly as in the proof of uniqueness)

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau} |X_s^{R_1} - X_s^{R_2}|\right] &= \mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau} \left|\int_0^s f_{R_1}(u,\cdot,X_{u-}^{R_1}) - f_{R_2}(u,\cdot,X_{u-}^{R_2})\,\mathrm{d}u\right|\right] \\ &\leq \mathbb{E}\left[\int_0^{t\wedge\tau} |f_{R_1}(u,\cdot,X_{u-}^{R_1}) - f_{R_2}(u,\cdot,X_{u-}^{R_2})|\,\mathrm{d}u\right] \\ &\leq \overline{K}_{R_2}\mathbb{E}\left[\int_0^{t\wedge\tau} |X_u^{R_1} - X_u^{R_2}|\,\mathrm{d}u\right] \\ &\leq \overline{K}_{R_2}\int_0^t \mathbb{E}\left[\sup_{0\leq s\leq u\wedge\tau} |X_s^{R_1} - X_s^{R_2}|\right]\mathrm{d}u. \end{aligned}$$

Again, we can apply Gronwall's Lemma and conclude

$$\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau} |X_s^{R_1} - X_s^{R_2}|\right] = 0 \qquad \text{for all } t.$$

Hence, $X_t^{R_1}$ and $X_t^{R_2}$ coincide almost surely for $t \leq \min(\tau_{R_1}, \tau_{R_2})$ and X^∞ is well defined. It remains to show that $\lim_{R\to\infty} \tau_R = \infty$ almost surely, since in this case X^∞ is a solution to (4.19) and therefore a semimartingale.

Let $h(x,t) := e^{-Bt}(1+x^2)$ with $B := 2B_1 + B_2$ and $Y_t^R := h(X_t^R, t)$, then by Itô's formula

$$
\begin{aligned}
Y_t^R - Y_0^R &= e^{-Bt}(1+(X_t^R)^2) - (1+(X_0)^2)\\
&= \int_0^t 2X_{s-}^R e^{-Bs}\,\mathrm{d}X_s^R - B\int_0^t e^{-Bs}\big(1+(X_{s-}^R)^2\big)\,\mathrm{d}s\\
&\quad + \int_0^t e^{-Bs}\langle\sigma(s), c_s\sigma(s)\rangle\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}^d} e^{-Bs}\langle\sigma(s), x\rangle^2\mu(\mathrm{d}s,\mathrm{d}x)\\
&= \text{local martingale}\\
&\quad + \int_0^t e^{-Bs}\Big(2X_{s-}^R f_R(s,\cdot,X_{s-}^R) - B\big(1+(X_{s-}^R)^2\big)\\
&\qquad\qquad + \langle\sigma(s), c_s\sigma(s)\rangle + \int_{\mathbb{R}^d}\langle\sigma(s), x\rangle^2 F_s(\mathrm{d}x)\Big)\,\mathrm{d}s.
\end{aligned}
$$

From condition 3 of the prerequisites and (4.18) we know that there is a localizing sequence $(T_n)_{n\geq 1}$ such that the stopped process $(Y^R)^{T_n}$ is a supermartingale for all n. By the optional stopping theorem, the stopped process $(Y^R)^{T_n\wedge\tau_R}$ is a supermartingale for all n. Hence,

$$
\begin{aligned}
1+(X_0)^2 &\geq \mathbb{E}\left[e^{-B(t\wedge T_n\wedge\tau_R)}\left(1+(X_{t\wedge T_n\wedge\tau_R}^R)^2\right)\right]\\
&\geq e^{-Bt}(1+R^2)\mathbb{P}(\{\tau_R\leq t\}\cap\{\tau_R\leq T_n\}).
\end{aligned}
$$

Taking the limes inferior (over n) on both sides and using Fatou's Lemma we obtain

$$1+(X_0)^2 \geq e^{-Bt}(1+R^2)\mathbb{P}(\{\tau_R\leq t\}).$$

From $\lim_{R\to\infty} 1+R^2 = \infty$ we get $\lim_{R\to\infty}\mathbb{P}(\{\tau_R\leq t\}) = 0$. Since for $R_1 < R_2$ we have $\{\tau_{R_2}\leq t\}\subset\{\tau_{R_1}\leq t\}$, there exists for $\mathbb{P}$-almost every ω and all $t>0$ a constant R_0 (which may depend on ω and t) such that $\tau_R(\omega)\geq t$ for all $R\geq R_0$. This is equivalent to $\lim_{R\to\infty}\tau_R = \infty$ almost surely. □

We can use the previous proposition to check that, at least in case the driving process L is one-dimensional (d=1), the SDE (4.16) admits a unique non-exploding solution:

Proposition 4.11 *Assume $d=1$. Suppose that $\gamma(\cdot, T_i)$ is a càglàd function for each $i\in\{1,\dots,n-1\}$ and that the characteristics of L^{T^*} are chosen in such a way that $f_1^i(\cdot,\cdot,x),\dots,f_4^i(\cdot,\cdot,x)$ have càglàd paths for each $x\in\mathbb{R}$. Then the stochastic differential equation (4.16) admits a unique (non-exploding) solution for each $i\in\{1,\dots,n-1\}$.*

Proof: We use proposition 4.10 with

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) := (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \mathbb{P}_{T_{i+1}}),$$

$$S_t := \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{T_{i+1}} + \int_0^t \int_{\mathbb{R}} x(\mu - \nu^{T_{i+1}})(\mathrm{d}s, \mathrm{d}x),$$

and $\sigma(s) := \gamma(s, T_i)$. Assumptions ($\mathbb{SUP}$) and ($\mathbb{DLR}$.1) imply (4.18). It remains to verify the conditions 1-3 of proposition 4.10 for f^i. Condition 1 is satisfied by assumption. Conditions 2 and 3 can be checked separately for $f_1^i, \ldots, f_4^i$. Again, ($\mathbb{SUP}$) and ($\mathbb{DLR}$.1) yield that condition 2 holds for $f_1^i, \ldots, f_4^i$ and that condition 3 holds for f_1^i and f_4^i. It remains to show that condition 3 is also satisfied for f_2^i and f_3^i. For this purpose, it is sufficient to prove that there are constants C_2, C_3 such that for all (t, ω) and all $x \in \mathbb{R}$

$$0 \leq \widetilde{f}_j^i(t, \omega, x) \leq C_j \qquad (j \in \{2, 3\}),$$

where $\widetilde{f}_j^i(t, \omega, x) := \frac{\delta_i e^x}{1+\delta_i e^x} f_j^i(t, \omega, x)$. The existence of the upper bound once again follows from ($\mathbb{SUP}$) and ($\mathbb{DLR}$.1). Moreover, $\widetilde{f}_2^i$ and $\widetilde{f}_3^i$ are nonnegative since $\alpha(\cdot, T_i, T_{i+1})$ and the integrand in (4.17) are nonnegative (at this point, the assumption $d = 1$ is needed). $\square$

Remark: To prove the existence of a solution to (4.16) for $d > 1$ we have to put further restrictions on the characteristics of L to meet the growth condition (condition 3) of proposition 4.10. For example, in the case of a multivariate Gaussian model (i.e. $F_s = 0$ for all $s \in [0, T^*]$) assuming that $\langle \gamma(s, T_j), c_s \lambda(s, T_i) \rangle$ is nonnegative for all $1 \leq j < i \leq n-1$ will do the job. This can be achieved by requiring that all entries in the matrices c_s are nonnegative.

In the subsequent sections, we assume that the drift terms $b^H(\cdot, T_i, T_{i+1})$ are chosen as described above and do not distinguish between $\widehat{H}$ and H anymore.

4.4 Defaultable forward measures

It is well known that pricing of derivatives in default-free interest rate models can often be facilitated considerably by changing numeraires, i.e. changing measures, in a suitable way. In particular, forward measures prove to be useful in many situations. Similarly, valuation of contingent claims in our model can be simplified by using two counterparts to the default-free forward measures. The first definition traces back to Schönbucher (1999a):

Definition 4.12 *The defaultable forward (martingale) measure or survival measure $\overline{\mathbb{Q}}_{T_i}$ for the settlement day T_i is defined on $(\Omega, \mathcal{G}_{T_i})$ by*

$$\frac{\mathrm{d}\overline{\mathbb{Q}}_{T_i}}{\mathrm{d}\mathbb{Q}_{T_i}} := \frac{B(0, T_i)}{B^0(0, T_i)} B^0(T_i, T_i) = \frac{B(0, T_i)}{\overline{B}(0, T_i)} 1\!\!1_{\{\tau > T_i\}}.$$

Equation (4.6) ensures that the preceding expression is indeed a density. $\overline{\mathbb{Q}}_{T_i}$ corresponds to the choice of $B^0(\cdot, T_i)$ as a "numeraire". We use quotation marks since $B^0(\cdot, T_i)$ is not a strictly positive process with probability one. Consequently, $\overline{\mathbb{Q}}_{T_i}$ is absolutely continuous with respect to $\mathbb{Q}_{T_i}$, but the two measures are not mutually equivalent. In particular, the set $A = \{\tau \leq t\}$ for $t \in (0, T_i]$ has a strictly positive probability under $\mathbb{Q}_{T_i}$ but zero probability under $\overline{\mathbb{Q}}_{T_i}$. The term "survival measure" is justified by the fact that

$$\overline{\mathbb{Q}}_{T_i}(A) = \frac{\mathbb{Q}_{T_i}(A \cap \{\tau > T_i\})}{\mathbb{Q}_{T_i}(\{\tau > T_i\})} = \mathbb{Q}_{T_i}(A | \{\tau > T_i\}) \qquad (A \in \mathcal{G}_{T_i}),$$

i.e. $\overline{\mathbb{Q}}_{T_i}$ can be regarded as the forward measure $\mathbb{Q}_{T_i}$ conditioned on survival until T_i. Once restricted to the σ-field $\mathcal{G}_t$, the defaultable forward measure becomes

$$\left.\frac{d\overline{\mathbb{Q}}_{T_i}}{d\mathbb{Q}_{T_i}}\right|_{\mathcal{G}_t} = \frac{B(0, T_i)}{\overline{B}(0, T_i)} \frac{\overline{B}(t, T_i)}{B(t, T_i)} 1\!\!1_{\{\tau > t\}} = \frac{B(0, T_i)}{\overline{B}(0, T_i)} 1\!\!1_{\{\tau > t\}} \frac{\mathbb{Q}_{T_i}(\{\tau > T_i\} | \mathcal{F}_t)}{\mathbb{Q}_{T_i}(\{\tau > t\} | \mathcal{F}_t)}.$$

The first equality follows from the fact that $\frac{B^0(\cdot, T_i)}{B(\cdot, T_i)}$ is a $\mathbb{Q}_{T_i}$-martingale, the second equality from (4.7).

Another very useful tool in the context of derivative pricing is the restricted defaultable forward measure, which has already been used in Bielecki and Rutkowski (2002, Section 15.2). Note that the defaultable forward measure restricted to the σ-field $\mathcal{F}_t$ is given by

$$\left.\frac{d\overline{\mathbb{Q}}_{T_i}}{d\mathbb{Q}_{T_i}}\right|_{\mathcal{F}_t} = \frac{B(0, T_i)}{\overline{B}(0, T_i)} \mathbb{Q}_{T_i}(\{\tau > T_i\} | \mathcal{F}_t)$$

and denote by $\mathbb{P}_{T_i}$ the restriction of $\mathbb{Q}_{T_i}$ to the σ-field $\mathcal{F}_{T_i}$. This notation differs slightly from the notation in the default-free part of the model where $\mathbb{P}_{T_i}$ was defined on $\widetilde{\mathcal{F}}_{T_i}$. However, this should not cause any confusion since $\mathcal{F}_{T_i}$ is the trivial extension of $\widetilde{\mathcal{F}}_{T_i}$.

Definition 4.13 *The restricted defaultable forward (martingale) measure $\overline{\mathbb{P}}_{T_i}$ for the settlement day T_i is defined on $(\Omega, \mathcal{F}_{T_i})$ by*

$$\frac{d\overline{\mathbb{P}}_{T_i}}{d\mathbb{P}_{T_i}} = \frac{B(0, T_i)}{\overline{B}(0, T_i)} \mathbb{Q}_{T_i}(\{\tau > T_i\} | \mathcal{F}_{T_i}).$$

We have an explicit expression for this density, namely

$$\frac{d\overline{\mathbb{P}}_{T_i}}{d\mathbb{P}_{T_i}} = \frac{B(0, T_i)}{\overline{B}(0, T_i)} e^{-\Gamma_{T_i}} = \frac{B(0, T_i)}{\overline{B}(0, T_i)} \prod_{k=0}^{i-1} \frac{1}{1 + \delta_k H(T_k, T_k)}. \tag{4.20}$$

Restricted to the the σ-field $\mathcal{F}_t$ this becomes (since $\prod_{k=0}^{i-1} \frac{1}{1+\delta_k H(\cdot, T_k)}$ is a $\mathbb{P}_{T_i}$-martingale)

$$\left.\frac{d\overline{\mathbb{P}}_{T_i}}{d\mathbb{P}_{T_i}}\right|_{\mathcal{F}_t} = \frac{B(0, T_i)}{\overline{B}(0, T_i)} \prod_{k=0}^{i-1} \frac{1}{1 + \delta_k H(t, T_k)}. \tag{4.21}$$

From equation (A.1) we get the representation

$$\frac{\mathrm{d}\overline{\mathbb{P}}_{T_i}}{\mathrm{d}\mathbb{P}_{T_i}} = \mathcal{E}_{T_{i-1}}\Bigg(\int_0^{\bullet} -\sum_{l=1}^{i-1} Y_{s-}^l \sqrt{c_s}\gamma(s,T_l)\,\mathrm{d}W_s^{T_i} + \int_0^{\bullet}\int_{\mathbb{R}^d}\left(\prod_{l=1}^{i-1}\left(1+Y_{s-}^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)^{-1}-1\right)(\mu-\nu^{T_i})(\mathrm{d}s,\mathrm{d}x)\Bigg)$$

with

$$Y_s^l := \frac{\delta_l H(s,T_l)}{1+\delta_l H(s,T_l)}.$$

Hence, the two predictable processes in Girsanov's Theorem for semimartingales (see Jacod and Shiryaev (2003, Theorem III.3.24)) associated with this change of measure are

$$\beta(s) = -\sum_{l=1}^{i-1}\left(Y_{s-}^l\gamma(s,T_l)\right) \qquad \text{and}$$

$$Y(s,x) = \prod_{l=1}^{i-1}\left(1+Y_{s-}^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)^{-1}.$$

We can conclude that

$$\overline{W}_t^{T_i} := W_t^{T_i} + \int_0^t \sum_{l=1}^{i-1} Y_{s-}^l\sqrt{c_s}\gamma(s,T_l)\,\mathrm{d}s \tag{4.22}$$

is a $\overline{\mathbb{P}}_{T_i}$-standard Brownian motion and the $\overline{\mathbb{P}}_{T_i}$-compensator of μ is given by

$$\overline{\nu}^{T_i}(\mathrm{d}s,\mathrm{d}x) = \prod_{l=1}^{i-1}\left(1+Y_{s-}^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)^{-1}\nu^{T_i}(\mathrm{d}s,\mathrm{d}x) =: \overline{F}_s^{T_i}(\mathrm{d}x)\,\mathrm{d}s. \tag{4.23}$$

Similar to the default-free part of the model, we have the following connection between restricted defaultable forward measures for different settlement days:

Lemma 4.14 *The defaultable Libor rate $(\overline{L}(t,T_i))_{0\le t\le T_i}$ is a $\overline{\mathbb{P}}_{T_{i+1}}$-martingale and*

$$\left.\frac{\mathrm{d}\overline{\mathbb{P}}_{T_i}}{\mathrm{d}\overline{\mathbb{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t} = \frac{\overline{B}(0,T_{i+1})}{\overline{B}(0,T_i)}(1+\delta_i\overline{L}(t,T_i)) \qquad (0\le t\le T_i).$$

Proof: From equation (4.1) we get

$$\begin{aligned}(1+\delta_i\overline{L}(t,T_i)) &= (1+\delta_i H(t,T_i))(1+\delta_i L(t,T_i))\\ &= \prod_{k=0}^{i}(1+\delta_k H(t,T_k))(1+\delta_i L(t,T_i))\prod_{k=0}^{i-1}(1+\delta_k H(t,T_k))^{-1}.\end{aligned}$$

Applying equations (3.17) and (4.21) yields

$$\begin{aligned}(1+\delta_i\overline{L}(t,T_i)) &= \frac{B(0,T_{i+1})}{\overline{B}(0,T_{i+1})}\left.\frac{\mathrm{d}\mathbb{P}_{T_{i+1}}}{\mathrm{d}\overline{\mathbb{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t}\frac{B(0,T_i)}{B(0,T_{i+1})}\left.\frac{\mathrm{d}\mathbb{P}_{T_i}}{\mathrm{d}\mathbb{P}_{T_{i+1}}}\right|_{\mathcal{F}_t}\frac{\overline{B}(0,T_i)}{B(0,T_i)}\left.\frac{\mathrm{d}\overline{\mathbb{P}}_{T_i}}{\mathrm{d}\mathbb{P}_{T_i}}\right|_{\mathcal{F}_t}\\ &= \frac{\overline{B}(0,T_i)}{\overline{B}(0,T_{i+1})}\left.\frac{\mathrm{d}\overline{\mathbb{P}}_{T_i}}{\mathrm{d}\overline{\mathbb{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t}\end{aligned}$$

and both statements are established. □

As mentioned above, (restricted) defaultable forward measures can be used to determine prices of contingent claims. Consider a defaultable claim with a promised payoff of X at the settlement day T_i and zero recovery upon default. Then its time-t value is given by

$$\pi_t^X := 1\!\!1_{\{\tau>t\}}B(t,T_i)\mathbb{E}_{\mathbb{Q}_{T_i}}[X1\!\!1_{\{\tau>T_i\}}|\mathcal{G}_t] \qquad (t\in[0,T_i]).$$

Consider the general case in which X is $\mathcal{G}_{T_i}$-measurable and the common case of an $\mathcal{F}_{T_i}$-measurable promised payoff X. The following proposition is a corrected version of Bielecki and Rutkowski (2002, Proposition 15.2.3):

Proposition 4.15 *Assume that the promised payoff X is $\mathcal{G}_{T_i}$-measurable and integrable with respect to $\overline{\mathbb{Q}}_{T_i}$. Then*

$$\pi_t^X = 1\!\!1_{\{\tau>t\}}\overline{B}(t,T_i)\mathbb{E}_{\overline{\mathbb{Q}}_{T_i}}[X|\mathcal{G}_t] = B^0(t,T_i)\mathbb{E}_{\overline{\mathbb{Q}}_{T_i}}[X|\mathcal{G}_t].$$

If X is $\mathcal{F}_{T_i}$-measurable, then

$$\pi_t^X = 1\!\!1_{\{\tau>t\}}\overline{B}(t,T_i)\mathbb{E}_{\overline{\mathbb{P}}_{T_i}}[X|\mathcal{F}_t] = B^0(t,T_i)\mathbb{E}_{\overline{\mathbb{P}}_{T_i}}[X|\mathcal{F}_t].$$

Proof: The first statement can be proved along the lines of Bielecki and Rutkowski (2002, Proposition 15.2.3). For the second statement observe that

$$\begin{aligned}\pi_t^X &= 1\!\!1_{\{\tau>t\}}B(t,T_i)\mathbb{E}_{\mathbb{Q}_{T_i}}[X1\!\!1_{\{\tau>T_i\}}|\mathcal{G}_t]\\ &= 1\!\!1_{\{\tau>t\}}B(t,T_i)\frac{\mathbb{E}_{\mathbb{Q}_{T_i}}[X1\!\!1_{\{\tau>T_i\}}|\mathcal{F}_t]}{\mathbb{Q}_{T_i}\{\tau>t|\mathcal{F}_t\}}\\ &= 1\!\!1_{\{\tau>t\}}\overline{B}(t,T_i)\frac{\mathbb{E}_{\mathbb{Q}_{T_i}}[X1\!\!1_{\{\tau>T_i\}}|\mathcal{F}_t]}{\mathbb{Q}_{T_i}\{\tau>T_i|\mathcal{F}_t\}}\\ &= 1\!\!1_{\{\tau>t\}}\overline{B}(t,T_i)\frac{\mathbb{E}_{\mathbb{P}_{T_i}}[X\mathbb{Q}_{T_i}\{\tau>T_i|\mathcal{F}_{T_i}\}|\mathcal{F}_t]}{\mathbb{Q}_{T_i}\{\tau>T_i|\mathcal{F}_t\}}\\ &= 1\!\!1_{\{\tau>t\}}\overline{B}(t,T_i)\mathbb{E}_{\overline{\mathbb{P}}_{T_i}}[X|\mathcal{F}_t].\end{aligned}$$

We used Bielecki and Rutkowski (2002, (5.2)) for the second equality, equation (4.7) for the third and the abstract Bayes rule for the last equality. □

4.5 Recovery rules and bond prices

In the previous sections we specified the evolution of (ratios of pre-default values of) defaultable zero coupon bonds with zero recovery. In real markets however, dafaultable bonds usually have a positive recovery. In order to adapt our model to this fact, we have to incorporate suitable recovery rules for bonds. An overview on different kinds of recovery rules can be found in Bielecki and Rutkowski (2002) and Schönbucher (1999b).

For defaultable zero coupon bonds we adopt the *fractional recovery of treasury value* scheme, i.e. in case of a default before or at maturity T of the bond, the owner receives a fixed fraction $\pi \in [0, 1)$ of a default-free zero coupon bond with the same notional (which we set equal to one for simplicity) and the same maturity. To put it differently, the owner receives an amount of π at T. Consequently, the time-T value of this bond is

$$B^{\pi}(T,T) := 1\!\!1_{\{\tau>T\}} + \pi 1\!\!1_{\{\tau\leq T\}} = \pi + (1-\pi)1\!\!1_{\{\tau>T\}}.$$

Its time-t value ($t \leq T$) is thus

$$B^{\pi}(t,T) = \pi B(t,T) + (1-\pi)1\!\!1_{\{\tau>t\}}\overline{B}(t,T).$$

In default-free interest rate models, a coupon bearing bond can be considered as a portfolio of zero coupon bonds. For defaultable coupon bonds the situation is not quite as simple. A coupon bond can still be decomposed into a series of zero coupon bonds, but it does not make much sense to assume the same recovery rate π for all. Schönbucher (1999a, p. 14) remarks: "The claim of a creditor on the defaulted debtor's assets is only determined by the outstanding principal and accrued interest payments of the defaulted loan or bond, any future coupon payments do not enter the consideration." We adopt his *recovery of par* scheme for coupon bearing bonds:

Assumption (recovery of par). *The recovery of a defaultable coupon bond that defaults in the time interval* $(T_k, T_{k+1}]$ *is given by the recovery rate* $\pi \in [0, 1)$ *times the sum of the notional and the accrued interest over* $(T_k, T_{k+1}]$. *It is paid at* T_{k+1}.

Note that this assumption restricts recovery payments to the tenor dates. This restriction is not strong for a number of reasons. We refer to Schönbucher (1999a, Section 6.2) for a discussion.

Let us denote by $e_k^X(t)$ the time-t value of receiving an amount of X at T_{k+1} if and only if a default occurred in the time interval $(T_k, T_{k+1}]$.

Lemma 4.16 *Let X be $\mathcal{F}_{T_k}$-measurable. Then, for $t \leq T_k$*

$$e_k^X(t) = 1\!\!1_{\{\tau>t\}}\overline{B}(t,T_{k+1})\delta_k \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[XH(T_k,T_k)|\mathcal{F}_t].$$

Proof: We have

$$e_k^X(T_{k+1}) = X1\!\!1_{\{\tau>T_k\}} - X1\!\!1_{\{\tau>T_{k+1}\}}.$$

Receiving an amount of $X\mathbb{1}_{\{\tau>T_k\}}$ at T_{k+1} is equivalent to receiving an amount of $X\mathbb{1}_{\{\tau>T_k\}}B(T_k,T_{k+1})$ at T_k. Combining this fact with proposition 4.15 yields for $t \le T_k$

$$\begin{aligned} e_k^X(t) &= \mathbb{1}_{\{\tau>t\}}\left(\overline{B}(t,T_k)\mathbb{E}_{\overline{\mathbb{P}}_{T_k}}[XB(T_k,T_{k+1})|\mathcal{F}_t] - \overline{B}(t,T_{k+1})\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[X|\mathcal{F}_t]\right) \\ &= \mathbb{1}_{\{\tau>t\}}\overline{B}(t,T_{k+1}) \\ &\quad\times\left(\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[(1+\delta_k\overline{L}(T_k,T_k))XB(T_k,T_{k+1})|\mathcal{F}_t] - \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[X|\mathcal{F}_t]\right) \\ &= \mathbb{1}_{\{\tau>t\}}\overline{B}(t,T_{k+1})\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[XH(T_k,T_k)|\mathcal{F}_t]. \end{aligned}$$

The second equality follows from the abstract Bayes rule, the third follows by using equation (4.1). □

With the help of the preceding lemma we can deduce the time-0 price of a defaultable coupon bond with m coupons of c that are promised to be paid at the dates $T_1,\ldots,T_m$ as

$$\begin{aligned} B^\pi_{\text{fixed}}(0;c,m) :&= \overline{B}(0,T_m) + \sum_{k=0}^{m-1} c\overline{B}(0,T_{k+1}) + \sum_{k=0}^{m-1} \pi(1+c)e_k^1(0) \\ &= \overline{B}(0,T_m) + \sum_{k=0}^{m-1} \overline{B}(0,T_{k+1})\left(c + \pi(1+c)\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)]\right). \end{aligned}$$

Similarly, the price of a defaultable floating coupon bond that pays an interest rate composed of the default-free Libor rate plus a constant spread x can be obtained. Suppose that the bond has m coupons, i.e. the bondholder is promised to receive an amount of $\delta_k(L(T_k,T_k)+x)$ at the dates T_{k+1} for $0 \le k \le m-1$, then its time-0 price is given by (using proposition 4.15)

$$\begin{aligned} B^\pi_{\text{floating}}(0;x,m) &:= \overline{B}(0,T_m) + \sum_{k=0}^{m-1} \delta_k\overline{B}(0,T_{k+1})\left(x + \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)]\right) \\ &\qquad + \sum_{k=0}^{m-1} \pi\left((1+\delta_k x)e_k^1(0) + \delta_k e_k^{L(T_k,T_k)}(0)\right) \\ &= \overline{B}(0,T_m) + \sum_{k=0}^{m-1} \delta_k\overline{B}(0,T_{k+1})\Big(x + \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)] \\ &\qquad + \pi(1+\delta_k x)\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)] \\ &\qquad + \pi\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)L(T_k,T_k)]\Big). \end{aligned}$$

Consequently, in order to price defaultable coupon bonds we need to evaluate $\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)]$, $\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)]$ and $\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)L(T_k,T_k)]$:

Let us use the abbreviations

$$V_t^i := \frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)} \qquad \text{and} \qquad Y_t^i := \frac{\delta_i H(t,T_i)}{1+\delta_i H(t,T_i)}.$$

Combining the equations (4.3), (4.15), (3.13), (3.15), (4.22), and (4.23) yields

$$\begin{aligned} H(t,T_k) \;=\; H(0,T_k)\exp\Bigg(&\int_0^t \overline{b}^H(s,T_k,T_{k+1})\,\mathrm{d}s + \int_0^t \sqrt{c_s}\gamma(s,T_k)\,\mathrm{d}\overline{W}_s^{T_{k+1}} \\ &+ \int_0^t\int_{\mathbb{R}^d} \langle\gamma(s,T_k),x\rangle(\mu-\overline{\nu}^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\Bigg), \end{aligned}$$

where

$$\begin{aligned} \overline{b}^H(s,T_k,T_{k+1}) \;=\; &-\frac{1}{2}\langle\gamma(s,T_k),c_s\gamma(s,T_k)\rangle + \sum_{l=1}^{k-1}\frac{Y_{s-}^l V_{s-}^k}{Y_{s-}^k}\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle \\ &- \int_{\mathbb{R}^d}\Big(e^{\langle\gamma(s,T_k),x\rangle}-1-\langle\gamma(s,T_k),x\rangle\Big)\,\overline{F}_s^{T_{k+1}}(\mathrm{d}x) \\ &+ \int_{\mathbb{R}^d}\frac{V_{s-}^k}{Y_{s-}^k}\Big(e^{\langle\lambda(s,T_k),x\rangle}-1\Big)\Big(1+Y_{s-}^k\Big(e^{\langle\gamma(s,T_k),x\rangle}-1\Big)\Big) \\ &\qquad\times\left(\prod_{l=1}^{k-1}\Big(1+Y_{s-}^l\Big(e^{\langle\gamma(s,T_l),x\rangle}-1\Big)\Big)-1\right)\overline{F}_s^{T_{k+1}}(\mathrm{d}x) \end{aligned}$$

as well as

$$\begin{aligned} L(t,T_k) \;=\; L(0,T_k)\exp\Bigg(&\int_0^t \overline{b}^L(s,T_k,T_{k+1})\,\mathrm{d}s + \int_0^t \sqrt{c_s}\lambda(s,T_k)\,\mathrm{d}\overline{W}_s^{T_{k+1}} \\ &+ \int_0^t\int_{\mathbb{R}^d} \langle\lambda(s,T_k),x\rangle(\mu-\overline{\nu}^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\Bigg) \end{aligned}$$

with

$$\begin{aligned} &\overline{b}^L(s,T_k,T_{k+1}) = \\ &\quad -\frac{1}{2}\langle\lambda(s,T_k),c_s\lambda(s,T_k)\rangle - \sum_{l=1}^{k} Y_{s-}^l\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle \\ &\quad - \int_{\mathbb{R}^d}\Big(e^{\langle\lambda(s,T_k),x\rangle}-1-\langle\lambda(s,T_k),x\rangle\Big)\,\overline{F}_s^{T_{k+1}}(\mathrm{d}x) \\ &\quad - \int_{\mathbb{R}^d}\Big(e^{\langle\lambda(s,T_k),x\rangle}-1\Big)\left(\prod_{l=1}^{k}\Big(1+Y_{s-}^l\Big(e^{\langle\gamma(s,T_l),x\rangle}-1\Big)\Big)-1\right)\overline{F}_s^{T_{k+1}}(\mathrm{d}x). \end{aligned}$$

Making use of Kallsen and Shiryaev (2002, Lemma 2.6) we get

$$
\begin{aligned}
&H(t,T_k) = \qquad (4.24)\\
&H(0,T_k)\exp\Bigg(\int_0^t\sum_{l=1}^{k-1}\frac{Y^l_{s-}V^k_{s-}}{Y^k_{s-}}\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle\,\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{R}^d}\frac{V^k_{s-}}{Y^k_{s-}}\left(e^{\langle\lambda(s,T_k),x\rangle}-1\right)\left(1+Y^k_{s-}\left(e^{\langle\gamma(s,T_k),x\rangle}-1\right)\right)\\
&\quad\times\left(\prod_{l=1}^{k-1}\left(1+Y^l_{s-}\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)-1\right)\overline{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x)\Bigg)\\
&\times\mathcal{E}_t\Bigg(\int_0^\bullet\sqrt{c_s}\gamma(s,T_k)\,\mathrm{d}\overline{W}^{T_{k+1}}_s+\int_0^\bullet\int_{\mathbb{R}^d}\left(e^{\langle\gamma(s,T_k),x\rangle}-1\right)(\mu-\overline{\nu}^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)\Bigg).
\end{aligned}
$$

To obtain an expression for $\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)]$ we approximate the stochastic terms V^i_{s-} and Y^i_{s-} by their deterministic initial values V^i_0 and Y^i_0. Similar approximations have been used by Brace, Gatarek, and Musiela (1997), Rebonato (1998), Schönbucher (1999a), and Schlögl (2002). This yields

$$
\begin{aligned}
\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)] \approx H(0,T_k)\exp\Bigg(&\int_0^t\sum_{l=1}^{k-1}\frac{Y^l_0V^k_0}{Y^k_0}\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle\,\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{R}^d}\frac{V^k_0}{Y^k_0}\left(e^{\langle\lambda(s,T_k),x\rangle}-1\right)\left(1+Y^k_0\left(e^{\langle\gamma(s,T_k),x\rangle}-1\right)\right)\\
&\times\left(\prod_{l=1}^{k-1}\left(1+Y^l_0\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)-1\right)\widetilde{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x)\Bigg),
\end{aligned}
$$

where $\widetilde{\nu}^{T_{k+1}}$ is an approximation for $\overline{\nu}^{T_{k+1}}$ given by

$$
\begin{aligned}
\widetilde{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) = &\prod_{l=1}^{k}\left(1+Y^l_0\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)^{-1} \qquad (4.25)\\
&\times\prod_{l=k+1}^{n-1}\left(1+V^l_0\left(e^{\langle\lambda(s,T_l),x\rangle}-1\right)\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).
\end{aligned}
$$

Proceeding similarly yields

$$\begin{aligned}
\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)] \approx L(0,T_k)\exp\Bigg(&-\int_0^t \sum_{l=1}^{k} Y_0^l \langle\gamma(s,T_l), c_s\lambda(s,T_k)\rangle\,\mathrm{d}s \\
&-\int_0^t\int_{\mathbb{R}^d}\left(e^{\langle\lambda(s,T_k),x\rangle}-1\right) \\
&\times\left(\prod_{l=1}^{k}\left(1+Y_0^l\left(e^{\langle\gamma(s,T_l),x\rangle}-1\right)\right)-1\right)\widetilde{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x)\Bigg).
\end{aligned}$$

To obtain $\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)H(T_k,T_k)]$ observe that, since $\overline{L}(\cdot,T_k)$ is a $\overline{\mathbb{P}}_{T_{k+1}}$-martingale,

$$\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[1+\delta_k\overline{L}(T_k,T_k)] = 1+\delta_k\overline{L}(0,T_k).$$

On the other hand, by equation (4.1),

$$\begin{aligned}
\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[1+\delta_k\overline{L}(T_k,T_k)] &= \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[(1+\delta_k L(T_k,T_k))(1+\delta_k H(T_k,T_k))] \\
&= 1+\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)]+\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)] \\
&\quad + (\delta_k)^2\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)H(T_k,T_k)].
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)H(T_k,T_k)] = \\
&\qquad \frac{1}{\delta_k}\left(\overline{L}(0,T_k)-\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)]-\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)]\right).
\end{aligned}$$

4.6 Credit-sensitive swap contracts

The market for credit derivatives has increased enormously in volume since the first of these contracts have been introduced in the early 1990s. Their success is due to the fact that they allow to transfer credit risk from one party to another and therewith to manage the risk exposure. Long before 1990 products with credit derivative-like features (as e.g. *letters of credit* or *bond insurances*) have been traded. However, in contrast to the modern products, these contracts did not allow to trade the credit risk protection separately from the risky underlying. There are many publications describing various credit derivatives in detail, among which are Schönbucher (2000), Bielecki and Rutkowski (2002) and Schmid (2004). Information about the size of the credit derivatives' market as well as on the market share that different products have can also be found in Schmid (2004).

The aim of this and the following section is to derive valuation formulae, in our model framework, for some of the most popular and heavily traded credit derivatives. Credit-sensitive swaps are considered now and credit options in the next section. We use the notational convention that the credit derivative contract is signed between two parties A (who will usually receive a payment if a default occurs) and B (who pays in case of a default). The reference entity (e.g. a corporate bond) is issued by a third party C.

If credit derivatives are traded over-the-counter, each party of the contract is exposed to the risk that the other party cannot fulfill its obligations. In the following, we assume that this counterparty risk can be neglected, i.e. only the risk that the reference entity defaults is considered.

4.6.1 Credit default swaps

Credit default swaps can be used to insure defaultable assets against default. The protection buyer A agrees to pay a fixed amount to the protection seller B periodically until a pre-specified credit event (as e.g. the default of a bond issued by a reference party C) occurs or the contract terminates. In turn, B promises to make a specified payment to A that covers his loss if the credit event happens. There are various types of default swaps differing in the specification of the credit event as well as in the specification of the default payment.

Let us consider a standard default swap with maturity date T_m whose credit event is the default of a coupon bond issued by C. The default payment is chosen such that it covers the loss of A. More precisely, A receives an amount of

$$1 - \pi(1 + c) \qquad \text{(fixed coupon bond)}$$

or

$$1 - \pi(1 + \delta_k(L(T_k, T_k) + x)) \qquad \text{(floating coupon bond)}$$

at T_{k+1} if a default happens in $(T_k, T_{k+1}]$ for $k \in \{0, \ldots, m-1\}$. For this protection A pays a fee s at the dates $T_0, \ldots, T_{m-1}$ until default. Our goal is to determine the default swap rate, i.e. the level of s that makes the initial value of the contract equal to zero.
The time-0 value of the fee payments is

$$s \sum_{k=1}^{m} \overline{B}(0, T_{k-1}).$$

The initial value of the default payment equals

$$\sum_{k=1}^{m} (1 - \pi(1 + c)) e^1_{k-1}(0)$$

for an underlying fixed coupon bond and

$$\sum_{k=1}^{m} \Big((1 - \pi(1 + \delta_{k-1} x)) e^1_{k-1}(0) - \pi \delta_{k-1} e^{L(T_{k-1}, T_{k-1})}_{k-1}(0) \Big)$$

for a floating coupon bond. Consequently, the default swap rates are

$$s_{\text{fixed}} = \frac{1 - \pi(1 + c)}{\sum_{k=1}^{m} \overline{B}(0, T_{k-1})} \sum_{k=1}^{m} \Big(\overline{B}(0, T_k) \delta_{k-1} \mathbb{E}_{\overline{\mathbb{P}}_{T_k}} [H(T_{k-1}, T_{k-1})] \Big)$$

and

$$s_{\text{floating}} = \frac{1}{\sum_{k=1}^{m} \overline{B}(0,T_{k-1})} \sum_{k=1}^{m} \Big(\overline{B}(0,T_k)\delta_{k-1}\Big((1-\pi(1+\delta_{k-1}x)) \\ \times \mathbb{E}_{\overline{\mathbb{P}}_{T_k}}[H(T_{k-1},T_{k-1})] - \pi\delta_{k-1}\mathbb{E}_{\overline{\mathbb{P}}_{T_k}}[H(T_{k-1},T_{k-1})L(T_{k-1},T_{k-1})]\Big)\Big).$$

The expectations in these equations can be obtained as in the previous section.

4.6.2 Total rate of return swaps

Total rate of return swaps (sometimes also called *total return swaps*) belong to the class of *synthetic securitizations*. The total return of some reference entity (e.g. a defaultable coupon bond, a basket of assets, etc.) is exchanged for periodic fixed or floating payments.

As an example, let us consider a total return swap with maturity date T_m and a fixed coupon bond issued by some reference party C that matures at T_M ($m \leq M \leq n$) as reference entity. The payer party A agrees to pay the total return of the bond to the receiver party B in return for periodic fixed payments during the lifetime of the contract. The total return consists of the coupons as well as of the change in the value of the bond, i.e. the difference between the bond's price at T_0 and maturity date of the contract or time of default of the bond (whatever comes first). The change in the bond's value can (and in case of a default before the swap terminates usually will) be negative. If the reference entity defaults, the swap contract terminates. Note that B carries the price risk (including credit risk) of the reference coupon bond. From his point of view, the swap is similar to a synthetic purchase of the bond.
The payment streams of this swap are as follows:

- If no default occurs in $(T_k, T_{k+1}]$ $(0 \leq k \leq m-1)$, B receives an amount of $(c-s)$ at T_{k+1}, where s is the fixed periodic payment of B and c is the coupon of the underlying bond.

- If a default occurs in $(T_k, T_{k+1}]$ $(0 \leq k \leq m-1)$, B receives an amount of $\pi(1+c) - B^{\pi}_{\text{fixed}}(0;c,M)$ at T_{k+1} and the swap then terminates.

- If no default occurs until T_m, B receives an amount of $B^{\pi}_{\text{fixed}}(T_m;c,M) - B^{\pi}_{\text{fixed}}(0;c,M)$ at T_m.

From B's point of view, the initial value of the contract is

$$(c-s)\sum_{k=1}^{m}\overline{B}(0,T_k) + (\pi(1+c) - B^{\pi}_{\text{fixed}}(0;c,M))\sum_{k=1}^{m} e^1_{k-1}(0) \\ +v^{\pi}_0(m,M,c) - \overline{B}(0,T_m)B^{\pi}_{\text{fixed}}(0;c,M).$$

Here, $v^{\pi}_0(m,M,c)$ denotes the value at time-0 of receiving, at T_m, an amount of $B^{\pi}_{\text{fixed}}(T_m;c,M)$ if no default occurs until T_m. Note that

$$\begin{aligned}\mathbb{1}_{\{\tau>T_m\}}B^{\pi}_{\text{fixed}}(T_m;c,M) = \mathbb{1}_{\{\tau>T_m\}}\Bigg(&\overline{B}(T_m,T_M)\\ &+\sum_{k=m}^{M-1}\overline{B}(T_m,T_{k+1})\left(c+\pi(1+c)\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)|\mathcal{F}_{T_m}]\right)\Bigg).\end{aligned}$$

Hence, by proposition 4.15,

$$\begin{aligned}v_0^{\pi}(m,M,c) = \overline{B}(0,T_m)\mathbb{E}_{\overline{\mathbb{P}}_{T_m}}\Bigg[&\overline{B}(T_m,T_M)\\ &+\sum_{k=m}^{M-1}\overline{B}(T_m,T_{k+1})\left(c+\pi(1+c)\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)|\mathcal{F}_{T_m}]\right)\Bigg].\end{aligned}$$

Using lemma 4.14 we get for $k\in\{m,\ldots,M-1\}$

$$\begin{aligned}\mathbb{E}_{\overline{\mathbb{P}}_{T_m}}[\overline{B}(T_m,T_{k+1})] &= \frac{\overline{B}(0,T_{k+1})}{\overline{B}(0,T_m)}\mathbb{E}_{\overline{\mathbb{P}}_{T_m}}\left[\prod_{l=m}^{k}\frac{1+\delta_l\overline{L}(0,T_l)}{1+\delta_l\overline{L}(T_m,T_l)}\right]\\ &= \frac{\overline{B}(0,T_{k+1})}{\overline{B}(0,T_m)}\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[1] = \frac{\overline{B}(0,T_{k+1})}{\overline{B}(0,T_m)}\end{aligned}$$

and similarly

$$\mathbb{E}_{\overline{\mathbb{P}}_{T_m}}\left[\overline{B}(T_m,T_{k+1})\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)|\mathcal{F}_{T_m}]\right] = \frac{\overline{B}(0,T_{k+1})}{\overline{B}(0,T_m)}\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)].$$

Thus,

$$v_0^{\pi}(m,M,c) = \overline{B}(0,T_M)+\sum_{k=m}^{M-1}\overline{B}(0,T_{k+1})\left(c+\pi(1+c)\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)]\right).$$

We can determine the fixed periodic payment s that makes the initial value of the contract equal to zero as

$$\begin{aligned}s = c-\left(\sum_{k=1}^{m}\overline{B}(0,T_k)\right)^{-1}\Bigg(&(B^{\pi}_{\text{fixed}}(0;c,M)-\pi(1+c))\sum_{k=1}^{m}e^1_{k-1}(0)\\ &+\overline{B}(0,T_m)B^{\pi}_{\text{fixed}}(0;c,M)-v_0^{\pi}(m,M,c)\Bigg).\end{aligned}$$

Similar formulae can be derived for the cases that the reference entity is a floating coupon bond or the periodic payment is floating.

4.6.3 Asset swaps

An *asset swap package* or *asset swap* is a combination of a fixed coupon bond issued by some reference party C and a fixed-for-floating interest rate swap. Holding an asset swap is thus similar to a position in a floating coupon bond issued by C. However, the interest rate swap usually remains in force even if the bond defaults. For more details and a pricing formula that can be employed in our model framework we refer to Schönbucher (1999a).

4.7 Credit options

The term *credit option* refers to calls and puts on floating rate or fixed rate bonds issued by corporations as well as to options on asset swap packages, credit default swaps and on many other credit derivatives. The purpose of this section is to price some popular credit options within our model framework under the following restriction on the volatility functions:

Assumption ($\mathbb{DLR}.\mathbb{VOL}$). *The volatility structures factorize in the following way: for $i \in \{1, \dots, n-1\}$*

$$\lambda(s, T_i) = \lambda_i \sigma(s) \qquad \textit{and} \qquad \gamma(s, T_i) = \gamma_i \sigma(s) \qquad (0 \le s \le T_i)$$

where λ_i and γ_i are positive constants and where $\sigma : [0, T^*] \to \mathbb{R}^d_+$ does not depend on i.

This condition allows us to derive approximate pricing formulae that can numerically be evaluated fast. As in the previous section we neglect the counterparty risk.

4.7.1 Options on defaultable bonds

Options on defaultable bonds are indeed traded, as Schmid (2004, p. 147) remarks: "For quite a big number of liquid bonds, such as some of the Latin American Brady bonds and bonds of large US corporates, there is a well-developed bond option market". To keep formulae (relatively) simple, we examine European call options on defaultable zero coupon bonds. The employed techniques can also be used to price options on coupon bearing bonds.

Let us consider a call with maturity T_i and strike $K \in (0, 1)$ on a defaultable zero coupon bond with maturity T_m ($i < m \le n$). The payoff at maturity is either $1\!\!1_{\{\tau > T_i\}} (B^\pi(T_i, T_m) - K)^+$ or $(B^\pi(T_i, T_m) - K)^+$ depending on whether or not the option is knocked out at default. In the following, we only consider the situation that the option is knocked out at default. The other case can be treated similarly.

The time-T_i payoff of the option is given by

$$\begin{aligned}
\pi^{\mathrm{CO}}_{T_i}(K, T_i, T_m) &:= 1\!\!1_{\{\tau > T_i\}} (B^\pi(T_i, T_m) - K)^+ \\
&= 1\!\!1_{\{\tau > T_i\}} \left(\pi B(T_i, T_m) + (1-\pi)\overline{B}(T_i, T_m) - K\right)^+ \\
&= 1\!\!1_{\{\tau > T_i\}} \Bigg(\pi \prod_{l=i}^{m-1} (1 + \delta_l L(T_i, T_l))^{-1} \\
&\qquad + (1-\pi) \prod_{l=i}^{m-1} (1 + \delta_l \overline{L}(T_i, T_l))^{-1} - K\Bigg)^+ .
\end{aligned}$$

To price the call we use Laplace transform methods and derive a convolution representation of the option value first. Proposition 4.15 yields

$$
\begin{aligned}
\pi_0^{\mathrm{CO}} &:= \pi_0^{\mathrm{CO}}(K, T_i, T_m) \\
&= \overline{B}(0, T_i) \mathbb{E}_{\overline{\mathbb{P}}_{T_i}} \Bigg[\Bigg(\pi \prod_{l=i}^{m-1} (1 + \delta_l L(T_i, T_l))^{-1} \\
&\qquad\qquad + (1-\pi) \prod_{l=i}^{m-1} (1 + \delta_l \overline{L}(T_i, T_l))^{-1} - K \Bigg)^+ \Bigg] \\
&= \overline{B}(0, T_m) \mathbb{E}_{\overline{\mathbb{P}}_{T_m}} \Bigg[\prod_{l=i}^{m-1} (1 + \delta_l \overline{L}(T_i, T_l)) \Bigg(\pi \prod_{l=i}^{m-1} (1 + \delta_l L(T_i, T_l))^{-1} \\
&\qquad\qquad + (1-\pi) \prod_{l=i}^{m-1} (1 + \delta_l \overline{L}(T_i, T_l))^{-1} - K \Bigg)^+ \Bigg] \\
&= \overline{B}(0, T_m) \mathbb{E}_{\overline{\mathbb{P}}_{T_m}} \Bigg[\Bigg(\pi \prod_{l=i}^{m-1} (1 + \delta_l H(T_i, T_l)) + (1-\pi) \\
&\qquad\qquad - K \prod_{l=i}^{m-1} \Big((1 + \delta_l L(T_i, T_l))(1 + \delta_l H(T_i, T_l)) \Big) \Bigg)^+ \Bigg].
\end{aligned}
$$

Combining the equations (3.13), (3.15), (4.3), and (4.15) with (3.18) and (3.20) and again using the abbreviations $V_t^i := \frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}$ and $Y_t^i := \frac{\delta_i H(t,T_i)}{1+\delta_i H(t,T_i)}$ yields for $k \in \{1, \ldots, n-1\}$

$$
L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b^L(s, T_k, T^*) \,\mathrm{d}s + \int_0^t \lambda(s, T_k) \,\mathrm{d}L_s^{T^*} \right)
$$

with

$$
\begin{aligned}
&b^L(s, T_k, T^*) = \\
&\quad - \frac{1}{2} \langle \lambda(s, T_k), c_s \lambda(s, T_k) \rangle - \sum_{j=k+1}^{n-1} V_{s-}^j \langle \lambda(s, T_j), c_s \lambda(s, T_k) \rangle \\
&\quad - \int_{\mathbb{R}^d} \left(\left(e^{\langle \lambda(s, T_k), x \rangle} - 1 \right) \prod_{j=k+1}^{n-1} \beta(s, x, T_j, T_{j+1}) - \langle \lambda(s, T_k), x \rangle \right) F_s^{T^*}(\mathrm{d}x)
\end{aligned}
$$

and

$$
H(t, T_k) = H(0, T_k) \exp\left(\int_0^t b^H(s, T_k, T^*) \,\mathrm{d}s + \int_0^t \gamma(s, T_k) \,\mathrm{d}L_s^{T^*} \right)
$$

with

$$
\begin{aligned}
&b^H(s,T_k,T^*) = \qquad (4.26)\\
&\quad \sum_{j=1}^{k} Y_{s-}^j \langle \gamma(s,T_j), c_s\gamma(s,T_k)\rangle - \frac{1}{2}\langle \gamma(s,T_k), c_s\gamma(s,T_k)\rangle \\
&\quad + \sum_{j=1}^{k-1}\left(\frac{Y_{s-}^j V_{s-}^k}{Y_{s-}^k}\langle \gamma(s,T_j), c_s\lambda(s,T_k)\rangle\right) - \sum_{j=k+1}^{n-1} V_{s-}^j \langle \lambda(s,T_j), c_s\gamma(s,T_k)\rangle \\
&\quad - \int_{\mathbb{R}^d}\left(\frac{\left(e^{\langle\gamma(s,T_k),x\rangle}-1\right)\prod_{j=k+1}^{n-1}\beta(s,x,T_j,T_{j+1})}{\prod_{j=1}^{k}\left(1+Y_{s-}^j\left(e^{\langle\gamma(s,T_j),x\rangle}-1\right)\right)} - \langle\gamma(s,T_k),x\rangle\right) F_s^{T^*}(\mathrm{d}x) \\
&\quad + \int_{\mathbb{R}^d} (Y_{s-}^k)^{-1}(\beta(s,x,T_k,T_{k+1})-1)\prod_{j=k+1}^{n-1}\beta(s,x,T_j,T_{j+1}) \\
&\qquad\qquad \times\left(1-\prod_{j=1}^{k-1}\left(1+Y_{s-}^j\left(e^{\langle\gamma(s,T_j),x\rangle}-1\right)\right)^{-1}\right) F_s^{T^*}(\mathrm{d}x).
\end{aligned}
$$

As in the previous section, we approximate the stochastic terms V_{s-}^i and Y_{s-}^i in the drift terms $b^L(s,T_k,T^*)$ and $b^H(s,T_k,T^*)$ by their deterministic initial values and call the resulting (deterministic) drifts $b_0^L(s,T_k,T^*)$ and $b_0^H(s,T_k,T^*)$ respectively (remember that V_{s-}^i is also contained in $\beta(s,x,T_i,T_{i+1})$). Then, due to the assumption on the volatility structure, we get

$$
\begin{aligned}
\pi_0^{\mathrm{CO}} = \overline{B}(0,T_m)\mathbb{E}_{\overline{\mathbb{P}}_{T_m}}\Bigg[\Bigg(&\pi\prod_{l=i}^{m-1}\left(1+\delta_l H(0,T_l)\exp\left(\frac{\gamma_l}{\sigma_{\mathrm{sum}}}X_{T_i}+B_l^H\right)\right) \\
&+(1-\pi) - K\prod_{l=i}^{m-1}\left(\left(1+\delta_l L(0,T_l)\exp\left(\frac{\lambda_l}{\sigma_{\mathrm{sum}}}X_{T_i}+B_l^L\right)\right)\right. \\
&\left.\left(1+\delta_l H(0,T_l)\exp\left(\frac{\gamma_l}{\sigma_{\mathrm{sum}}}X_{T_i}+B_l^H\right)\right)\right)\Bigg)^+\Bigg]
\end{aligned}
$$

with

$$
\begin{aligned}
\sigma_{\mathrm{sum}} &:= \sum_{l=i}^{m-1}(\lambda_l+\gamma_l), \\
X_{T_i} &:= \int_0^{T_i}\sum_{l=i}^{m-1}(\lambda(s,T_l)+\gamma(s,T_l))\,\mathrm{d}L_s^{T^*} = \sigma_{\mathrm{sum}}\int_0^{T_i}\sigma(s)\,\mathrm{d}L_s^{T^*},
\end{aligned}
$$

and

$$
B_l^L := \int_0^{T_i} b_0^L(s,T_l,T^*)\,\mathrm{d}s, \qquad B_l^H := \int_0^{T_i} b_0^H(s,T_l,T^*)\,\mathrm{d}s.
$$

Note that the option price depends on the distribution of one random variable only, namely on the distribution of X_{T_i} with respect to $\overline{\mathbb{P}}_{T_m}$. Assume that this distribution possesses a Lebesgue-density φ (see section 2.2 for a discussion on this assumption). The option price can then be written as a convolution, namely

$$\pi_0^{\mathrm{CO}} = \overline{B}(0,T_m)\int_{\mathbb{R}} g(-x)\varphi(x)\,\mathrm{d}x = \overline{B}(0,T_m)(g*\varphi)(0) \tag{4.27}$$

with $g(x) := (v(x))^+$ and

$$\begin{aligned} v(x) := \pi \prod_{l=i}^{m-1} &\Big(1+\delta_l H(0,T_l)\exp\Big(-\frac{\gamma_l}{\sigma_{\mathrm{sum}}}x + B_l^H\Big)\Big) \\ &+ (1-\pi) - K\prod_{l=i}^{m-1}\Big(\Big(1+\delta_l L(0,T_l)\exp\Big(-\frac{\lambda_l}{\sigma_{\mathrm{sum}}}x+B_l^L\Big)\Big) \\ &\qquad\qquad \Big(1+\delta_l H(0,T_l)\exp\Big(-\frac{\gamma_l}{\sigma_{\mathrm{sum}}}x+B_l^H\Big)\Big)\Big). \end{aligned}$$

The next step is to determine the bilateral Laplace transform of g. Observe that we can write v as

$$\begin{aligned} v(x) = \prod_{l=i}^{m-1} &\Big(\Big(1+\delta_l L(0,T_l)\exp\Big(-\frac{\lambda_l}{\sigma_{\mathrm{sum}}}x+B_l^L\Big)\Big) \\ &\quad \Big(1+\delta_l H(0,T_l)\exp\Big(-\frac{\gamma_l}{\sigma_{\mathrm{sum}}}x+B_l^H\Big)\Big)\Big) \\ \times\Bigg(&\pi\prod_{l=i}^{m-1}\Big(1+\delta_l L(0,T_l)\exp\Big(-\frac{\lambda_l}{\sigma_{\mathrm{sum}}}x+B_l^L\Big)\Big)^{-1} \\ &+(1-\pi)\prod_{l=i}^{m-1}\Big(\Big(1+\delta_l L(0,T_l)\exp\Big(-\frac{\lambda_l}{\sigma_{\mathrm{sum}}}x+B_l^L\Big)\Big) \\ &\qquad \Big(1+\delta_l H(0,T_l)\exp\Big(-\frac{\gamma_l}{\sigma_{\mathrm{sum}}}x+B_l^H\Big)\Big)\Big)^{-1} - K\Bigg). \end{aligned}$$

v has a unique zero Z since the first $m-i$ factors on the right-hand side are positive and the last factor is continuous, strictly increasing, takes positive as well as negative values, and hence has a unique zero. Consequently,

$$g(x) = v(x)\mathbb{1}_{[Z,\infty)}(x).$$

Note that $v(x)$ can also be written as a finite sum of expressions of the type "$c_1\exp(-c_2 x)$" with $c_1 \in \mathbb{R}$ and $c_2 \in [0,1]$. For $z \in \mathbb{C}$ with $\Re z > 0$ we get

$$\int_{\mathbb{R}} e^{-zx}(c_1 e^{-c_2 x})\mathbb{1}_{[Z,\infty)}(x)\,\mathrm{d}x = \frac{c_1}{z+c_2}e^{-Z(z+c_2)}.$$

Hence, the Laplace transform of g exists for all $z \in \mathbb{C}$ with $\Re z > 0$ and a closed form expression (depending on Z) can be derived. However, since the number of summands of the above form in v increases exponentially as $(m-i)$ increases, a numerical evaluation of the Laplace transform is (at least for large values of $(m-i)$) more appropriate. Putting pieces together, we obtain the following formula for the option price:

Proposition 4.17 *Suppose that the distribution of X_{T_i} possesses a Lebesgue-density. Denote by $\overline{M}_{T_m}^{X_{T_i}}$ the $\overline{\mathbb{P}}_{T_m}$-moment generating function of X_{T_i}. Choose an $R > 0$ such that $\overline{M}_{T_m}^{X_{T_i}}(-R) < \infty$. Then the price of the call is approximately given by*

$$\pi_0^{CO}(K, T_i, T_m) = \overline{B}(0, T_m)\frac{1}{\pi}\int_0^\infty \Re\Big(L[g](R + \mathrm{i}u)\overline{M}_{T_m}^{X_{T_i}}(-R - \mathrm{i}u)\Big)\,\mathrm{d}u, \quad (4.28)$$

where $L[g]$ denotes the bilateral Laplace transform of g. Furthermore, we have

$$\overline{M}_{T_m}^{X_{T_i}}(-R - \mathrm{i}u) \approx \exp\int_0^{T_i}\Big(\theta_s(f^m(s) - (R + \mathrm{i}u)\sigma_{sum}\sigma(s)) - \theta_s(f^m(s))\Big)\,\mathrm{d}s$$

with

$$f^m(s) := -\sum_{l=1}^{m-1}\frac{\delta_l H(0, T_l)}{1 + \delta_l H(0, T_l)}\gamma(s, T_l) + \sum_{l=m}^{n-1}\frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)}\lambda(s, T_l).$$

PROOF: Using the convolution representation (4.27) and performing Laplace and inverse Laplace transformations, we get

$$\pi_0^{\mathrm{CO}} = \overline{B}(0, T_m)\frac{1}{\pi}\int_0^\infty \Re\Big(L[g](R + \mathrm{i}u)\overline{M}_{T_m}^{X_{T_i}}(-R - \mathrm{i}u)\Big)\,\mathrm{d}u.$$

It remains to derive an expression for the moment generating function. Observe that

$$\begin{aligned}\overline{M}_{T_m}^{X_{T_i}}(z) &= \mathbb{E}_{\overline{\mathbb{P}}_{T_m}}\left[\exp\left(z\sigma_{\mathrm{sum}}\int_0^{T_i}\sigma(s)\,\mathrm{d}L_s^{T^*}\right)\right] \\ &= \mathbb{E}_{\mathbb{P}_{T^*}}\left[Z_{T_i}^m\exp\left(z\sigma_{\mathrm{sum}}\int_0^{T_i}\sigma(s)\,\mathrm{d}L_s^{T^*}\right)\right],\end{aligned}$$

where $(Z_t^m)_{0\le t\le T_m}$ denotes the density process of $\overline{\mathbb{P}}_{T_m}$ with respect to $\mathbb{P}_{T^*}$ (which is of course a $\mathbb{P}_{T^*}$-martingale), given by

$$\begin{aligned}Z_t^m &:= \prod_{l=1}^{m-1}\frac{1 + \delta_l H(0, T_l)}{1 + \delta_l H(t, T_l)}\prod_{l=m}^{n-1}\frac{1 + \delta_l L(t, T_l)}{1 + \delta_l L(0, T_l)} \\ &= \prod_{l=1}^{m-1}\frac{1 + \delta_l H(0, T_l)}{1 + \delta_l H(0, T_l)\exp\left(\int_0^t \gamma(s, T_l)\,\mathrm{d}L_s^{T^*} + \mathrm{drift}\right)} \\ &\quad\times\prod_{l=m}^{n-1}\frac{1 + \delta_l L(0, T_l)\exp\left(\int_0^t \lambda(s, T_l)\,\mathrm{d}L_s^{T^*} + \mathrm{drift}\right)}{1 + \delta_l L(0, T_l)}.\end{aligned}$$

We use an approximation that has already been employed in the derivation of caplet prices in the default-free Lévy Libor model. Note that $1+\varepsilon\exp(x) \approx (1+\varepsilon)\exp\left(\frac{\varepsilon}{1+\varepsilon}x\right)$ for small absolute values of x and approximate (compare (3.24))

$$Z_t^m \approx \exp\left(\int_0^t f^m(s)\,\mathrm{d}L_s^{T^*} + D_t^m\right),$$

where

$$f^m(s) := -\sum_{l=1}^{m-1} \frac{\delta_l H(0,T_l)}{1+\delta_l H(0,T_l)}\gamma(s,T_l) + \sum_{l=m}^{n-1} \frac{\delta_l L(0,T_l)}{1+\delta_l L(0,T_l)}\lambda(s,T_l)$$

and the drift term D^m is chosen in such a way that the $\mathbb{P}_{T^*}$-martingale property of Z^m is preserved, i.e.

$$D_t^m := \log\left(\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^t f^m(s)\,\mathrm{d}L_s^{T^*}\right]^{-1}\right).$$

Hence, for $z \in \mathbb{C}$ with $\Re z = -R$ we obtain

$$\begin{aligned}
\overline{M}_{T_m}^{X_{T_i}}(z) &\approx \mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^{T_i} f^m(s)\,\mathrm{d}L_s^{T^*}\right]^{-1} \\
&\qquad \times \mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\int_0^{T_i}(f^m(s)+z\sigma_{\mathrm{sum}}\sigma(s))\,\mathrm{d}L_s^{T^*}\right] \\
&= \exp\int_0^{T_i}\Big(\theta_s(f^m(s)+z\sigma_{\mathrm{sum}}\sigma(s)) - \theta_s(f^m(s))\Big)\,\mathrm{d}s,
\end{aligned}$$

where the last line follows from proposition 1.9. □

4.7.2 Credit spread options

The following definition is taken from Schmid (2004, p. 148):

Definition 4.18 *A credit spread call (put) option with maturity T and strike spread K on a defaultable bond $B^\pi(\cdot,U)$ with maturity $U \geq T$ gives the holder the right to buy (sell) the defaultable bond at time T at a price that corresponds to a yield spread of K above the yield of an otherwise identical non-defaultable bond $B(\cdot,U)$.*

Let us consider a call that is knocked out at default with maturity T_i and strike spread K on the defaultable bond $B^\pi(\cdot,T_m)$ $(i < m \leq n)$. Its time-T_i value is given by

$$\begin{aligned}\pi_{T_i}^{\text{CSO}}(K,T_i,T_m) &:= \mathbb{1}_{\{\tau>T_i\}}\Big(B^\pi(T_i,T_m)-e^{-(T_m-T_i)K}B(T_i,T_m)\Big)^+\\ &= \mathbb{1}_{\{\tau>T_i\}}\Big((1-\pi)\overline{B}(T_i,T_m)-(e^{-(T_m-T_i)K}-\pi)B(T_i,T_m)\Big)^+\end{aligned}$$

In the following, only the case $\pi < e^{-(T_m-T_i)K}$ is considered. In any other case, the call will always be exercised (and is therefore no real option). Proceeding similarly as in the previous section we get

$$\pi_0^{\text{CSO}}(K,T_i,T_m) = \overline{B}(0,T_m)(g*\varphi)(0)$$

with $g(x) := (v(x))^+$ and

$$\begin{aligned}&v(x) :=\\ &\quad(1-\pi)-(e^{-(T_m-T_i)K}-\pi)\prod_{l=i}^{m-1}\left(1+\delta_l H(0,T_l)\exp\Big(-\frac{\gamma_l}{\sigma_{\text{sum}}}x+B_l^H\Big)\right).\end{aligned}$$

Here, φ, σ_{sum}, and B_l^H are defined as in section 4.7.1. Since v is continuous, strictly increasing and takes negative as well as positive values it has a unique zero. We can conclude, as in the previous section, that the bilateral Laplace transform of g exists for all $z \in \mathbb{C}$ with $\Re z > 0$. By applying exactly the same arguments as before we arrive at formula (4.28) for the price of the call. Hence, the only difference in the pricing formulae for a call on a defaultable bond and a credit spread call lies in the different Laplace transforms $L[g]$.

4.7.3 Credit default swaptions

A *credit default swaption* gives its holder the right to enter a credit default swap at some pre-specified time and swap rate. These options are often embedded in other credit derivatives (e.g. as an extension option in a credit default swap). For more details we refer to Schönbucher (1999a).

Let us consider a credit default swaption that is knocked out at default with strike rate S and maturity T_i on a default swap that terminates at T_m ($i < m \leq n$) with an underlying fixed coupon bond. Its time-T_i value is

$$\pi_{T_i}^{\text{CDS}}(S,T_i,T_m) := \mathbb{1}_{\{\tau>T_i\}}\left((s(T_i;T_i,T_m)-S)^+\sum_{k=i}^{m-1}\overline{B}(T_i,T_k)\right),$$

where $s(t;T_i,T_m)$ denotes the forward default swap rate at time t. Note that

$$\begin{aligned}&\mathbb{1}_{\{\tau>T_i\}}s(T_i,T_i,T_m)\sum_{k=i}^{m-1}\overline{B}(T_i,T_k) =\\ &\qquad\mathbb{1}_{\{\tau>T_i\}}(1-\pi(1+c))\sum_{k=i}^{m-1}\Big(\overline{B}(T_i,T_{k+1})\delta_k\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)|\mathcal{F}_{T_i}]\Big).\end{aligned}$$

As before, we approximate the stochastic terms V^i_{s-} and Y^i_{s-} in (4.24) by their deterministic initial values V^i_0 and Y^i_0 and obtain

$$\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)|\mathcal{F}_{T_i}] \approx C^{i,k} H(T_i,T_k)$$

with

$$\begin{aligned} C^{i,k} := \exp\Bigg(& \int_{T_i}^{T_k} \sum_{l=1}^{k-1} \frac{Y^l_0 V^k_0}{Y^k_0} \langle \gamma(s,T_l), c_s \lambda(s,T_k)\rangle \,\mathrm{d}s \\ & + \int_{T_i}^{T_k}\int_{\mathbb{R}^d} \frac{V^k_0}{Y^k_0} \left(e^{\langle \lambda(s,T_k),x\rangle} - 1\right)\left(1 + Y^k_0\left(e^{\langle \gamma(s,T_k),x\rangle} - 1\right)\right) \\ & \qquad \times \left(\prod_{l=1}^{k-1}\left(1 + Y^l_0\left(e^{\langle \gamma(s,T_l),x\rangle} - 1\right)\right) - 1\right) \widetilde{\nu}^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x)\Bigg) \end{aligned}$$

and $\widetilde{\nu}^{T_{k+1}}$ given by (4.25). By proposition 4.15

$$\begin{aligned} \pi^{\mathrm{CDS}}_0 &:= \pi^{\mathrm{CDS}}_0(S,T_i,T_m) \\ &= \overline{B}(0,T_i)\mathbb{E}_{\overline{\mathbb{P}}_{T_i}}\Bigg[\Bigg((1-\pi(1+c))\sum_{k=i}^{m-1}\left(\overline{B}(T_i,T_{k+1})\delta_k C^{i,k} H(T_i,T_k)\right) \\ &\qquad\qquad - S\sum_{k=i}^{m-1}\overline{B}(T_i,T_k)\Bigg)^+\Bigg] \\ &= \overline{B}(0,T_i)\mathbb{E}_{\overline{\mathbb{P}}_{T_i}}\Bigg[\Bigg(\frac{(1-\pi(1+c))\delta_{m-1}C^{i,m-1}H(T_i,T_{m-1})}{\prod_{l=i}^{m-1}(1+\delta_l L(T_i,T_l))(1+\delta_l H(T_i,T_l))} \\ &\qquad + \sum_{k=i}^{m-2}\frac{(1-\pi(1+c))\delta_k C^{i,k}H(T_i,T_k) - S}{\prod_{l=i}^{k}(1+\delta_l L(T_i,T_l))(1+\delta_l H(T_i,T_l))} - S\Bigg)^+\Bigg]. \end{aligned}$$

We proceed similarly as in the derivation of the price for a call on a defaultable bond and define σ_{sum}, X_{T_i}, B^H_l, and B^L_l as before. Assuming that the distribution of X_{T_i} with respect to $\overline{\mathbb{P}}_{T_i}$ possesses a Lebesgue-density φ we obtain

$$\pi^{\mathrm{CDS}}_0 = \overline{B}(0,T_i)(g*\varphi)(0)$$

with $g(x) := (v(x))^+$ and

$$\begin{aligned} v(x) := & \frac{(1-\pi(1+c))\delta_{m-1}C^{i,m-1}H(0,T_{m-1})\exp\left(-\frac{\gamma_{m-1}}{\sigma_{\mathrm{sum}}}x + B^H_{m-1}\right)}{\prod_{l=i}^{m-1}\left(1+\delta_l L(0,T_l)\exp\left(-\frac{\lambda_l}{\sigma_{\mathrm{sum}}}x + B^L_l\right)\right)} \\ & \quad \times \frac{1}{\prod_{l=i}^{m-1}\left(1+\delta_l H(0,T_l)\exp\left(-\frac{\gamma_l}{\sigma_{\mathrm{sum}}}x + B^H_l\right)\right)} \\ & + \sum_{k=i}^{m-2}\Bigg(\frac{(1-\pi(1+c))\delta_k C^{i,k}H(0,T_k)\exp\left(-\frac{\gamma_k}{\sigma_{\mathrm{sum}}}x + B^H_k\right) - S}{\prod_{l=i}^{k}\left(1+\delta_l L(0,T_l)\exp\left(-\frac{\lambda_l}{\sigma_{\mathrm{sum}}}x + B^L_l\right)\right)} \\ & \qquad \times \frac{1}{\prod_{l=i}^{k}\left(1+\delta_l H(0,T_l)\exp\left(-\frac{\gamma_l}{\sigma_{\mathrm{sum}}}x + B^H_l\right)\right)}\Bigg) - S. \end{aligned}$$

Note that v is continuous, tends to $-S$ as $x \to -\infty$ and to $-(m-i)S$ as $x \to \infty$. Consequently, g has compact support and the bilateral Laplace transform of g exists for all $z \in \mathbb{C}$. In a numerical evaluation of the Laplace transform, for large values of $m-i$, we can save computational time by applying the multiplication scheme

$$\sum_{k=i}^{m-2} c_k \prod_{l=i}^{k} d_l = d_i(c_i + d_{i+1}(c_{i+1} + d_{i+1}(\dots(c_{m-3} + d_{m-2}c_{m-2})))).$$

We obtain the following formula for the price of the swaption:

Proposition 4.19 *Suppose that the distribution of X_{T_i} possesses a Lebesgue-density. Denote by $\overline{M}_{T_i}^{X_{T_i}}$ the $\overline{\mathbb{P}}_{T_i}$-moment generating function of X_{T_i}. Choose an $R \in \mathbb{R}$ such that $\overline{M}_{T_i}^{X_{T_i}}(-R) < \infty$ (e.g. $R = 0$). Then the price of the credit default swaption is approximately given by*

$$\pi_0^{CDS}(K, T_i, T_m) = \overline{B}(0, T_i)\frac{1}{\pi}\int_0^\infty \Re\Big(L[g](R + \mathrm{i}u)\overline{M}_{T_i}^{X_{T_i}}(-R - \mathrm{i}u)\Big)\,\mathrm{d}u, \quad (4.29)$$

where $L[g]$ denotes the bilateral Laplace transform of g. Furthermore, we have

$$\overline{M}_{T_i}^{X_{T_i}}(-R - \mathrm{i}u) \approx \exp\int_0^{T_i}\Big(\theta_s(f^i(s) - (R + \mathrm{i}u)\sigma_{sum}\sigma(s)) - \theta_s(f^i(s))\Big)\,\mathrm{d}s$$

with

$$f^i(s) := -\sum_{l=1}^{i-1}\frac{\delta_l H(0, T_l)}{1 + \delta_l H(0, T_l)}\gamma(s, T_l) + \sum_{l=i}^{n-1}\frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)}\lambda(s, T_l).$$

Proof: As before, we use the convolution representation of the swaption price combined with Laplace and inverse Laplace transformation methods to get (4.29). The expression for the moment generating function follows as in the proof of proposition 4.17. □

4.8 Conclusion

A generalization of the *Libor market model with default risk* by Schönbucher (1999a) and extension of the *Lévy Libor model* due to Eberlein and Özkan (2005) has been introduced, the *Lévy Libor model with default risk*. A pricing formula for derivatives has been established which uses two counterparts to forward measures, namely *defaultable forward mesures* and *restricted defaultable forward measures*. Using this formula, we deduced approximate pricing solutions for some popular credit derivatives. A topic for future research is the extension of the model to rating classes.

Appendix A

Proof of Proposition 4.8

First, we derive a condition ensuring the martingale property of $\prod_{i=1}^{k-1} \frac{1}{1+\delta_i \widehat{H}(\cdot,T_i)}$ that involves the terms $b^H(\cdot, T_1, T_k), \dots, b^H(\cdot, T_{k-1}, T_k)$:

Lemma A.1 $\left(\prod_{i=1}^{k-1} \frac{1}{1+\delta_i \widehat{H}(t,T_i)}\right)_{0\le t\le T_{k-1}}$ *is a* $\mathbb{Q}_{T_k}$*-martingale if*

$$\begin{aligned}
&\sum_{i=1}^{k-1} Y^i_{s-} b^H(s, T_i, T_k) \\
&= -\frac{1}{2}\sum_{i=1}^{k-1} Y^i_{s-}\langle \gamma(s,T_i), c_s \gamma(s,T_i)\rangle + \sum_{\substack{i,j=1\\ j\ge i}}^{k-1} Y^i_{s-} Y^j_{s-} \langle \gamma(s,T_i), c_s\gamma(s,T_j)\rangle \\
&\quad + \int_{\mathbb{R}^d} \left(\sum_{i=1}^{k-1} \left(Y^i_{s-}\langle \gamma(s,T_i), x\rangle\right) - 1 + \prod_{i=1}^{k-1}\left(1 + Y^i_{s-}\left(e^{\langle \gamma(s,T_i),x\rangle} - 1\right)\right)^{-1}\right) F^{T_k}_s(\mathrm{d}x).
\end{aligned}$$

where $Y^i_s := \frac{\delta_i \widehat{H}(s,T_i)}{1+\delta_i \widehat{H}(s,T_i)}$.

Proof: Let us denote $X^i_t := 1 + \delta_i \widehat{H}(t,T_i)$, then using lemma 4.6, the fact that $\mathrm{d}X^i_t = \delta_i \,\mathrm{d}\widehat{H}(t,T_i)$, and Kallsen and Shiryaev (2002, Lemma 2.6) we get

$$\begin{aligned}
X^i_t &= X^i_0\, \mathcal{E}_t\Bigg(\int_0^{\bullet} Y^i_{s-} a(s,T_i,T_k)\,\mathrm{d}s + \int_0^{\bullet} Y^i_{s-}\sqrt{c_s}\gamma(s,T_i)\,\mathrm{d}W^{T_k}_s \\
&\qquad + \int_0^{\bullet}\int_{\mathbb{R}^d} Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle} - 1\right)(\mu - \nu^{T_k})(\mathrm{d}s,\mathrm{d}x)\Bigg) \\
&= X^i_0 \exp\Bigg(\int_0^t D(s,T_i,T_k)\,\mathrm{d}s + \int_0^t Y^i_{s-}\sqrt{c_s}\gamma(s,T_i)\,\mathrm{d}W^{T_k}_s \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} \log\left(1 + Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle} - 1\right)\right)(\mu - \nu^{T_k})(\mathrm{d}s,\mathrm{d}x)\Bigg),
\end{aligned}$$

where

$$
\begin{aligned}
&D(s,T_i,T_k)\\
&\quad:= \; Y^i_{s-}a(s,T_i,T_k) - \frac{1}{2}(Y^i_{s-})^2\langle\gamma(s,T_i),c_s\gamma(s,T_i)\rangle\\
&\qquad + \int\limits_{\mathbb{R}^d}\left(\log\left(1+Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle}-1\right)\right) - Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle}-1\right)\right)F^{T_k}_s(\mathrm{d}x)\\
&\quad\overset{(4.10)}{=} \; Y^i_{s-}b^H(s,T_i,T_k) + \frac{1}{2}\left(Y^i_{s-}-(Y^i_{s-})^2\right)\langle\gamma(s,T_i),c_s\gamma(s,T_i)\rangle\\
&\qquad + \int\limits_{\mathbb{R}^d}\left(\log\left(1+Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle}-1\right)\right) - Y^i_{s-}\langle\gamma(s,T_i),x\rangle\right)F^{T_k}_s(\mathrm{d}x).
\end{aligned}
$$

Consequently, using Kallsen and Shiryaev (2002, Lemma 2.6) once again,

$$
\begin{aligned}
&\left(\prod_{i=1}^{k-1}\frac{X^i_t}{X^i_0}\right)^{-1}\\
&\quad = \exp\Bigg(-\int_0^t\sum_{i=1}^{k-1}D(s,T_i,T_k)\,\mathrm{d}s - \int_0^t\sum_{i=1}^{k-1}(Y^i_{s-}\sqrt{c_s}\gamma(s,T_i))\,\mathrm{d}W^{T_k}_s\\
&\qquad - \int_0^t\int\limits_{\mathbb{R}^d}\log\left(\prod_{i=1}^{k-1}\left(1+Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle}-1\right)\right)\right)(\mu-\nu^{T_k})(\mathrm{d}s,\mathrm{d}x)\Bigg)\\
&\quad = \mathcal{E}_t\Bigg(\int_0^{\bullet}A(s,T_k)\,\mathrm{d}s - \int_0^{\bullet}\sum_{i=1}^{k-1}(Y^i_{s-}\sqrt{c_s}\gamma(s,T_i))\,\mathrm{d}W^{T_k}_s \qquad\qquad (A.1)\\
&\qquad + \int_0^{\bullet}\int\limits_{\mathbb{R}^d}\left(\prod_{i=1}^{k-1}\left(1+Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle}-1\right)\right)^{-1}-1\right)(\mu-\nu^{T_k})(\mathrm{d}s,\mathrm{d}x)\Bigg)
\end{aligned}
$$

with

$$
\begin{aligned}
A(s,T_k) := & -\sum_{i=1}^{k-1}D(s,T_i,T_k) + \frac{1}{2}\sum_{i,j=1}^{k-1}Y^i_{s-}Y^j_{s-}\langle\gamma(s,T_i),c_s\gamma(s,T_j)\rangle\\
& + \int\limits_{\mathbb{R}^d}\Bigg(\prod_{i=1}^{k-1}\left(1+Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle}-1\right)\right)^{-1}-1\\
&\qquad + \log\left(\prod_{i=1}^{k-1}\left(1+Y^i_{s-}\left(e^{\langle\gamma(s,T_i),x\rangle}-1\right)\right)\right)\Bigg)F^{T_k}_s(\mathrm{d}x).
\end{aligned}
$$

$\prod_{i=1}^{k-1}(1+\delta_i\widehat{H}(t,T_i))^{-1}$ is a $\mathbb{Q}_{T_k}$-local martingale if $A(s,T_k)=0$ for all s. In this case it is also a martingale since it is bounded by 0 and 1 and therefore of class $[D]$ (compare Jacod and Shiryaev (2003, I.1.47c)). Plugging in the expression for $D(\cdot, T_i, T_k)$ yields

$$
\begin{aligned}
&A(s,T_k) = \\
&-\sum_{i=1}^{k-1} Y_{s-}^i b^H(s,T_i,T_k) - \frac{1}{2}\sum_{i=1}^{k-1} Y_{s-}^i \langle \gamma(s,T_i), c_s\gamma(s,T_i)\rangle \\
&+\sum_{\substack{i,j=1\\ j\geq i}}^{k-1} Y_{s-}^i Y_{s-}^j \langle \gamma(s,T_i), c_s\gamma(s,T_j)\rangle \\
&+\int_{\mathbb{R}^d} \left(\sum_{i=1}^{k-1} \left(Y_{s-}^i \langle \gamma(s,T_i), x\rangle\right) - 1 + \prod_{i=1}^{k-1} \left(1 + Y_{s-}^i \left(e^{\langle \gamma(s,T_i),x\rangle} - 1\right)\right)^{-1} \right) F_s^{T_k}(\mathrm{d}x).
\end{aligned}
$$

□

Now assume that the drift terms $b^H(\cdot, T_i, T_{i+1})$ satisfy (4.15). Then, using equations (4.11) and (3.20) we obtain

$$
\begin{aligned}
&b^H(s,T_i,T_k) = \\
&\quad \sum_{j=1}^{i} Y_{s-}^j \langle \gamma(s,T_j), c_s\gamma(s,T_i)\rangle - \frac{1}{2}\langle \gamma(s,T_i), c_s\gamma(s,T_i)\rangle \\
&\quad +\sum_{j=1}^{i-1} \left(\frac{Y_{s-}^j}{Y_{s-}^i} \langle \gamma(s,T_j), c_s\alpha(s,T_i,T_{i+1})\rangle \right) - \left\langle \gamma(s,T_i), c_s \left(\sum_{l=i+1}^{k-1} \alpha(s,T_l,T_{l+1}) \right) \right\rangle \\
&\quad +\int_{\mathbb{R}^d} \left(\langle \gamma(s,T_i), x\rangle - \frac{\left(e^{\langle \gamma(s,T_i),x\rangle} - 1\right) \prod_{l=i+1}^{k-1} \beta(s,x,T_l,T_{l+1})}{\prod_{j=1}^{i} \left(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j),x\rangle} - 1\right)\right)} \right) F_s^{T_k}(\mathrm{d}x) \\
&\quad + (Y_{s-}^i)^{-1} \int_{\mathbb{R}^d} (\beta(s,x,T_i,T_{i+1}) - 1) \prod_{l=i+1}^{k-1} \beta(s,x,T_l,T_{l+1}) \\
&\qquad\qquad\qquad \left(1 - \prod_{j=1}^{i-1} \left(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j),x\rangle} - 1\right)\right)^{-1} \right) F_s^{T_k}(\mathrm{d}x) \\
&= b_1^H(s,T_i,T_k) + b_2^H(s,T_i,T_k) + b_3^H(s,T_i,T_k),
\end{aligned}
$$

where

$$
\begin{aligned}
b_1^H(s,T_i,T_k) &:= \sum_{j=1}^{i} Y_{s-}^j \langle \gamma(s,T_j), c_s\gamma(s,T_i)\rangle - \frac{1}{2}\langle \gamma(s,T_i), c_s\gamma(s,T_i)\rangle, \\
b_2^H(s,T_i,T_k) &:= \sum_{j=1}^{i-1} \left(\frac{Y_{s-}^j}{Y_{s-}^i} \langle \gamma(s,T_j), c_s\alpha(s,T_i,T_{i+1})\rangle \right) \\
&\qquad\qquad - \left\langle \gamma(s,T_i), c_s \left(\sum_{l=i+1}^{k-1} \alpha(s,T_l,T_{l+1}) \right) \right\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
b_3^H(s,T_i,T_k) := (Y_{s-}^i)^{-1} \int\limits_{\mathbb{R}^d} \Bigg(& Y_{s-}^i \langle \gamma(s,T_i), x\rangle \\
& + \Bigg(\prod_{j=1}^{i} \Big(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j),x\rangle} - 1\right)\Big)^{-1} - 1\Bigg) \prod_{l=i+1}^{k-1} \beta(s,x,T_l,T_{l+1}) \\
& - \Bigg(\prod_{j=1}^{i-1} \Big(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j),x\rangle} - 1\right)\Big)^{-1} - 1\Bigg) \prod_{l=i}^{k-1} \beta(s,x,T_l,T_{l+1})\Bigg) F_s^{T_k}(\mathrm{d}x).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{i=1}^{k-1} Y_{s-}^i b_2^H(s,T_i,T_k) &= \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \mathbb{1}_{\{j\le i-1\}} Y_{s-}^j \langle \gamma(s,T_j), c_s\alpha(s,T_i,T_{i+1})\rangle \\
&\quad - \sum_{i=1}^{k-1}\sum_{l=1}^{k-1} \mathbb{1}_{\{i\le l-1\}} Y_{s-}^i \langle \gamma(s,T_i), c_s\alpha(s,T_l,T_{l+1})\rangle \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{k-1} Y_{s-}^i b_3^H(s,T_i,T_k) = \int\limits_{\mathbb{R}^d} \Bigg(& \sum_{i=1}^{k-1} \left(Y_{s-}^i \langle \gamma(s,T_i), x\rangle\right) - 1 \\
& + \prod_{j=1}^{k-1} \Big(1 + Y_{s-}^j \left(e^{\langle \gamma(s,T_j),x\rangle} - 1\right)\Big)^{-1} \Bigg) F_s^{T_k}(\mathrm{d}x).
\end{aligned}
$$

Hence, $\sum_{i=1}^{k-1} Y_{s-}^i b^H(s,T_i,T_k)$ satisfies the prerequisite of lemma A.1 and the claim is proven.

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